journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 2, Article 43, 14 pp.

A NEW SUBCLASS OF k-UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

H. M. SRIVASTAVA, T. N. SHANMUGAM, C. RAMACHANDRAN, AND S. SIVASUBRAMANIAN

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA V8W 3P4, CANADA
harimsri@math.uvic.ca

DEPARTMENT OF INFORMATION TECHNOLOGY
SALALAH COLLEGE OF TECHNOLOGY
POST OFFICE BOX 608
SALALAH PC211, SULTANATE OF OMAN
drtns2001@yahoo.com

DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING, ANNA UNIVERSITY
CHENNAI 600025, TAMILNADU, INDIA
crjsp2004@yahoo.com

DEPARTMENT OF MATHEMATICS
EASWARI ENGINEERING COLLEGE
RAMAPURAM, CHENNAI 600089
TAMILNADU, INDIA
sivasaisastha@rediffmail.com

Received 31 May, 2007; accepted 15 June, 2007
Communicated by Th.M. Rassias

ABSTRACT. The main object of this paper is to introduce and investigate a subclass $\mathcal{U}(\lambda, \alpha, \beta, k)$ of normalized analytic functions in the open unit disk Δ, which generalizes the familiar class of uniformly convex functions. The various properties and characteristics for functions belonging to the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ derived here include (for example) a characterization theorem, coefficient inequalities and coefficient estimates, a distortion theorem and a covering theorem, extreme points, and the radii of close-to-convexity, starlikeness and convexity. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

Key words and phrases: Analytic functions; Univalent functions; Coefficient inequalities and coefficient estimates; Starlike functions; Convex functions; Close-to-convex functions; k-Uniformly convex functions; k-Uniformly starlike functions; Uniformly starlike functions; Hadamard product (or convolution); Extreme points; Radii of close-to-convexity, starlikeness and convexity; Integral operators.

2000 *Mathematics Subject Classification.* Primary 30C45.

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

179-07

1. INTRODUCTION AND MOTIVATION

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are *univalent* in Δ. Suppose also that, for $0 \leqq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of functions in $\mathcal{A}$ which are, respectively, *starlike of order* α in Δ and *convex of order* α in Δ (see, for example, [11]). Finally, let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions f given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0) \tag{1.2}$$

with *negative* coefficients. Silverman [9] introduced and investigated the following subclasses of the function class $\mathcal{T}$:

$$\mathcal{T}^*(\alpha) := \mathcal{S}^*(\alpha) \cap \mathcal{T} \qquad \text{and} \qquad \mathcal{C}(\alpha) := \mathcal{K}(\alpha) \cap \mathcal{T} \qquad (0 \leqq \alpha < 1). \tag{1.3}$$

Definition 1. A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ if it satisfies the following inequality:

$$\Re\left(\frac{zF'(z)}{F(z)}\right) > k\left|\frac{zF'(z)}{F(z)} - 1\right| + \beta \tag{1.4}$$
$$(0 \leqq \alpha \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1;\ k \geqq 0),$$

where

$$F(z) := \lambda\alpha z^2 f''(z) + (\lambda - \alpha) z f'(z) + (1 - \lambda + \alpha) f(z). \tag{1.5}$$

The above-defined function class $\mathcal{U}(\lambda, \alpha, \beta, k)$ is of special interest and it contains many well-known as well as new classes of analytic univalent functions. In particular, $\mathcal{U}(\lambda, \alpha, \beta, 0)$ is the class of functions with negative coefficients, which was introduced and studied recently by Kamali and Kadıoğlu [3], and $\mathcal{U}(\lambda, 0, \beta, 0)$ is the function class which was introduced and studied by Srivastava *et al.* [12] (see also Aqlan *et al.* [1]). We note that the class of k-uniformly convex functions was introduced and studied recently by Kanas and Wiśniowska [4]. Subsequently, Kanas and Wiśniowska [5] introduced and studied the class of k-uniformly starlike functions. The various properties of the above two function classes were extensively investigated by Kanas and Srivastava [6]. Furthermore, we have [*cf.* Equation (1.3)]

$$\mathcal{U}(0, 0, \beta, 0) \equiv \mathcal{T}^*(\alpha) \qquad \text{and} \qquad \mathcal{U}(1, 0, \beta, 0) \equiv \mathcal{C}(\alpha). \tag{1.6}$$

We remark here that the classes of k-uniformly starlike functions and k-uniformly convex functions are an extension of the relatively more familiar classes of uniformly starlike functions and uniformly convex functions investigated earlier by (for example) Goodman [2], Rønning [8], and Ma and Minda [7] (see also the more recent contributions on these function classes by Srivastava and Mishra [10]).

In our present investigation of the function class $\mathcal{U}(\lambda, \alpha, \beta, k)$, we obtain a characterization theorem, coefficient inequalities and coefficient estimates, a distortion theorem and a covering theorem, extreme points, and the radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{U}(\lambda, \alpha, \beta, k)$.

2. A CHARACTERIZATION THEOREM AND RESULTING COEFFICIENT ESTIMATES

We employ the technique adopted by Aqlan *et al.* [1] to find the coefficient estimates for the function class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Our main characterization theorem for this function class is stated as Theorem 1 below.

Theorem 1. *A function $f \in \mathcal{T}$ given by (1.2) is in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if*

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\} \{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n \leqq 1 - \beta \tag{2.1}$$

$$(0 \leqq \alpha \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1;\ k \geqq 0).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}} z^n \qquad (n \geqq 2). \tag{2.2}$$

Proof. By Definition 1, $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if the condition (1.4) is satisfied. Since it is easily verified that

$$\Re(\omega) > k|\omega - 1| + \beta \iff \Re\big(\omega(1 + ke^{i\theta}) - ke^{i\theta}\big) > \beta$$
$$(-\pi \leqq \theta < \pi;\ 0 \leqq \beta < 1;\ k \geqq 0),$$

the inequality (1.4) may be rewritten as follows:

$$\Re\left(\frac{zF'(z)}{F(z)}(1 + ke^{i\theta}) - ke^{i\theta}\right) > \beta \tag{2.3}$$

or, equivalently,

$$\Re\left(\frac{zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta}}{F(z)}\right) > \beta. \tag{2.4}$$

Now, by setting

$$G(z) = zF'(z)(1 + ke^{i\theta}) - F(z)ke^{i\theta}, \tag{2.5}$$

the inequality (2.4) becomes equivalent to

$$|G(z) + (1-\beta)F(z)| > |G(z) - (1+\beta)F(z)| \qquad (0 \leqq \beta < 1),$$

where $F(z)$ and $G(z)$ are defined by (1.5) and (2.5), respectively. We thus observe that

$$\begin{aligned}
&|G(z) + (1-\beta)F(z)| \\
&\quad \geqq |(2-\beta)z| - \left|\sum_{n=2}^{\infty}(n - \beta + 1)\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n z^n\right| \\
&\qquad - \left|ke^{i\theta}\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n z^n\right| \\
&\quad \geqq (2-\beta)|z| - \sum_{n=2}^{\infty}(n - \beta + 1)\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n |z|^n \\
&\qquad - k\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n |z|^n \\
&\quad \geqq (2-\beta)|z| - \sum_{n=2}^{\infty}\{(n(k+1) - (k+\beta) + 1\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}a_n |z|^n.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&|G(z)-(1+\beta)F(z)|\\&\qquad\leqq \beta|z|+\sum_{n=2}^{\infty}\{(n(k+1)-(k+\beta)-1\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&|G(z)+(1-\beta)F(z)|-|G(z)-(1+\beta)F(z)|\\&\qquad\geqq 2(1-\beta)|z|-2\sum_{n=2}^{\infty}\{(n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_n|z|^n\\&\qquad\geqq 0,\end{aligned}$$

which implies the inequality (2.1) asserted by Theorem 1.

Conversely, by setting

$$0\leqq |z|=r<1,$$

and choosing the values of z on the *positive* real axis, the inequality (2.3) reduces to the following form:

$$\Re\left(\frac{(1-\beta)-\sum_{n=2}^{\infty}\{(n-\beta)-ke^{i\theta}(n-1)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_nr^{n-1}}\right)\geqq 0, \tag{2.6}$$

which, in view of the elementary identity:

$$\Re(-e^{i\theta})\geqq -|e^{i\theta}|=-1,$$

becomes

$$\Re\left(\frac{(1-\beta)-\sum_{n=2}^{\infty}\{(n-\beta)-k(n-1)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(n-1)\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}a_nr^{n-1}}\right)\geqq 0. \tag{2.7}$$

Finally, upon letting $r\to 1-$ in (2.7), we get the desired result. □

By taking $\alpha=0$ and $k=0$ in Theorem 1, we can deduce the following corollary.

Corollary 1. *Let* $f\in\mathcal{T}$ *be given by* (1.2). *Then* $f\in\mathcal{U}(\lambda,0,\beta,0)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\beta)\{(n-1)\lambda+1\}a_n\leqq 1-\beta.$$

By setting $\alpha=0$, $\lambda=1$ and $k=0$ in Theorem 1, we get the following corollary.

Corollary 2 (Silverman [9])**.** *Let* $f\in\mathcal{T}$ *be given by* (1.2). *Then* $f\in\mathcal{C}(\beta)$ *if and only if*

$$\sum_{n=2}^{\infty}n(n-\beta)a_n\leqq 1-\beta.$$

The following coefficient estimates for $f\in\mathcal{U}(\lambda,\alpha,\beta,k)$ is an immediate consequence of Theorem 1.

Theorem 2. *If* $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ *is given by* (1.2), *then*

(2.8) $$a_n \leqq \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \qquad (n \geqq 2)$$
$$(0 \leqq \alpha \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1;\ k \geqq 0).$$

Equality in (2.8) *holds true for the function* $f(z)$ *given by* (2.2).

By taking $\alpha = k = 0$ in Theorem 2, we obtain the following corollary.

Corollary 3. *Let* $f \in \mathcal{T}$ *be given by* (1.2). *Then* $f \in \mathcal{U}(\lambda, 0, \beta, 0)$ *if and only if*

(2.9) $$a_n \leqq \frac{1-\beta}{(n-\beta)\{(n-1)\lambda+1\}} \qquad (n \geqq 2).$$

Equality in (2.9) *holds true for the function* $f(z)$ *given by*

(2.10) $$f(z) = z - \frac{1-\beta}{(n-\beta)\{(n-1)\lambda+1\}}\, z^n \qquad (n \geqq 2).$$

Lastly, if we set $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 1, we get the following familiar result.

Corollary 4 (Silverman [9])**.** *Let* $f \in \mathcal{T}$ *be given by* (1.2). *Then* $f \in \mathcal{C}(\beta)$ *if and only if*

(2.11) $$a_n \leqq \frac{1-\beta}{n(n-\beta)} \qquad (n \geqq 2).$$

Equality in (2.11) *holds true for the function* $f(z)$ *given by*

(2.12) $$f(z) = z - \frac{1-\beta}{n(n-\beta)}\, z^n \qquad (n \geqq 2).$$

3. DISTORTION AND COVERING THEOREMS FOR THE FUNCTION CLASS $\mathcal{U}(\lambda, \alpha, \beta, k)$

Theorem 3. *If* $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, *then*

(3.1) $$r - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, r^2 \leqq |f(z)| \leqq r + \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, r^2$$
$$(|z| = r < 1).$$

Equality in (3.1) *holds true for the function* $f(z)$ *given by*

(3.2) $$f(z) = z - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, z^2.$$

Proof. We only prove the second part of the inequality in (3.1), since the first part can be derived by using similar arguments. Since $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, by using Theorem 1, we find that

$$\begin{aligned}
&(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)\sum_{n=2}^{\infty} a_n \\
&\quad = \sum_{n=2}^{\infty}(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)a_n \\
&\quad \leqq \sum_{n=2}^{\infty} \{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}\, a_n \\
&\quad \leqq 1-\beta,
\end{aligned}$$

which readily yields the following inequality:

$$\sum_{n=2}^{\infty} a_n \leqq \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)}. \tag{3.3}$$

Moreover, it follows from (1.2) and (3.3) that

$$\begin{aligned}|f(z)| &= \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leqq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leqq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leqq r + \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)}\, r^2,\end{aligned}$$

which proves the second part of the inequality in (3.1). □

Theorem 4. *If* $f \in \mathcal{U}(\lambda,\alpha,\beta,k)$, *then*

$$1 - \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, r \leqq |f'(z)| \leqq 1 + \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)}\, r \quad (|z| = r < 1). \tag{3.4}$$

Equality in (3.4) *holds true for the function* $f(z)$ *given by* (3.2).

Proof. Our proof of Theorem 4 is much akin to that of Theorem 3. Indeed, since $f \in \mathcal{U}(\lambda,\alpha,\beta,k)$, it is easily verified from (1.2) that

$$|f'(z)| \leqq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leqq 1 + r \sum_{n=2}^{\infty} n a_n \tag{3.5}$$

and

$$|f'(z)| \geqq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leqq 1 + r \sum_{n=2}^{\infty} n a_n. \tag{3.6}$$

The assertion (3.4) of Theorem 4 would now follow from (3.5) and (3.6) by means of a rather simple consequence of (3.3) given by

$$\sum_{n=2}^{\infty} n a_n \leqq \frac{2(1-\beta)}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha+1)}. \tag{3.7}$$

This completes the proof of Theorem 4. □

Theorem 5. *If* $f \in \mathcal{U}(\lambda,\alpha,\beta,k)$, *then* $f \in \mathcal{T}^*(\delta)$, *where*

$$\delta := 1 - \frac{1-\beta}{(2+k-\beta)(2\lambda\alpha+\lambda-\alpha)-(1-\beta)}.$$

The result is sharp with the extremal function $f(z)$ *given by* (3.2).

Proof. It is sufficient to show that (2.1) implies that

$$\sum_{n=2}^{\infty} (n-\delta) a_n \leqq 1-\delta, \tag{3.8}$$

that is, that

$$\frac{n-\delta}{1-\delta} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta} \qquad (n \geqq 2), \tag{3.9}$$

Since (3.9) is equivalent to the following inequality:

$$\delta \leqq 1 - \frac{(n-1)(1-\beta)}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}-(1-\beta)} \qquad (n \geqq 2)$$
$$=: \Psi(n),$$

and since

$$\Psi(n) \leqq \Psi(2) \qquad (n \geqq 2),$$

(3.9) holds true for

$$n \geqq 2,\ 0 \leqq \lambda \leqq 1,\ 0 \leqq \alpha \leqq 1,\ 0 \leqq \beta < 1 \quad \text{and} \quad k \geqq 0.$$

This completes the proof of Theorem 5. □

By setting $\alpha = k = 0$ in Theorem 5, we can deduce the following result.

Corollary 5. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then*

$$f \in \mathcal{T}^*\left(\frac{\lambda(2-\beta)+\beta}{\lambda(2-\beta)+1}\right).$$

This result is sharp for the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{(\lambda+1)(2-\beta)}\ z^2.$$

For the choices $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Theorem 5, we obtain the following result of Silverman [9].

Corollary 6. *If $f \in \mathcal{C}(\beta)$, then*

$$f \in \mathcal{T}^*\left(\frac{2}{3-\beta}\right).$$

This result is sharp for the extremal function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{2(2-\beta)}\ z^2.$$

4. EXTREME POINTS OF THE FUNCTION CLASS $\mathcal{U}(\lambda, \alpha, \beta, k)$

Theorem 6. *Let*

$$f_1(z) = z \quad \textit{and}$$
$$f_n(z) = z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\ z^n \qquad (n \geqq 2).$$

Then $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$ if and only if it can be represented in the form:

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \qquad \left(\mu_n \geqq 0;\ \sum_{n=1}^{\infty} \mu_n = 1\right). \tag{4.1}$$

Proof. Suppose that the function $f(z)$ can be written as in (4.1). Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n \left(z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\, z^n \right) \\ &= z - \sum_{n=2}^{\infty} \mu_n \left(\frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \right) z^n. \end{aligned}$$

Now

$$\begin{aligned} &\sum_{n=2}^{\infty} \mu_n \left(\frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}(1-\beta)}{(1-\beta)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \right) \\ &\qquad\qquad = \sum_{n=2}^{\infty} \mu_n \\ &\qquad\qquad = 1-\mu_1 \\ &\qquad\qquad \leqq 1, \end{aligned}$$

which implies that $f \in \mathcal{U}(\lambda,\alpha,\beta,k)$.

Conversely, we suppose that $f \in \mathcal{U}(\lambda,\alpha,\beta,k)$. Then, by Theorem 2, we have

$$a_n \leqq \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \qquad (n \geqq 2).$$

Therefore, we may write

$$\mu_n = \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta}\, a_n \qquad (n \geqq 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

Then

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

with $f_n(z)$ given as in (4.1). This completes the proof of Theorem 6. □

Corollary 7. *The extreme points of the function class $f \in \mathcal{U}(\lambda,\alpha,\beta,k)$ are the functions*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{1-\beta}{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\, z^n \qquad (n \geqq 2).$$

For $\alpha = k = 0$ in Corollary 7, we have the following result.

Corollary 8. *The extreme points of $f \in \mathcal{U}(\lambda,0,\beta,0)$ are the functions*

$$f_1(z) = z \quad \textit{and} \quad f_n(z) = z - \frac{1-\beta}{\{n-\beta\}\{(n-1)\lambda+1\}}\, z^n \qquad (n \geqq 2).$$

By setting $\alpha = 0$, $\lambda = 1$ and $k = 0$ in Corollary 7, we obtain the following result obtained by Silverman [9].

Corollary 9. *The extreme points of the class* $\mathcal{C}(\beta)$ *are the functions*

$$f_1(z) = z \qquad \textit{and} \qquad f_n(z) = z - \frac{1-\beta}{n(n-\beta)}\, z^n \qquad (n \geqq 2).$$

Theorem 7. *The class* $\mathcal{U}(\lambda, \alpha, \beta, k)$ *is a convex set.*

Proof. Suppose that each of the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqq 0;\ j = 1, 2) \tag{4.2}$$

is in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z) \qquad (0 \leqq \mu \leqq 1)$$

is also in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

with the aid of Theorem 1, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\}[\mu a_{n,1} + (1-\mu) a_{n,2}] \\ &\quad \leqq \mu \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_{n,1} \\ &\qquad + (1-\mu) \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_{n,2} \\ &\quad \leqq \mu(1-\beta) + (1-\mu)(1-\beta) \\ &\quad \leqq 1 - \beta, \end{aligned} \tag{4.3}$$

which implies that $g \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Hence $\mathcal{U}(\lambda, \alpha, \beta, k)$ is indeed a convex set. □

5. MODIFIED HADAMARD PRODUCTS (OR CONVOLUTION)

For functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $(f*g)(z)$ is defined, as usual, by

$$(f*g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n =: (g*f)(z). \tag{5.1}$$

On the other hand, for functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (j = 1, 2)$$

in the class $\mathcal{T}$, we define the *modified* Hadamard product (or convolution) as follows:

$$(f_1 \bullet f_2)(z) := z - \sum_{n=2}^{\infty} a_{n,1}\, a_{n,2} z^n =: (f_2 \bullet f_1)(z). \tag{5.2}$$

Then we have the following result.

Theorem 8. *If* $f_j(z) \in \mathcal{U}(\lambda, \alpha, \beta, k)$ $(j = 1, 2)$, *then*

$$(f_1 \bullet f_2)(z) \in \mathcal{U}(\lambda, \alpha, \beta, k, \xi),$$

where

$$\xi := \frac{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha+\lambda-\alpha+1\} - 2(1-\beta)^2}{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha+\lambda-\alpha+1\} - (1-\beta)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ *given as in* (3.2).

Proof. Since $f_j(z) \in \mathcal{U}(\lambda, \alpha, \beta, k)$ $(j = 1, 2)$, we have

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} a_{n,j} \leqq 1-\beta \qquad (j = 1, 2), \tag{5.3}$$

which, in view of the Cauchy-Schwarz inequality, yields

$$\sum_{n=2}^{\infty} \frac{\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta} \sqrt{a_{n,1}\, a_{n,2}} \leqq 1. \tag{5.4}$$

We need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{\{n(k+1) - (k+\xi)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\xi}\, a_{n,1}\, a_{n,2} \leqq 1. \tag{5.5}$$

Thus, in light of (5.4) and (5.5), whenever the following inequality:

$$\frac{n-\xi}{1-\xi} \sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{n-\beta}{1-\beta} \qquad (n \geqq 2)$$

holds true, the inequality (5.5) is satisfied. We find from (5.4) that

$$\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{1-\beta}{\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}} \qquad (n \geqq 2). \tag{5.6}$$

Thus, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left(\frac{1-\beta}{\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}\right) \leqq \frac{n-\beta}{1-\beta} \qquad (n \geqq 2),$$

or, if

$$\xi \leqq \frac{(n-\beta)\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} - n(1-\beta)^2}{(n-\beta)\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} - (1-\beta)^2} \qquad (n \geqq 2),$$

then (5.4) is satisfied. Setting

$$\Phi(n) := \frac{(n-\beta)\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} - n(1-\beta)^2}{(n-\beta)\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\} - (1-\beta)^2} \qquad (n \geqq 2),$$

we see that $\Phi(n)$ is an *increasing* function for $n \geqq 2$. This implies that

$$\xi \leqq \Phi(2) = \frac{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha+\lambda-\alpha+1\} - 2(1-\beta)^2}{(2-\beta)\{2+k-\beta\}\{2\lambda\alpha+\lambda-\alpha+1\} - (1-\beta)^2}.$$

Finally, by taking each of the functions $f_j(z)$ $(j = 1, 2)$ given as in (3.2), we see that the assertion of Theorem 8 is sharp. □

6. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem 9. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $\mathcal{U}(\lambda,\alpha,\beta,k)$. *Then* $f(z)$ *is close-to-convex of order* ρ $(0 \leqq \rho < 1)$ *in* $|z| < r_1(\lambda,\alpha,\beta,\rho,k)$, *where*

$$r_1(\lambda,\alpha,\beta,\rho,k) := \inf_n \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(1-\beta)}\right)^{\frac{1}{n-1}} \qquad (n \geqq 2).$$

The result is sharp for the function $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$|f'(z)-1| \leqq 1-\rho \qquad \big(0 \leqq \rho < 1;\ |z| < r_1(\lambda,\alpha,\beta,\rho,k)\big).$$

Since

$$|f'(z)-1| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leqq \sum_{n=2}^{\infty} n a_n |z|^{n-1}, \tag{6.1}$$

we have

$$|f'(z)-1| \leqq 1-\rho \qquad (0 \leqq \rho < 1),$$

if

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right) a_n |z|^{n-1} \leqq 1. \tag{6.2}$$

Hence, by Theorem 1, (6.2) will hold true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta},$$

that is, if

$$|z| \leqq \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(1-\beta)}\right)^{\frac{1}{n-1}} \qquad (n \geqq 2). \tag{6.3}$$

The assertion of Theorem 9 would now follow easily from (6.3). □

Theorem 10. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $\mathcal{U}(\lambda,\alpha,\beta,k)$. *Then* $f(z)$ *is starlike of order* ρ $(0 \leqq \rho < 1)$ *in* $|z| < r_2(\lambda,\alpha,\beta,\rho,k)$, *where*

$$r_2(\lambda,\alpha,\beta,\rho,k) := \inf_n \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}} \qquad (n \geqq 2).$$

The result is sharp for the function $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq 1-\rho \qquad \big(0 \leqq \rho < 1;\ |z| < r_2(\lambda,\alpha,\beta,\rho,k)\big).$$

Since

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}}, \tag{6.4}$$

we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1 - \rho \qquad (0 \leqq \rho < 1),$$

if

$$\sum_{n=2}^{\infty} \left(\frac{n-\rho}{1-\rho}\right) a_n |z|^{n-1} \leqq 1. \tag{6.5}$$

Hence, by Theorem 1, (6.5) will hold true if

$$\left(\frac{n-\rho}{1-\rho}\right) |z|^{n-1} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta},$$

that is, if

$$|z| \leqq \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}} \qquad (n \geqq 2). \tag{6.6}$$

The assertion of Theorem 10 would now follow easily from (6.6). □

Theorem 11. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $\mathcal{U}(\lambda, \alpha, \beta, k)$. *Then* $f(z)$ *is convex of order* ρ $(0 \leqq \rho < 1)$ *in* $|z| < r_3(\lambda, \alpha, \beta, \rho, k)$, *where*

$$r_3(\lambda, \alpha, \beta, \rho, k) := \inf_n \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}} \qquad (n \geqq 2).$$

The result is sharp for the function $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 - \rho \qquad \big(0 \leqq \rho < 1;\ |z| < r_3(\lambda, \alpha, \beta, \rho, k)\big).$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}}, \tag{6.7}$$

we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 - \rho \qquad (0 \leqq \rho < 1),$$

if

$$\sum_{n=2}^{\infty} \left(\frac{n(n-\rho)}{1-\rho}\right) a_n |z|^{n-1} \leqq 1. \tag{6.8}$$

Hence, by Theorem 1, (6.8) will hold true if

$$\left(\frac{n(n-\rho)}{1-\rho}\right) |z|^{n-1} \leqq \frac{\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{1-\beta},$$

that is, if

$$|z| \leqq \left(\frac{(1-\rho)\{n(k+1)-(k+\beta)\}\{(n-1)(n\lambda\alpha+\lambda-\alpha)+1\}}{n(n-\rho)(1-\beta)}\right)^{\frac{1}{n-1}} \qquad (n \geqq 2). \tag{6.9}$$

Theorem 11 now follows easily from (6.9). □

7. HADAMARD PRODUCTS AND INTEGRAL OPERATORS

Theorem 12. *Let $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$. Suppose also that*

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \qquad (0 \leqq g_n \leqq 1). \tag{7.1}$$

Then

$$f * g \in \mathcal{U}(\lambda, \alpha, \beta, k).$$

Proof. Since $0 \leqq g_n \leqq 1 \ \ (n \geqq 2)$,

$$\begin{aligned}\sum_{n=2}^{\infty} &\{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n g_n \\ &\leqq \sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}\{(n-1)(n\lambda\alpha + \lambda - \alpha) + 1\} a_n \\ &\leqq 1 - \beta, \end{aligned} \tag{7.2}$$

which completes the proof of Theorem 12. □

Corollary 10. *If $f \in \mathcal{U}(\lambda, \alpha, \beta, k)$, then the function $\mathcal{F}(z)$ defined by*

$$\mathcal{F}(z) := \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt \qquad (c > -1) \tag{7.3}$$

is also in the class $\mathcal{U}(\lambda, \alpha, \beta, k)$.

Proof. Since

$$\mathcal{F}(z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right) z^n \qquad \left(0 < \frac{c+1}{c+n} < 1\right),$$

the result asserted by Corollary 10 follows from Theorem 12. □

REFERENCES

[1] E. AQLAN, J.M. JAHANGIRI AND S.R. KULKARNI, Classes of k-uniformly convex and starlike functions, *Tamkang J. Math.*, **35** (2004), 1–7.

[2] A.W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.

[3] M. KAMALI AND E. KADIOĞLU, On a new subclass of certain starlike functions with negative coefficients, *Atti Sem. Mat. Fis. Univ. Modena*, **48** (2000), 31–44.

[4] S. KANAS AND A. WIŚNIOWSKA, Conic regions and k-uniform convexity, *J. Comput. Appl. Math.*, **105** (1999), 327–336.

[5] S. KANAS AND A. WIŚNIOWSKA, Conic domains and starlike functions, *Rev. Roumaine Math. Pures Appl.*, **45** (2000), 647–657.

[6] S. KANAS AND H.M. SRIVASTAVA, Linear operators associated with k-uniformly convex functions, *Integral Transform. Spec. Funct.*, **9** (2000), 121–132.

[7] W. MA AND D. MINDA, Uniformly convex functions. II, *Ann. Polon. Math.*, **58** (1993), 275–285.

[8] F. RØNNING, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189–196.

[9] H. SILVERMAN, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.

[10] H.M. SRIVASTAVA AND A.K. MISHRA, Applications of fractional calculus to parabolic starlike and uniformly convex functions, *Comput. Math. Appl.*, **39** (3–4) (2000), 57–69.

[11] H.M. SRIVASTAVA AND S. OWA (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[12] H.M. SRIVASTAVA, S. OWA AND S.K. CHATTERJEA, A note on certain classes of starlike functions, *Rend. Sem. Mat. Univ Padova*, **77** (1987), 115–124.