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INEQUALITIES INVOLVING BESSEL FUNCTIONS OF THE FIRST KIND

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Abstract

An inequality involving a function $f_\alpha(x) = \Gamma(\alpha+1)(2/x)^\alpha J_\alpha(x)$ $(\alpha > -\frac{1}{2})$ is obtained. The lower and upper bounds for this function are also derived.

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Contents

1. Introduction and Definitions

In this note we deal with the function

$$f_\alpha(x) = \Gamma(\alpha+1)\left(\frac{2}{x}\right)^\alpha J_\alpha(x), \tag{1.1}$$

$x \in \mathbb{R}$, $\alpha > -\frac{1}{2}$ and J_α stands for the Bessel function of the first kind of order α. It is known (see, e.g., [1, (9.1.69)]) that

$$f_\alpha(x) = {}_0F_1\left(-;\, \alpha+1;\, -\frac{x^2}{4}\right) = \sum_{n=0}^{\infty} \frac{1}{n!(\alpha+1)_n}\left(-\frac{x^2}{4}\right)^n,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ ($k = 0, 1, \ldots$). It is obvious from the above representation that $f_\alpha(-x) = f_\alpha(x)$ and also that $f_\alpha(0) = 1$. The function under discussion admits the integral representation

$$f_\alpha(x) = \int_{-1}^{1} \cos(xt)\, d\mu(t) \tag{1.2}$$

(see, e.g., [1, (9.1.20)]) where $d\mu(t) = \mu(t)dt$ with

$$\mu(t) = (1-t^2)^{\alpha-\frac{1}{2}} \Big/ \left(2^{2\alpha} B\left(\alpha+\frac{1}{2},\, \alpha+\frac{1}{2}\right)\right) \tag{1.3}$$

being the Dirichlet measure on the interval $[-1, 1]$ and $B(\cdot,\cdot)$ stands for the beta function. Clearly

$$\int_{-1}^{1} d\mu(t) = 1. \tag{1.4}$$

Thus $\mu(t)$ is the probability measure on the interval $[-1, 1]$.

In [2], R. Askey has shown that the following inequality

$$f_\alpha(x) + f_\alpha(y) \le 1 + f_\alpha(z) \tag{1.5}$$

holds true for all $\alpha \ge 0$ and $z^2 = x^2 + y^2$. This provides a generalization of Grünbaum's inequality ([4]) who has established (1.5) for $\alpha = 0$.

In this note we give a different upper bound for the sum $f_\alpha(x) + f_\alpha(y)$ (see (2.1)). Also, lower and upper bounds for the function in question are derived.

2. Main Results

Our first result reads as follows.

Theorem 2.1. *Let* $x, y \in \mathbb{R}$*. If* $\alpha > -\frac{1}{2}$*, then*

$$[f_\alpha(x) + f_\alpha(y)]^2 \le [1 + f_\alpha(x+y)][1 + f_\alpha(x-y)]. \tag{2.1}$$

Proof. Using (1.2), some elementary trigonometric identities, Cauchy-Schwarz inequality for integrals, and (1.4) we obtain

$$\begin{aligned}
|f_\alpha(x) + f_\alpha(y)| &\le \int_{-1}^{1} |\cos(xt) + \cos(yt)| d\mu(t) \\
&= 2\int_{-1}^{1} \left|\cos\frac{(x+y)t}{2}\cos\frac{(x-y)t}{2}\right| d\mu(t) \\
&\le 2\left[\int_{-1}^{1} \cos^2\frac{(x+y)t}{2} d\mu(t)\right]^{\frac{1}{2}} \left[\int_{-1}^{1} \cos^2\frac{(x-y)t}{2} d\mu(t)\right]^{\frac{1}{2}} \\
&= 2\left[\frac{1}{2}\int_{-1}^{1} (1 + \cos(x+y)t) d\mu(t)\right]^{\frac{1}{2}} \\
&\qquad \times \left[\frac{1}{2}\int_{-1}^{1} (1 + \cos(x-y)t) d\mu(t)\right]^{\frac{1}{2}} \\
&= [1 + f_\alpha(x+y)]^{\frac{1}{2}} [1 + f_\alpha(x-y)]^{\frac{1}{2}}.
\end{aligned}$$

Hence, the assertion follows. □

When $x = y$, inequality (2.1) simplifies to $2f_\alpha^2(x) \le 1 + f_\alpha(2x)$ which bears resemblance of the double-angle formula for the cosine function $2\cos^2 x = 1 + \cos 2x$.

Our next goal is to establish computable lower and upper bounds for the function f_α. We recall some well-known facts about Gegenbauer polynomials C_k^α ($\alpha > -\frac{1}{2}$, $k \in \mathbb{N}$) and the Gauss-Gegenbauer quadrature formulas. They are orthogonal on the interval $[-1, 1]$ with the weight function $w(t) = (1 - t^2)^{\alpha - \frac{1}{2}}$. The explicit formula for C_k^α is

$$C_k^\alpha(t) = \sum_{m=0}^{[k/2]} (-1)^m \frac{\Gamma(\alpha + k - m)}{\Gamma(\alpha) m! (k - 2m)!} (2t)^{k-2m}$$

(see, e.g., [1, (22.3.4)]). In particular,

$$C_2^\alpha(t) = 2\alpha(\alpha + 1)t^2 - \alpha, \quad C_3^\alpha(t) = \frac{2}{3}\alpha(\alpha + 1)[2(\alpha + 2)t^3 - 3t]. \tag{2.2}$$

The classical Gauss-Gegenbauer quadrature formula with the remainder is [3]

$$\int_{-1}^{1} (1 - t^2)^{\alpha - \frac{1}{2}} g(t)dt = \sum_{i=1}^{k} w_i g(t_i) + \gamma_k g^{(2k)}(\eta), \tag{2.3}$$

where $g \in C^{2k}([-1, 1])$, γ_k is a positive number and does not depend on g, and η is an intermediate point in the interval $(-1, 1)$. Recall that the nodes t_i $(1 \le i \le n)$ are the roots of C_k^α and the weights w_i are given explicitly by [5, (15.3.2)]

$$w_i = \pi\, 2^{2-2\alpha} \frac{\Gamma(2\alpha + k)}{k! [\Gamma(\alpha)]^2} \cdot \frac{1}{(1 - t_i^2)[(C_k^\alpha)'(t_i)]^2} \tag{2.4}$$

$(1 \le i \le k)$.

We are in a position to prove the following.

Theorem 2.2. *Let* $\alpha > -\frac{1}{2}$. *If* $|x| \le \frac{\pi}{2}$, *then*

$$\cos\left(\frac{x}{\sqrt{2(\alpha+1)}}\right) \le f_\alpha(x) \le \frac{1}{3(\alpha+1)}\left[2\alpha+1+(\alpha+2)\cos\left(\sqrt{\frac{3}{2(\alpha+2)}}x\right)\right]. \tag{2.5}$$

Equalities hold in (2.5) *if* $x = 0$.

Proof. In order to establish the lower bound in (2.5) we use the Gauss-Gegenbauer quadrature formula (2.3) with $g(t) = \cos(xt)$ and $k = 2$. Since $g^{(4)}(t) = x^4\cos(xt) \ge 0$ for $t \in [-1, 1]$ and $|x| \le \frac{\pi}{2}$,

$$w_1 g(t_1) + w_2 g(t_2) \le \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \cos(xt)dt. \tag{2.6}$$

Making use of (2.2) and (2.4) we obtain

$$-t_1 = t_2 = \frac{1}{\sqrt{2(\alpha+1)}}$$

and $w_1 = w_2 = \frac{1}{2}\,2^{2\alpha}B(\alpha+\frac{1}{2},\,\alpha+\frac{1}{2})$. This in conjunction with (2.6) gives

$$2^{2\alpha}B\left(\alpha+\frac{1}{2},\,\alpha+\frac{1}{2}\right)\cos\left(\frac{x}{\sqrt{2(\alpha+1)}}\right) \le \int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\cos(xt)dt.$$

Application of (1.3) together with the use of (1.2) gives the asserted result. In order to derive the upper bound in (2.5) we use again (2.3). Letting $g(t) = \cos(xt)$ and $k = 3$ one has $g^{(6)}(t) = -x^6\cos(xt) \le 0$ for $|t| \le 1$ and $|x| \le \frac{\pi}{2}$. Hence

$$\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\cos(xt)dt \le w_1 g(t_1) + w_2 g(t_2) + w_3 g(t_3). \tag{2.7}$$

It follows from (2.2) and (2.4) that

$$-t_1 = t_3 = \sqrt{\frac{3}{2(\alpha+2)}}, \qquad t_2 = 0$$

and

$$w_1 = w_3 = 2^{2\alpha}B\left(\alpha+\frac{1}{2},\ \alpha+\frac{1}{2}\right)\frac{\alpha+2}{6(\alpha+1)},$$
$$w_2 = 2^{2\alpha}B\left(\alpha+\frac{1}{2},\ \alpha+\frac{1}{2}\right)\frac{2\alpha+1}{3(\alpha+1)}.$$

This in conjunction with (2.7), (1.3), and (1.2) gives the desired result. The proof is complete. □

Sharper lower and upper bounds for f_α can be obtained using higher order quadrature formulas (2.3) with even and odd numbers of knots, respectively.

References

[1] M. ABRAMOWITZ AND I.A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1965.

[2] R. ASKEY, Grünbaum's inequality for Bessel functions, *J. Math. Anal. Appl.*, **41** (1973), 122–124.

[3] K.E. ATKINSON, *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989.

[4] F. GRÜNBAUM, A property of Legendre polynomials, *Proc. Nat. Acad. Sci., USA*, **67** (1970), 959–960.

[5] G. SZEGÖ, Orthogonal polynomials, *in* Colloquium Publications, Vol. 23, 4th ed., *American Mathematical Society*, Providence, RI, 1975.