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INEQUALITIES INVOLVING BESSEL FUNCTIONS OF THE FIRST KIND

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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

An inequality involving a function $f_\alpha(x) = \Gamma(\alpha + 1)(2/x)^\alpha J_\alpha(x)$ ($\alpha > -\frac{1}{2}$) is obtained. The lower and upper bounds for this function are also derived.

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Contents

1	Introduction and Definitions	3
2	Main Results	5
	References	



Inequalities Involving Bessel Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 2 of 9

1. Introduction and Definitions

In this note we deal with the function

$$(1.1) \quad f_\alpha(x) = \Gamma(\alpha + 1) \left(\frac{2}{x}\right)^\alpha J_\alpha(x),$$

$x \in \mathbb{R}$, $\alpha > -\frac{1}{2}$ and J_α stands for the Bessel function of the first kind of order α . It is known (see, e.g., [1, (9.1.69)]) that

$$f_\alpha(x) = {}_0F_1\left(-; \alpha + 1; -\frac{x^2}{4}\right) = \sum_{n=0}^{\infty} \frac{1}{n!(\alpha + 1)_n} \left(-\frac{x^2}{4}\right)^n,$$

where $(a)_k = \Gamma(a + k)/\Gamma(a)$ ($k = 0, 1, \dots$). It is obvious from the above representation that $f_\alpha(-x) = f_\alpha(x)$ and also that $f_\alpha(0) = 1$. The function under discussion admits the integral representation

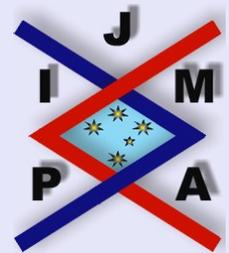
$$(1.2) \quad f_\alpha(x) = \int_{-1}^1 \cos(xt) d\mu(t)$$

(see, e.g., [1, (9.1.20)]) where $d\mu(t) = \mu(t)dt$ with

$$(1.3) \quad \mu(t) = (1 - t^2)^{\alpha - \frac{1}{2}} \Big/ \left(2^{2\alpha} B\left(\alpha + \frac{1}{2}, \alpha + \frac{1}{2}\right)\right)$$

being the Dirichlet measure on the interval $[-1, 1]$ and $B(\cdot, \cdot)$ stands for the beta function. Clearly

$$(1.4) \quad \int_{-1}^1 d\mu(t) = 1.$$



Title Page

Contents



Go Back

Close

Quit

Page 3 of 9

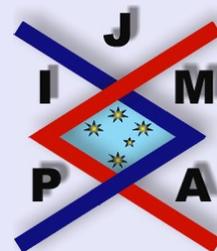
Thus $\mu(t)$ is the probability measure on the interval $[-1, 1]$.

In [2], R. Askey has shown that the following inequality

$$(1.5) \quad f_\alpha(x) + f_\alpha(y) \leq 1 + f_\alpha(z)$$

holds true for all $\alpha \geq 0$ and $z^2 = x^2 + y^2$. This provides a generalization of Grünbaum's inequality ([4]) who has established (1.5) for $\alpha = 0$.

In this note we give a different upper bound for the sum $f_\alpha(x) + f_\alpha(y)$ (see (2.1)). Also, lower and upper bounds for the function in question are derived.



Inequalities Involving Bessel Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 4 of 9

2. Main Results

Our first result reads as follows.

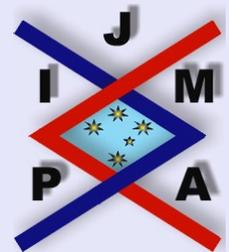
Theorem 2.1. *Let $x, y \in \mathbb{R}$. If $\alpha > -\frac{1}{2}$, then*

$$(2.1) \quad [f_\alpha(x) + f_\alpha(y)]^2 \leq [1 + f_\alpha(x + y)][1 + f_\alpha(x - y)].$$

Proof. Using (1.2), some elementary trigonometric identities, Cauchy-Schwarz inequality for integrals, and (1.4) we obtain

$$\begin{aligned} |f_\alpha(x) + f_\alpha(y)| &\leq \int_{-1}^1 |\cos(xt) + \cos(yt)| d\mu(t) \\ &= 2 \int_{-1}^1 \left| \cos \frac{(x+y)t}{2} \cos \frac{(x-y)t}{2} \right| d\mu(t) \\ &\leq 2 \left[\int_{-1}^1 \cos^2 \frac{(x+y)t}{2} d\mu(t) \right]^{\frac{1}{2}} \left[\int_{-1}^1 \cos^2 \frac{(x-y)t}{2} d\mu(t) \right]^{\frac{1}{2}} \\ &= 2 \left[\frac{1}{2} \int_{-1}^1 (1 + \cos(x+y)t) d\mu(t) \right]^{\frac{1}{2}} \\ &\quad \times \left[\frac{1}{2} \int_{-1}^1 (1 + \cos(x-y)t) d\mu(t) \right]^{\frac{1}{2}} \\ &= [1 + f_\alpha(x + y)]^{\frac{1}{2}} [1 + f_\alpha(x - y)]^{\frac{1}{2}}. \end{aligned}$$

Hence, the assertion follows. \square



Inequalities Involving Bessel
Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 5 of 9

When $x = y$, inequality (2.1) simplifies to $2f_\alpha^2(x) \leq 1 + f_\alpha(2x)$ which bears resemblance of the double-angle formula for the cosine function $2\cos^2 x = 1 + \cos 2x$.

Our next goal is to establish computable lower and upper bounds for the function f_α . We recall some well-known facts about Gegenbauer polynomials C_k^α ($\alpha > -\frac{1}{2}$, $k \in \mathbb{N}$) and the Gauss-Gegenbauer quadrature formulas. They are orthogonal on the interval $[-1, 1]$ with the weight function $w(t) = (1 - t^2)^{\alpha - \frac{1}{2}}$. The explicit formula for C_k^α is

$$C_k^\alpha(t) = \sum_{m=0}^{[k/2]} (-1)^m \frac{\Gamma(\alpha + k - m)}{\Gamma(\alpha)m!(k - 2m)!} (2t)^{k-2m}$$

(see, e.g., [1, (22.3.4)]). In particular,

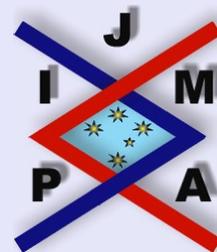
$$(2.2) \quad C_2^\alpha(t) = 2\alpha(\alpha + 1)t^2 - \alpha, \quad C_3^\alpha(t) = \frac{2}{3}\alpha(\alpha + 1)[2(\alpha + 2)t^3 - 3t].$$

The classical Gauss-Gegenbauer quadrature formula with the remainder is [3]

$$(2.3) \quad \int_{-1}^1 (1 - t^2)^{\alpha - \frac{1}{2}} g(t) dt = \sum_{i=1}^k w_i g(t_i) + \gamma_k g^{(2k)}(\eta),$$

where $g \in C^{2k}([-1, 1])$, γ_k is a positive number and does not depend on g , and η is an intermediate point in the interval $(-1, 1)$. Recall that the nodes t_i ($1 \leq i \leq n$) are the roots of C_k^α and the weights w_i are given explicitly by [5, (15.3.2)]

$$(2.4) \quad w_i = \pi 2^{2-2\alpha} \frac{\Gamma(2\alpha + k)}{k![\Gamma(\alpha)]^2} \cdot \frac{1}{(1 - t_i^2)[(C_k^\alpha)'(t_i)]^2}$$



Inequalities Involving Bessel Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 6 of 9

$(1 \leq i \leq k)$.

We are in a position to prove the following.

Theorem 2.2. *Let $\alpha > -\frac{1}{2}$. If $|x| \leq \frac{\pi}{2}$, then*

$$(2.5) \quad \begin{aligned} \cos\left(\frac{x}{\sqrt{2(\alpha+1)}}\right) &\leq f_\alpha(x) \\ &\leq \frac{1}{3(\alpha+1)} \left[2\alpha + 1 + (\alpha + 2) \cos\left(\sqrt{\frac{3}{2(\alpha+2)}}x\right) \right]. \end{aligned}$$

Equalities hold in (2.5) if $x = 0$.

Proof. In order to establish the lower bound in (2.5) we use the Gauss-Gegenbauer quadrature formula (2.3) with $g(t) = \cos(xt)$ and $k = 2$. Since $g^{(4)}(t) = x^4 \cos(xt) \geq 0$ for $t \in [-1, 1]$ and $|x| \leq \frac{\pi}{2}$,

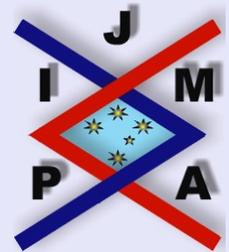
$$(2.6) \quad w_1 g(t_1) + w_2 g(t_2) \leq \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(xt) dt.$$

Making use of (2.2) and (2.4) we obtain

$$-t_1 = t_2 = \frac{1}{\sqrt{2(\alpha+1)}}$$

and $w_1 = w_2 = \frac{1}{2} 2^{2\alpha} B(\alpha + \frac{1}{2}, \alpha + \frac{1}{2})$. This in conjunction with (2.6) gives

$$2^{2\alpha} B\left(\alpha + \frac{1}{2}, \alpha + \frac{1}{2}\right) \cos\left(\frac{x}{\sqrt{2(\alpha+1)}}\right) \leq \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(xt) dt.$$



Inequalities Involving Bessel Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 7 of 9

Application of (1.3) together with the use of (1.2) gives the asserted result. In order to derive the upper bound in (2.5) we use again (2.3). Letting $g(t) = \cos(xt)$ and $k = 3$ one has $g^{(6)}(t) = -x^6 \cos(xt) \leq 0$ for $|t| \leq 1$ and $|x| \leq \frac{\pi}{2}$. Hence

$$(2.7) \quad \int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(xt) dt \leq w_1 g(t_1) + w_2 g(t_2) + w_3 g(t_3).$$

It follows from (2.2) and (2.4) that

$$-t_1 = t_3 = \sqrt{\frac{3}{2(\alpha+2)}}, \quad t_2 = 0$$

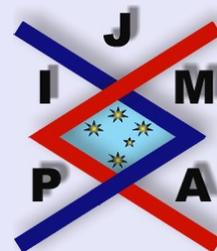
and

$$w_1 = w_3 = 2^{2\alpha} B\left(\alpha + \frac{1}{2}, \alpha + \frac{1}{2}\right) \frac{\alpha + 2}{6(\alpha + 1)},$$

$$w_2 = 2^{2\alpha} B\left(\alpha + \frac{1}{2}, \alpha + \frac{1}{2}\right) \frac{2\alpha + 1}{3(\alpha + 1)}.$$

This in conjunction with (2.7), (1.3), and (1.2) gives the desired result. The proof is complete. \square

Sharper lower and upper bounds for f_α can be obtained using higher order quadrature formulas (2.3) with even and odd numbers of knots, respectively.



Inequalities Involving Bessel Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

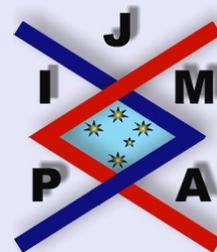
Close

Quit

Page 8 of 9

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Inequalities Involving Bessel Functions of the First Kind

Edward Neuman

Title Page

Contents



Go Back

Close

Quit

Page 9 of 9