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A SHARP INEQUALITY OF OSTROWSKI-GRÜSS TYPE

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

The main purpose of this paper is to use a Grüss type inequality for Riemann-Stieltjes integrals to obtain a sharp integral inequality of Ostrowski-Grüss type for functions whose first derivative are functions of Lipschitzian type and precisely characterize the functions for which equality holds.

2000 Mathematics Subject Classification: 26D15.

Key words: Ostrowski-Grüss type inequality, Grüss type inequality for Riemann-Stieltjes integrals, Lipschitzian type function, Sharp bound.

Contents

1	Introduction	3
2	The Results	7
	References	



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 2 of 21

1. Introduction

In 1935, G. Grüss (see [4, p. 296]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

Theorem A. *Let $h, g : [a, b] \rightarrow \mathbb{R}$ be two integrable functions such that $\phi \leq h(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b]$, where $\phi, \Phi, \gamma, \Gamma$ are real numbers. Then we have*

$$(1.1) \quad |T(h, g)| \\ := \left| \frac{1}{b-a} \int_a^b h(x)g(x)dx - \frac{1}{b-a} \int_a^b h(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \\ \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma),$$

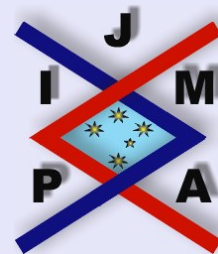
and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

It is clear that the constant $\frac{1}{4}$ is achieved for

$$h(x) = g(x) = \operatorname{sgn} \left(x - \frac{a+b}{2} \right).$$

From then on, (1.1) has been known in the literature as the Grüss inequality.

In 1998, S.S. Dragomir and I. Fedotov [2] established the following Grüss type inequality for Riemann-Stieltjes integrals:



A Sharp Inequality of
Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 3 of 21

Theorem B. Let $h, u : [a, b] \rightarrow \mathbb{R}$ be so that u is L -Lipschitzian on $[a, b]$, i.e.,

$$|u(x) - u(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$, h is Riemann integrable on $[a, b]$ and there exists the real numbers m, M so that $m \leq h(x) \leq M$ for all $x \in [a, b]$. Then we have the inequality

$$(1.2) \quad \left| \int_a^b h(x) du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b h(t) dt \right| \leq \frac{1}{2} L(M - m)(b - a)$$

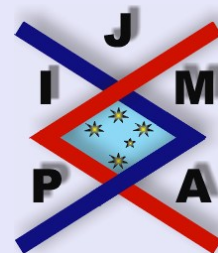
and the constant $\frac{1}{2}$ is sharp.

In a recent paper [3], the inequality (1.2) has been improved and refined as follows:

Theorem C. Let $h, u : [a, b] \rightarrow \mathbb{R}$ be so that u is L -Lipschitzian on $[a, b]$, h is Riemann integrable on $[a, b]$ and there exist the real numbers m, M so that $m \leq h(x) \leq M$ for all $x \in [a, b]$. Then we have

$$(1.3) \quad \begin{aligned} \left| \int_a^b h(x) du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b h(t) dt \right| & \\ & \leq L \int_a^b \left| h(x) - \frac{1}{b - a} \int_a^b h(t) dt \right| dx \\ & \leq L(b - a) \sqrt{T(h, h)} \\ & \leq \frac{1}{2} L(M - m)(b - a). \end{aligned}$$

All the inequalities in (1.3) are sharp and the constant $\frac{1}{2}$ is the best possible one.



A Sharp Inequality of
Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 4 of 21

Theorem D. Let $h, u : [a, b] \rightarrow \mathbb{R}$ be so that u is (l, L) -Lipschitzian on $[a, b]$, i.e., it satisfies the condition

$$l(x_2 - x_1) \leq u(x_2) - u(x_1) \leq L(x_2 - x_1)$$

for $a \leq x_1 \leq x_2 \leq b$ with $l < L$, h is Riemann integral on $[a, b]$ and there exist the real numbers m, M so that $m \leq h(x) \leq M$ for all $x \in [a, b]$. Then we have the inequality

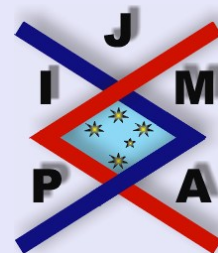
$$\begin{aligned} (1.4) \quad & \left| \int_a^b h(x) du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b h(t) dt \right| \\ & \leq \frac{L - l}{2} \int_a^b \left| h(x) - \frac{1}{b - a} \int_a^b h(t) dt \right| dx \\ & \leq \frac{L - l}{2} (b - a) \sqrt{T(h, h)} \\ & \leq \frac{1}{4} (L - l) (M - m) (b - a). \end{aligned}$$

All the inequalities in (1.4) are sharp and the constant $\frac{1}{4}$ is the best possible one.

In [1], L.J. Dedić et al. have proved the following Ostrowski type inequality as

Theorem E. If u' is L -Lipschitzian on $[a, b]$, then for every $x \in [a, b]$ we have

$$\begin{aligned} (1.5) \quad & \left| \int_a^b u(t) dt - \frac{b - a}{2} \left[u(x) + \frac{u(a) + u(b)}{2} + \left(x - \frac{a + b}{2} \right) u'(x) \right] \right| \\ & \leq L \left(\frac{1}{3} \left| x - \frac{a + b}{2} \right|^3 + \frac{(b - a)^3}{48} \right). \end{aligned}$$



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



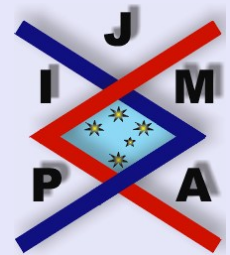
Go Back

Close

Quit

Page 5 of 21

In this paper, we will use Theorem **C** and Theorem **D** to obtain some sharp integral inequalities of Ostrowski-Grüss type for functions whose first derivative are functions of Lipschitzian type. Thus a further generalization of the Ostrowski type inequality and a perturbed version of the inequality (1.5) is obtained.



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 6 of 21

2. The Results

Theorem 2.1. Let $u : [a, b] \rightarrow \mathbb{R}$ be a differentiable function so that u' is (l, L) -Lipschitzian on $[a, b]$, i.e., satisfies the condition

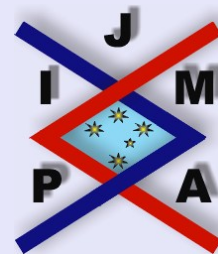
$$(2.1) \quad l(x_2 - x_1) \leq u'(x_2) - u'(x_1) \leq L(x_2 - x_1)$$

for $a \leq x_1 \leq x_2 \leq b$ with $l < L$. Then for all $x \in [a, b]$ we have

$$(2.2) \quad \left| \int_a^b u(t) dt - \frac{b-a}{2} \left[u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2} \right) u'(x) \right] - \frac{u'(b) - u'(a)}{4} \left[\left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right] \right| \leq \frac{L-l}{4} I(a, b, x),$$

where

$$(2.3) \quad I(a, b, x) = \begin{cases} \frac{1}{6} \left(\frac{a+b}{2} - x \right) \left(\frac{a+3b}{4} - x \right) [3(x-a) + (b-x)] + \frac{4}{3} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}, & a \leq x \leq \xi, \\ \frac{1}{6} \left(\frac{a+b}{2} - x \right) \left(x - \frac{3a+b}{4} \right) [(x-a) + 3(b-x)] + 4 \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}, & \xi < x < \zeta, \\ \frac{16}{3} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}, & \zeta \leq x \leq \theta, \\ \frac{1}{6} \left(x - \frac{a+b}{2} \right) \left(\frac{a+3b}{4} - x \right) [3(x-a) + (b-x)] + 4 \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}, & \theta < x < \eta, \\ \frac{1}{6} \left(x - \frac{a+b}{2} \right) \left(x - \frac{3a+b}{4} \right) [(x-a) + 3(b-x)] + \frac{4}{3} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}, & \eta \leq x \leq b \end{cases}$$



A Sharp Inequality of
Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 7 of 21

with

$$\xi = \frac{a+b}{2} - \frac{\sqrt{3}(b-a)}{6}, \quad \eta = \frac{a+b}{2} + \frac{\sqrt{3}(b-a)}{6},$$

$$\zeta = a + \frac{\sqrt{6}(b-a)}{6}, \quad \theta = b - \frac{\sqrt{6}(b-a)}{6}$$

and $a < \xi < \frac{3a+b}{4} < \zeta < \frac{a+b}{2} < \theta < \frac{a+3b}{4} < \eta < b$.

Proof. Integrating by parts produces the identity

$$(2.4) \quad \int_a^b K(x, t) du'(t)$$

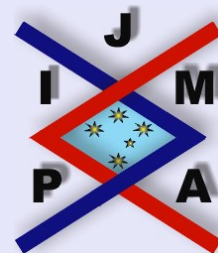
$$= \int_a^b u(t) dt - \frac{1}{2}(b-a) \left[u(x) + \frac{u(a) + u(b)}{2} + \left(x - \frac{a+b}{2} \right) u'(x) \right],$$

where

$$(2.5) \quad K(x, t) = \begin{cases} \frac{1}{2}(t-a) \left(t - \frac{a+b}{2} \right), & t \in [a, x], \\ \frac{1}{2}(t-b) \left(t - \frac{a+b}{2} \right), & t \in (x, b]. \end{cases}$$

Moreover,

$$(2.6) \quad \frac{1}{b-a} \int_a^b K(x, t) dt = \frac{1}{4} \left[\left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right].$$



**A Sharp Inequality of
Ostrowski-Grüss Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 8 of 21

Applying the Grüss type inequality (1.4) gives

$$\left| \int_a^b K(x, t) du'(t) - \frac{u'(b) - u'(a)}{b - a} \int_a^b K(x, t) dt \right| \leq \frac{L - l}{2} \int_a^b \left| K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) ds \right| dt.$$

Then for any fixed $x \in [a, b]$ we can derive from (2.4), (2.5) and (2.6) that

$$(2.7) \quad \left| \int_a^b u(t) dt - \frac{b - a}{2} \left[u(x) + \frac{u(a) + u(b)}{2} + \left(x - \frac{a + b}{2} \right) u'(x) \right] - \frac{u'(b) - u'(a)}{4} \left[\left(x - \frac{a + b}{2} \right)^2 - \frac{(b - a)^2}{12} \right] \right| \leq \frac{L - l}{4} I(a, b, x),$$

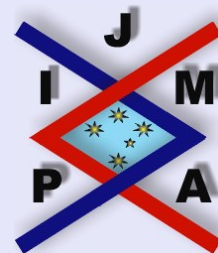
where

$$I(a, b, x) = \int_a^x \left| (t - a) \left(t - \frac{a + b}{2} \right) - \frac{1}{2} \left[\left(x - \frac{a + b}{2} \right)^2 - \frac{(b - a)^2}{12} \right] \right| dt + \int_x^b \left| (t - b) \left(t - \frac{a + b}{2} \right) - \frac{1}{2} \left[\left(x - \frac{a + b}{2} \right)^2 - \frac{(b - a)^2}{12} \right] \right| dt.$$

The last two integrals can be calculated as follows:

For brevity, we put

$$p_1(t) := (t - a) \left(t - \frac{a + b}{2} \right) - \frac{1}{2} \left[\left(x - \frac{a + b}{2} \right)^2 - \frac{(b - a)^2}{12} \right], \quad t \in [a, x],$$



**A Sharp Inequality of
Ostrowski-Grüss Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 9 of 21

$$p_2(t) := (t - b) \left(t - \frac{a + b}{2} \right) - \frac{1}{2} \left[\left(x - \frac{a + b}{2} \right)^2 - \frac{(b - a)^2}{12} \right], \quad t \in [x, b].$$

Then we have

$$p_1(a) = p_2(b) = \frac{1}{2} \left[\frac{(b - a)^2}{12} - \left(x - \frac{a + b}{2} \right)^2 \right];$$

$$p_1(x) = \frac{1}{2} \left(x + \frac{b - a}{2} \right) \left(x - \frac{a + b}{2} \right) + \frac{(b - a)^2}{24},$$

$$p_2(x) = \frac{1}{2} \left(x - \frac{b - a}{2} \right) \left(x - \frac{a + b}{2} \right) + \frac{(b - a)^2}{24}.$$

Set

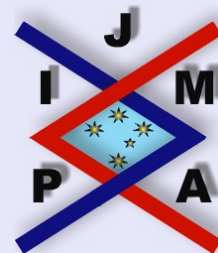
$$\xi = \frac{a + b}{2} - \frac{\sqrt{3}(b - a)}{6}, \quad \eta = \frac{a + b}{2} + \frac{\sqrt{3}(b - a)}{6},$$

$$\zeta = a + \frac{\sqrt{6}(b - a)}{6}, \quad \theta = b - \frac{\sqrt{6}(b - a)}{6}.$$

It is easy to find that $p_1(a) = p_2(b) \leq 0$ for $x \in [a, \xi] \cup [\eta, b]$, $p_1(a) = p_2(b) > 0$ for $x \in (\xi, \eta)$ and $p_1(x) \leq 0$ for $x \in [a, \zeta]$, $p_1(x) > 0$ for $x \in (\zeta, b]$, $p_2(x) > 0$ for $x \in [a, \theta]$, $p_2(x) \leq 0$ for $x \in [\theta, b]$. Notice that

$$a < \xi < \frac{3a + b}{4} < \zeta < \frac{a + b}{2} < \theta < \frac{a + 3b}{4} < \eta < b,$$

we see that there are five possible cases to be determined.



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 10 of 21

(i) In case $x \in [\zeta, \theta]$. $p_1(a) = p_2(b) > 0$, $p_1(x) \geq 0$, $p_2(x) \geq 0$ and it is easy to find by elementary calculus that the function $p_1(t)$ is strictly decreasing in $(a, \frac{3a+b}{4})$ and strictly increasing in $(\frac{3a+b}{4}, x)$, also, as the function $p_2(t)$ is strictly decreasing in $(x, \frac{a+3b}{4})$ and strictly increasing in $(\frac{a+3b}{4}, b)$. Moreover, $p_1(\frac{3a+b}{4}) = p_2(\frac{a+3b}{4}) < 0$. So, $p_1(t)$ has two zeros in (a, x) at the points

$$t_1 = \frac{3a+b}{4} - \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{1}{2}}$$

and

$$t_2 = \frac{3a+b}{4} + \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{1}{2}}.$$

Also $p_2(t)$ has two zeros in (x, b) at the points

$$t_3 = \frac{a+3b}{4} - \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{1}{2}}$$

and

$$t_4 = \frac{a+3b}{4} + \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{1}{2}}.$$

Thus we have

$$(2.8) \quad I(a, b, x) = \int_a^{t_1} \left[(t-a) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt$$



A Sharp Inequality of
Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents

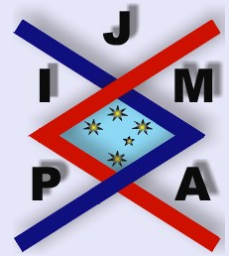


Go Back

Close

Quit

Page 11 of 21



Title Page

Contents



Go Back

Close

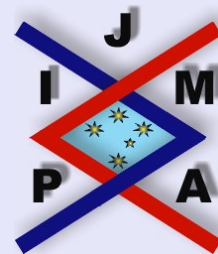
Quit

Page 12 of 21

$$\begin{aligned}
 & + \int_{t_1}^{t_2} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-a) \left(t - \frac{a+b}{2} \right) \right] dt \\
 & + \int_{t_2}^x \left[(t-a) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
 & + \int_x^{t_3} \left[(t-b) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
 & + \int_{t_3}^{t_4} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-b) \left(t - \frac{a+b}{2} \right) \right] dt \\
 & + \int_{t_4}^b \left[(t-b) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
 & = \frac{16}{3} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}.
 \end{aligned}$$

(ii) In case $x \in [a, \xi]$, $p_1(a) = p_2(b) \leq 0$, $p_1(x) < 0, p_2(x) > 0$ and $p_1(t)$ is strictly decreasing in (a, x) as well as $p_2(t)$ is strictly decreasing in $(x, \frac{a+3b}{4})$ and strictly increasing in $(\frac{a+3b}{4}, b)$ with $t_3 \in (x, \frac{a+3b}{4})$ such that $p_2(t_3) = 0$. Thus we have

$$\begin{aligned}
 (2.9) \quad I(a, b, x) & = \int_a^x \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-a) \left(t - \frac{a+b}{2} \right) \right] dt
 \end{aligned}$$



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 13 of 21

$$\begin{aligned}
 & + \int_x^{t_3} \left[(t-b) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
 & + \int_{t_3}^b \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-b) \left(t - \frac{a+b}{2} \right) \right] dt \\
 = & \frac{1}{6} \left(\frac{a+b}{2} - x \right) \left(\frac{a+3b}{4} - x \right) [3(x-a) + (b-x)] \\
 & + \frac{4}{3} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}.
 \end{aligned}$$

(iii) In case (ξ, ζ) , $p_1(a) = p_2(b) > 0$, $p_1(x) < 0, p_2(x) > 0$ and $p_1(t)$ has a unique zero $t_1 \in (a, x)$, $p_2(t)$ has two zeros $t_3, t_4 \in (x, b)$. Thus we have

(2.10) $I(a, b, x)$

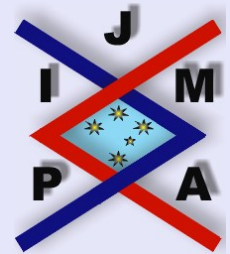
$$\begin{aligned}
 = & \int_a^{t_1} \left[(t-a) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
 & + \int_{t_1}^x \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-a) \left(t - \frac{a+b}{2} \right) \right] dt \\
 & + \int_x^{t_3} \left[(t-b) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
 & + \int_{t_3}^{t_4} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-b) \left(t - \frac{a+b}{2} \right) \right] dt
 \end{aligned}$$

$$\begin{aligned}
& + \int_{t_4}^b \left[(t-b) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
& = \frac{1}{6} \left(\frac{a+b}{2} - x \right) \left(x - \frac{3a+b}{4} \right) [(x-a) + 3(b-x)] \\
& \quad + 4 \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}.
\end{aligned}$$

(iv) In case $x \in (\theta, \eta)$, $p_1(a) = p_2(b) > 0$, $p_1(x) > 0$, $p_2(x) < 0$ and $p_1(t)$ has two zeros $t_1, t_2 \in (a, x)$, $p_2(t)$ has a unique zero $t_4 \in (x, b)$. Thus we have

(2.11) $I(a, b, x)$

$$\begin{aligned}
& = \int_a^{t_1} \left[(t-a) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
& \quad + \int_{t_1}^{t_2} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-a) \left(t - \frac{a+b}{2} \right) \right] dt \\
& \quad + \int_{t_2}^x \left[(t-a) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\
& \quad + \int_x^{t_4} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-b) \left(t - \frac{a+b}{2} \right) \right] dt \\
& \quad + \int_{t_4}^b \left[(t-b) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt
\end{aligned}$$



**A Sharp Inequality of
Ostrowski-Grüss Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 14 of 21

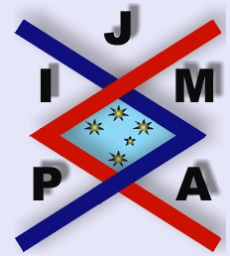
$$= \frac{1}{6} \left(x - \frac{a+b}{2} \right) \left(\frac{a+3b}{4} - x \right) [3(x-a) + (b-x)] \\ + 4 \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}.$$

(v) In case $x \in [\eta, b]$, $p_1(a) = p_2(b) \leq 0$, $p_1(x) > 0$, $p_2(x) < 0$ and $p_1(t)$ has a unique zero $t_2 \in (a, x)$, $p_2(t) \leq 0$ for $t \in [x, b]$. Thus we have

$$(2.12) \quad I(a, b, x) \\ = \int_a^{t_2} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-a) \left(t - \frac{a+b}{2} \right) \right] dt \\ + \int_{t_2}^x \left[(t-a) \left(t - \frac{a+b}{2} \right) - \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{24} \right] dt \\ + \int_x^b \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{24} - (t-b) \left(t - \frac{a+b}{2} \right) \right] dt \\ = \frac{1}{6} \left(x - \frac{a+b}{2} \right) \left(x - \frac{3a+b}{4} \right) [(x-a) + 3(b-x)] \\ + \frac{4}{3} \left[\frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{48} \right]^{\frac{3}{2}}.$$

Consequently, the inequality (2.2) with (2.3) follows from (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12).

The proof is completed. \square



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 15 of 21

Remark 1. *It is not difficult to prove that the inequality (2.2) with (2.3) is sharp in the sense that we can construct the function u to attain the equality in (2.2) with (2.3). Indeed, we may choose u such that*

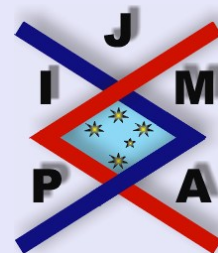
$$u(t) = \begin{cases} \frac{1}{2}(t-a)^2, & a \leq t < x, \\ \frac{l}{2}(t-x)^2 + \frac{l}{2}[2(x-a)t - (x^2 - a^2)], & x \leq t < t_3, \\ \frac{l}{2}[(t-t_3)^2 + 2(x-a)t - (x^2 - a^2)] \\ \quad + \frac{l}{2}[2(t_3-x)t - (t_3^2 - x^2)], & t_3 \leq t \leq b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} l(t-a), & a \leq t < x, \\ L(t-x) + (x-a)l, & x \leq t < t_3, \\ l(t-t_3+x-a) + (t_3-x)L, & t_3 \leq t \leq b, \end{cases}$$

for any $x \in [a, \xi]$, and

$$u(t) = \begin{cases} \frac{l}{2}(t-a)^2, & a \leq t < t_1, \\ \frac{l}{2}(t-t_1)^2 + \frac{l}{2}[2(t_1-a)t - (t_1^2 - a^2)], & t_1 \leq t < x, \\ \frac{l}{2}[(t-x)^2 + 2(t_1-a)t - (t_1^2 - a^2)] \\ \quad + \frac{l}{2}[2(x-t_1)t - (x^2 - t_1^2)], & x \leq t < t_3, \\ \frac{l}{2}[(t-t_3)^2 + 2(x-t_1)t - (x^2 - t_1^2)] \\ \quad + \frac{l}{2}[2(t_3-x+t_1-a)t - (t_3^2 - x^2 + t_1^2 - a^2)], & t_3 \leq t < t_4, \\ \frac{l}{2}[(t-t_4)^2 + 2(t_3-x+t_1-a)t - (t_3^2 - x^2 + t_1^2 - a^2)] \\ \quad + \frac{l}{2}[2(t_4-t_3+x-t_1)t - (t_4^2 - t_3^2 + x^2 - t_1^2)], & t_4 \leq t \leq b, \end{cases}$$



A Sharp Inequality of
Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

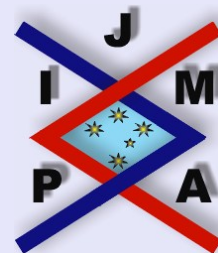
Page 16 of 21

which follows

$$u'(t) = \begin{cases} L(t-a), & a \leq t < t_1, \\ l(t-t_1) + (t_1-a)L, & t_1 \leq t < x, \\ L(t-x+t_1-a) + (x-t_1)l, & x \leq t < t_3, \\ l(t-t_3+x-t_1) + (t_3-x+t_1-a)L, & t_3 \leq t < t_4, \\ L(t-t_4+t_3-x+t_1-a) + (t_4-t_3+x-t_1)l, & t_4 \leq t \leq b, \end{cases}$$

for any $x \in (\xi, \zeta)$, and

$$u(t) = \begin{cases} \frac{L}{2}(t-a)^2, & a \leq t < t_1, \\ \frac{l}{2}(t-t_1)^2 + \frac{L}{2}[2(t_1-a)t - (t_1^2 - a^2)], & t_1 \leq t < t_2, \\ \frac{L}{2}[(t-t_2)^2 + 2(t_1-a)t - (t_1^2 - a^2)] \\ \quad + \frac{l}{2}[2(t_2-t_1)t - (t_2^2 - t_1^2)], & t_2 \leq t < t_3, \\ \frac{l}{2}[(t-t_3)^2 + 2(t_2-t_1)t - (t_2^2 - t_1^2)] \\ \quad + \frac{L}{2}[2(t_3-t_2+t_1-a)t - (t_3^2 - t_2^2 + t_1^2 - a^2)], & t_3 \leq t < t_4, \\ \frac{L}{2}[(t-t_4)^2 + 2(t_3-t_2+t_1-a)t - (t_3^2 - t_2^2 + t_1^2 - a^2)] \\ \quad + \frac{l}{2}[2(t_4-t_3+t_2-t_1)t - (t_4^2 - t_3^2 + t_2^2 - t_1^2)], & t_4 \leq t \leq b, \end{cases}$$



**A Sharp Inequality of
Ostrowski-Grüss Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

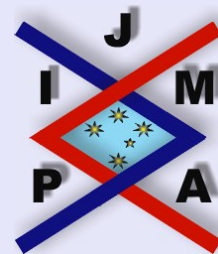
Page 17 of 21

which follows

$$u'(t) = \begin{cases} L(t-a), & a \leq t < t_1, \\ l(t-t_1) + (t_1-a)L, & t_1 \leq t < t_2, \\ L(t-t_2+t_1-a) + (t_2-t_1)l, & t_2 \leq t < t_3, \\ l(t-t_3+t_2-t_1) + (t_3-t_2+t_1-a)L, & t_3 \leq t < t_4, \\ L(t-t_4+t_3-t_2+t_1-a) + (t_4-t_3+t_2-t_1)l, & t_4 \leq t \leq b, \end{cases}$$

for any $x \in (\xi, \zeta)$, and

$$u(t) = \begin{cases} \frac{L}{2}(t-a)^2, & a \leq t < t_1, \\ \frac{l}{2}(t-t_1)^2 + \frac{L}{2}[2(t_1-a)t - (t_1^2 - a^2)], & t_1 \leq t < t_2, \\ \frac{L}{2}[(t-t_2)^2 + 2(t_1-a)t - (t_1^2 - a^2)] \\ \quad + \frac{l}{2}[2(t_2-t_1)t - (t_2^2 - t_1^2)], & t_2 \leq t < x, \\ \frac{l}{2}[(t-x)^2 + 2(t_2-t_1)t - (t_2^2 - t_1^2)] \\ \quad + \frac{L}{2}[2(x-t_2+t_1-a)t - (x^2 - t_2^2 + t_1^2 - a^2)], & x \leq t < t_4, \\ \frac{L}{2}[(t-t_4)^2 + 2(x-t_2+t_1-a)t - (x^2 - t_2^2 + t_1^2 - a^2)] \\ \quad + \frac{l}{2}[2(t_4-x+t_2-t_1)t - (t_4^2 - x^2 + t_2^2 - t_1^2)], & t_4 \leq t \leq b, \end{cases}$$



**A Sharp Inequality of
Ostrowski-Grüss Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 18 of 21

which follows

$$u'(t) = \begin{cases} L(t-a), & a \leq t < t_1, \\ l(t-t_1) + (t_1-a)L, & t_1 \leq t < t_2, \\ L(t-t_2+t_1-a) + (t_2-t_1)l, & t_2 \leq t < x, \\ l(t-x+t_2-t_1) + (x-t_2+t_1-a)L, & x \leq t < t_4, \\ L(t-t_4+x-t_2+t_1-a) + (t_4-x+t_2-t_1)l, & t_4 \leq t \leq b, \end{cases}$$

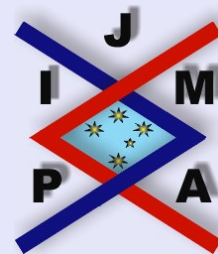
for any $x \in (\theta, \eta)$, and

$$u(t) = \begin{cases} \frac{1}{2}(t-a)^2, & a \leq t < t_2, \\ \frac{L}{2}(t-t_2)^2 + \frac{l}{2}[2(t_2-a)t - (t_2^2 - a^2)], & t_2 \leq t < x, \\ \frac{l}{2}[(t-x)^2 + 2(t_2-a)t - (t_2^2 - a^2)] \\ \quad + \frac{L}{2}[2(x-t_2)t - (x^2 - t_2^2)], & x \leq t \leq b, \end{cases}$$

which follows

$$u'(t) = \begin{cases} l(t-a), & a \leq t < t_2, \\ L(t-t_2) + (t_2-a)l, & t_2 \leq t < x, \\ l(t-x+t_2-a) + (x-t_2)L, & x \leq t \leq b. \end{cases}$$

for any $x \in [\eta, b]$.



**A Sharp Inequality of
Ostrowski-Grüss Type**

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 19 of 21

It is clear that all the above $u'(t)$ satisfy the condition (2.1) on $[a, b]$.

Remark 2. For $x = \frac{a+b}{2}$, we have

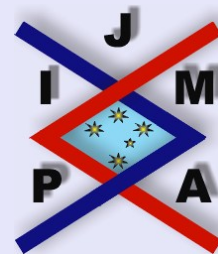
$$\left| \int_a^b u(t)dt - \frac{b-a}{2} \left[u\left(\frac{a+b}{2}\right) + \frac{u(a)+u(b)}{2} \right] + \frac{(b-a)^2}{48} [u'(b) - u'(a)] \right| \leq \frac{(L-l)(b-a)^3}{144\sqrt{3}}.$$

Corollary 2.2. If u' is L -Lipschitzian on $[a, b]$, then for all $x \in [a, b]$ we have

$$(2.13) \quad \left| \int_a^b u(t)dt - \frac{b-a}{2} \left[u(x) + \frac{u(a)+u(b)}{2} + \left(x - \frac{a+b}{2}\right) u'(x) \right] - \frac{u'(b) - u'(a)}{4} \left[\left(x - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12} \right] \right| \leq \frac{L}{2} I(a, b, x),$$

where $I(a, b, x)$ is as defined in (2.3).

Proof. It is immediate by taking $l = -L$ in the theorem. □



A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

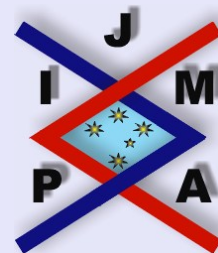
Close

Quit

Page 20 of 21

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A Sharp Inequality of Ostrowski-Grüss Type

Zheng Liu

Title Page

Contents



Go Back

Close

Quit

Page 21 of 21