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AN ELEMENTARY PROOF OF THE PRESERVATION OF LIPSCHITZ CONSTANTS BY THE MEYER-KÖNIG AND ZELLER OPERATORS

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Abstract

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Given the real numbers $A \geq 0$ and $0 < \alpha \leq 1$, we denote by $\mathrm{Lip}_A\alpha$ the set of all functions $f : [0,1] \to \mathbb{R}$, satisfying

$$|f(x_2) - f(x_1)| \leq A|x_2 - x_1|^{\alpha} \qquad \text{for all} \quad x_1, x_2 \in [0,1].$$

The main purpose of this note is to present an elementary proof of the following result:

Given the continuous function $f : [0,1] \to \mathbb{R}$, it holds that

$$f \in \mathrm{Lip}_A\alpha \tag{1}$$

if and only if

$$M_n f \in \mathrm{Lip}_A\alpha \qquad \text{for all} \quad n \geq 1, \tag{2}$$

where $(M_n)_{n\geq 1}$ is the sequence of Meyer-König and Zeller operators.

It should be mentioned that similar proofs for other operators are to be found in [2] and [3]. On the other hand, the equivalence (1) $\Leftrightarrow$ (2) is a special case of a much more general result [1, Theorem 1]. However, the proof presented in [1] is completely different and does not have an elementary character.

Proof. Let $f : [0,1] \to \mathbb{R}$ be a continuous function and let n be a positive integer. Recall that the nth Meyer-König and Zeller power series associated to f is defined by (see [4])

$$M_n f(1) = f(1),$$

$$M_n f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \qquad x \in [0,1[,$$

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}, \qquad k = 0, 1, 2, \ldots .$$

That (2) implies (1) follows from the fact that the sequence $(M_n f)_{n\geq 1}$ converges uniformly to f on $[0,1]$. Thus it remains to prove that (1) implies (2). To this end, let n be an arbitrary positive integer and let $0 \leq x_1 < x_2 < 1$ (since $M_n f$ is continuous at 1, it suffices to consider only the case $x_2 < 1$). Then we have

$$\begin{aligned}
&M_n f(x_2)\\
&= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} x_2^j (1-x_2)^{n+1}\\
&= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} (1-x_2)^{n+1}\left(\frac{x_2 - x_1 + x_1 - x_1 x_2}{1-x_1}\right)^j\\
&= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} \frac{(1-x_2)^{n+1}}{(1-x_1)^j}\sum_{k=0}^{j}\binom{j}{k} x_1^k (1-x_2)^k (x_2-x_1)^{j-k}\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{j} f\left(\frac{j}{n+j}\right)\frac{(n+j)!}{n!k!(j-k)!}\cdot\frac{x_1^k (x_2-x_1)^{j-k}(1-x_2)^{n+k+1}}{(1-x_1)^j}\\
&= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} f\left(\frac{j}{n+j}\right)\frac{(n+j)!}{n!k!(j-k)!}\cdot\frac{x_1^k (x_2-x_1)^{j-k}(1-x_2)^{n+k+1}}{(1-x_1)^j}\\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} f\left(\frac{k+\ell}{n+k+\ell}\right)\frac{(n+k+\ell)!}{n!k!\ell!}\cdot\frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}},
\end{aligned}$$

where the change of index $j-k=\ell$ was used for the last equality. We have also

$$
\begin{aligned}
M_n f(x_1) \\
&= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x_1^k (1-x_1)^{n+1} \\
&= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x_1^k \cdot \frac{(1-x_2)^{n+k+1}}{(1-x_1)^k} \cdot \frac{1}{\left(1-\frac{x_2-x_1}{1-x_1}\right)^{n+k+1}} \\
&= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} \frac{x_1^k (1-x_2)^{n+k+1}}{(1-x_1)^k} \sum_{\ell=0}^{\infty} \binom{n+k+\ell}{\ell}\left(\frac{x_2-x_1}{1-x_1}\right)^{\ell} \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} f\left(\frac{k}{n+k}\right) \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}}.
\end{aligned}
$$

In particular, the above equalities show that

(3) $$\sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = 1,$$

(4) $$\sum_{k,\ell=0}^{\infty} \frac{k+\ell}{n+k+\ell} \cdot \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_2,$$

(5) $$\sum_{k,\ell=0}^{\infty} \frac{k}{n+k} \cdot \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_1.$$

Since $f \in \mathrm{Lip}_A \alpha$, we have

$$
\begin{aligned}
&|M_n f(x_2) - M_n f(x_1)| \\
&\le \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2 - x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left| f\left(\frac{k+\ell}{n+k+\ell}\right) - f\left(\frac{k}{n+k}\right)\right| \\
&\le A \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2 - x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k}\right)^{\alpha}.
\end{aligned}
$$

Taking into account (3) and the fact that the function $t \in [0,\infty[\longmapsto t^\alpha \in [0,\infty[$ is concave, we deduce that

$$
\begin{aligned}
&|M_n f(x_2) - M_n f(x_1)| \\
&\le A \left[\sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2 - x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k}\right)\right]^{\alpha}.
\end{aligned}
$$

Using now (4) and (5) we get

$$
|M_n f(x_2) - M_n f(x_1)| \le A(x_2 - x_1)^\alpha,
$$

i.e., $M_n f \in \mathrm{Lip}_A \alpha$. This completes the proof. □

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