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INEQUALITIES FOR THE MAXIMUM MODULUS OF THE DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. Let $P(z)$ be a polynomial of degree n and $M(P,t) = \text{Max}_{|z|=t} |P(z)|$. In this paper we shall estimate $M(P^{'},\rho)$ in terms of $M(P,r)$ where $P(z)$ does not vanish in the disk $|z| \leq K,\ K \geq 1,\ 0 \leq r < \rho < K$ and obtain an interesting refinement of some result of Dewan and Malik. We shall also obtain an interesting generalization as well as a refinement of well-known result of P. Turan for polynomials not vanishing outside the unit disk.

Key words and phrases: Polynomial, Derivative, Bernstein Inequality, Maximum Modulus.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $P(z)$ be a polynomial of degree n and let $M(P,r) = \text{Max}_{|z|=r} |P(z)|$ and $m(P,t) = \min_{|z|=t} |P(z)|$ concerning the estimate of $\max |P^{'}(z)|$ in terms of the $\max |P(z)|$ on the unit circle $|z| = 1$, we have

$$\underset{|z|=1}{\text{Max}} |P^{'}(z)| \leq n \underset{|z|=1}{\text{Max}} |P(z)|. \tag{1.1}$$

Inequality (1.1) is a famous result known as Bernstein's Inequality (for reference see [4], [5], [10], [11]). Equality in (1.1) holds if and only if $P(z)$ has all its zeros at the origin. So it is natural to seek improvements under appropriate assumptions on the zeros of $P(z)$.

If $P(z)$ does not vanish in $|z| < 1$,then the inequality (1.1) can be replaced by

$$\underset{|z|=1}{\text{Max}} |P^{'}(z)| \leq \frac{n}{2} \underset{|z|=1}{\text{Max}} |P(z)| \tag{1.2}$$

Inequality (1.2) was conjectured by Erdos and later proved by Lax [8]. On the other hand, it was shown by Turan [12] that if all the zeros of $P(z)$ lie in $|z| < 1$, then

$$\underset{|z|=1}{\text{Max}} |P^{'}(z)| \geq \frac{n}{2} \underset{|z|=1}{\text{Max}} |P(z)|. \tag{1.3}$$

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As an extension of (1.2), Malik [9] showed that if $P(z)$ does not vanish in $|z| < K,\ K \geq 1$, then

$$\underset{|z|=1}{\operatorname{Max}} |P'(z)| \leq \frac{n}{1+K} \underset{|z|=1}{\operatorname{Max}} |P(z)| \tag{1.4}$$

Recently Dewan and Abdullah [6] have obtained the following generalization of inequality (1.4).

Theorem A. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z| < K,\ K \geq 1$, then for $0 \leq r < \rho \leq K$,*

$$\underset{|z|=\rho}{\operatorname{Max}} |P'(z)| \leq \frac{n(\rho+K)^{n-1}}{(K+r)^n} \left\{ 1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)n}{(K^2-\rho^2)n|a_0| + 2K^2\rho|a_1|} \times \left(\frac{\rho - r}{K+\rho}\right) \left(\frac{K+r}{K+\rho}\right)^{n-1} \right\} \underset{|z|=r}{\operatorname{Max}} |P(z)| \tag{1.5}$$

Inequality (1.3) was generalized by Aziz and Shah [2] by proving the following interesting result.

Theorem B. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in the disk $|z| \leq K \leq 1$ with $s-$fold zeros at origin, then for $|z| = 1$,*

$$\underset{|z|=1}{\operatorname{Max}} |P'(z)| \geq \frac{n+Ks}{1+K} \underset{|z|=1}{\operatorname{Max}} |P(z)|. \tag{1.6}$$

The result is sharp and the extremal polynomial is

$$P(z) = z^s (z+K)^{n-s}, \quad 0 < s \leq n.$$

Here in this paper,we shall first obtain the following interesting improvement of Theorem A which is also a generalization of inequality (1.4).

Theorem 1.1. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree $n > 1$, having no zeros in $|z| < K,\ K \geq 1$,then for $0 \leq r \leq \rho \leq K$,*

$$\begin{aligned} M(P',\rho) \leq{} & \frac{n(\rho+K)^{n-1}}{(K+r)^n} \\ & \times \left[1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)n}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|} \left(\frac{\rho-r}{K+\rho}\right) \left(\frac{K+r}{K+\rho}\right)^{n-1} \right] M(P,r) \\ & - n\left(\frac{r+K}{\rho+K}\right) \left[\frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|} \right. \\ & \left. \times \left\{ \left(\left(\frac{\rho+K}{r+K}\right)^n - 1 \right) - n(\rho - r) \right\} \right] m(P,K). \end{aligned} \tag{1.7}$$

The result is best possible and equality holds for the polynomial

$$P(z) = (z+K)^n.$$

Next we prove the following result which is a refinement of Theorem B.

Theorem 1.2. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in the disk $|z|\le K,\ K\le 1$ with $t-$fold zeros at the origin, then,*

$$M(P^{'},1)\ge \frac{n+Kt}{1+K}M(P,1)+\frac{n-t}{(1+K)K^t}m(P,K). \tag{1.8}$$

The result is sharp and equality holds for the polynomial

$$P(z)=z^t(z+K)^{n-t},\ 0<t\le n.$$

The following result immediately follows by taking $K=1$ in Theorem 1.2.

Corollary 1.3. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in $|z|\le 1$, with $t-$fold zeros at the origin, then for $|z|=1$,*

$$M(P^{'},1)\ge \frac{n+t}{2}M(P,1)+\frac{n-t}{2}m(P,1). \tag{1.9}$$

The result is best possible and equality holds for the polynomial $P(z)=(z+K)^n$.

Remark 1.4. For $t=0$, Corollary 1.3 reduces to a result due to Aziz and Dawood [1].

2. LEMMAS

For the proofs of these theorems, we require the following lemmas. The first result is due to Govil, Rahman and Schmeisser [7].

Lemma 2.1. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in $|z|\ge K\ge 1$, then*

$$\operatorname*{Max}_{|z|=1}|P^{'}(z)|\le n\frac{(n|a_0|+K^2|a_1|)}{(1+K^2)n|a_0|+2K^2|a_1|}\operatorname*{Max}_{|z|=1}|P(z)|. \tag{2.1}$$

Lemma 2.2. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n which does not vanish in $|z|<K$ where $K>0$,then for $0\le rR\le K^2$ and $r\le R$, we have*

$$\operatorname*{Max}_{|z|=r}|P(z)|\ge\left(\frac{r+K}{R+K}\right)^n\operatorname*{Max}_{|z|=R}|P(z)|+\left[1-\left(\frac{r+K}{R+K}\right)^n\right]\operatorname{Min}_{|z|=K}|P(z)|. \tag{2.2}$$

Here the result is best possible and equality in (2.2) holds for the polynomial $P(z)=(z+K)^n$.

Lemma 2.2 is due to Aziz and Zargar [3].

Lemma 2.3. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zeros in $|z|<K,\ K\ge 1$, then for $0\le r\le\rho\le K$,*

$$\begin{aligned} M(P,\rho) &\le\left(\frac{K+\rho}{K+r}\right)^n\left[1-\frac{K(K-\rho)(n|a_0|-K|a_1|)n}{(K^2+\rho^2)n|a_0|+2K^2\rho|a_1|}\left(\frac{\rho-r}{K+\rho}\right)\left(\frac{K+r}{K+\rho}\right)^{n-1}\right]M(P,r)\\ &\quad-\left[\frac{(n|a_0|\rho+K^2|a_1|)(r+K)}{(\rho^2+K^2)n|a_0|+2K^2\rho|a_1|}\left\{\left(\left(\frac{\rho+K}{r+K}\right)^n-1\right)-n(\rho-r)\right\}\right]m(P,K). \end{aligned} \tag{2.3}$$

The result is best possible with equality for the polynomial $P(z)=(z+K)^n$.

Proof of Lemma 2.3. Since $P(z)$ has no zeros in $|z| < K,\ K \geq 1$, therefore the polynomial $T(z) = P(tz)$ has no zeros in $|z| < \frac{K}{t}$, where $0 \leq t \leq K$. Using Lemma 2.1 for the polynomial $T(z)$, with K replaced by $\frac{K}{t} \geq 1$, we get

$$\underset{|z|=1}{\text{Max}}|T^{'}(z)| \leq n\left\{\frac{(n|a_0| + \frac{K^2}{t^2}|ta_1|)}{(1+\frac{K^2}{t^2})n|a_0| + 2\frac{K^2}{t^2}|ta_1|}\right\}\underset{|z|=1}{\text{Max}}|T(z)|,$$

which implies

$$\underset{|z|=t}{\text{Max}}|P^{'}(z)| \leq n\left\{\frac{(n|a_0|t + K^2|a_1|)}{(t^2+K^2)n|a_0| + 2K^2t|a_1|}\right\}\underset{|z|=t}{\text{Max}}|P(z)|. \tag{2.4}$$

Now for $0 \leq r \leq \rho \leq K$ and $0 \leq \theta < 2\pi$, by (2.4) we have

$$\begin{aligned} \left|P(\rho e^{i\theta}) - P(re^{i\theta})\right| &\leq \int_r^\rho |P^{'}(te^{i\theta})|dt \\ &\leq \int_r^\rho n\left\{\frac{(n|a_0|t + K^2|a_1|)}{(t^2+K^2)n|a_0| + 2K^2t|a_1|}\right\}\underset{|z|=t}{\text{Max}}|P(z)|. \end{aligned} \tag{2.5}$$

Using Lemma 2.2 with $R = t$ and noting that $0 \leq r \leq t \leq \rho \leq K$ and $0 \leq rt \leq K^2$, it follows that

$$|P(\rho e^{i\theta}) - P(re^{i\theta})| \leq \int_r^\rho n\left\{\frac{(n|a_0|t + K^2|a_1|)}{(t^2+K^2)n|a_0| + 2K^2t|a_1|}\right\},$$

$$\begin{aligned} &\left(\frac{t+K}{r+K}\right)^n\left\{M(P,r) - \left(1 - \left(\frac{r+K}{t+K}\right)^n\right)m(P,K)\right\}dt \\ &\qquad\leq n\left\{\frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right\} \\ &\qquad\quad\times\int_r^\rho \left(\frac{t+K}{r+K}\right)^n\left\{M(P,r) - \left(1 - \left(\frac{r+K}{t+K}\right)^n\right)m(P,K)\right\}dt. \end{aligned}$$

This gives for $0 \leq r \leq \rho \leq K$,

$$\begin{aligned} &M(P,\rho) \\ &\leq \left[1 + \frac{n(K+\rho)}{(K+r)^n}\left\{\frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right\}\int_r^\rho (K+t)^{n-1}dt\right]M(P,r) \\ &\quad - n\left\{\frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right\}\int_r^\rho\left(\left(\frac{t+K}{r+K}\right)^n - 1\right)dt\, m(P,k) \\ &\leq \left[1 - \left\{\frac{(K+\rho)(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right\}\right. \\ &\quad\left. + \left\{\frac{(K+\rho)(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right\}\left(\frac{K+\rho}{K+r}\right)^n\right]M(P,r) \\ &\quad - n\left[\left\{\frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right\}\int_r^\rho\left(\frac{(t+K)^{n-1}}{(r+K)^{n-1}} - 1\right)dt\right]m(P,k) \end{aligned}$$

$$
\begin{aligned}
&< \left[\frac{K(K-\rho)(n|a_0|-K|a_1|)}{(K^2+\rho^2)n|a_0|+2K^2\rho|a_1|}\right.\\
&\qquad \left. + \left\{1-\frac{K(K-\rho)(n|a_0|-K|a_1|)}{(K^2+\rho^2)n|a_0|+2K^2\rho|a_1|}\right\}\left(\frac{K+\rho}{K+r}\right)^n\right] M(P,r)\\
&\qquad - n\left\{\frac{(n|a_0|\rho+K^2|a_1|)}{(\rho^2+K^2)n|a_0|+2K^2\rho|a_1|}\int_r^\rho\left(\left(\frac{t+K}{r+K}\right)^{n-1}-1\right)dt\right\} m(P,k)\\
&= \left(\frac{K+\rho}{K+r}\right)^n\left[1-\frac{K(K-\rho)(n|a_0|-K|a_1|)}{(K^2+\rho^2)n|a_0|+2K^2\rho|a_1|}\left\{1-\left(\frac{K+r}{K+\rho}\right)^n\right\}\right] M(P,r)\\
&\qquad - n\left\{\frac{(n|a_0|\rho+K^2|a_1|)}{(\rho^2+K^2)n|a_0|+2K^2\rho|a_1|}\right\}\frac{1}{(r+K)^{n-1}}\\
&\qquad \times\left\{\frac{(\rho+K)^n-(r+K)^n}{n}-(\rho-r)\right\} m(P,k)\\
&= \left(\frac{K+\rho}{K+r}\right)^n\left[1-\frac{K(K-\rho)(n|a_0|-K|a_1|)}{(K^2+\rho^2)n|a_0|+2K^2\rho|a_1|}\right.\\
&\qquad \left.\times\frac{(\rho-r)}{(K+\rho)\left\{1-\frac{K+r}{K+\rho}\right\}}\left\{1-\left(\frac{K+r}{K+\rho}\right)^n\right\}\right] M(P,r)\\
&\qquad - \left[\frac{(n|a_0|\rho+K^2|a_1|)}{(\rho^2+K^2)n|a_0|+2K^2\rho|a_1|}\right.\\
&\qquad \left.\times(r+K)\left\{\left\{\left(\frac{\rho+K}{r+K}\right)^n-1\right\}-n(\rho-r)\right\}\right] m(P,k)\\
&\le \left(\frac{K+\rho}{K+r}\right)^n\left[1-\frac{K(K-\rho)(n|a_0|-K|a_1|)n}{(K^2+\rho^2)n|a_0|+2K^2\rho|a_1|}\left(\frac{\rho-r}{K+\rho}\right)\left(\frac{K+r}{K+\rho}\right)^{n-1}\right] M(P,r)\\
&\qquad - \left[\frac{(n|a_0|\rho+K^2|a_1|)(r+K)}{(\rho^2+K^2)n|a_0|+2K^2\rho|a_1|}\left\{\left(\left(\frac{\rho+K}{r+K}\right)^n-1\right)-n(\rho-r)\right\}\right] m(P,K)
\end{aligned}
$$

which proves Lemma 2.3. □

3. PROOF OF THE THEOREMS

Proof of Theorem 1.1. Since the polynomial $P(z)=\sum_{j=0}^n a_j z^j$ has no zeros in $|z|<K$, where $K\ge 1$, therefore it follows that $F(z)=P(\rho z)$ has no zero in $|z|<\frac{K}{\rho}$, $\frac{K}{\rho}\ge 1$. Applying inequality (1.4) to the polynomial $F(z)$, we get

$$\operatorname*{Max}_{|z|=1}|F'(z)|\le\frac{n}{1+\frac{K}{\rho}}\operatorname*{Max}_{|z|=1}|F(z)|,$$

which gives

$$\operatorname*{Max}_{|z|=\rho}|P'(z)|\le\frac{n}{\rho+K}\operatorname*{Max}_{|z|=\rho}|P(z)|. \tag{3.1}$$

Now if $0 \le r \le \rho \le K$, then from (3.1) it follows with the help of Lemma 2.3 that

$$\underset{|z|=\rho}{\operatorname{Max}} |P'(z)| \le \frac{n(K+\rho)^{n-1}}{(K+r)^n} \left[1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)n}{(K^2+\rho^2)n|a_0| + 2K^2\rho|a_1|} \right.$$
$$\left. \times \left(\frac{\rho - r}{K+\rho}\right)\left(\frac{K+r}{K+\rho}\right)^{n-1}\right] M(P,r) - n\left(\frac{r+K}{\rho+K}\right)\left[\frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2+K^2)n|a_0| + 2K^2\rho|a_1|}\right.$$
$$\left. \times \left\{\left(\left(\frac{\rho+K}{r+K}\right)^n - 1\right) - n(\rho - r)\right\}\right] m(P,K),$$

which completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. If $m = \operatorname{Min}_{|z|=K} |P(z)|$, then $m \le |P(z)|$ for $|z| = K$, which gives $m|\frac{z}{K}|^t \le |P(z)|$ for $|z| = K$. Since all the zeros of $P(z)$ lie in $|z| \le K \le 1$, with $t-$fold zeros at the origin,therefore for every complex number α such that $|\alpha| < 1$, it follows (by Rouches' Theorem for $m > 0$) that the polynomial $G(z) = P(z) + \frac{\alpha m}{K^t} z^t$ has all its zeros in $|z| \le K,\ K \le 1$ with $t-$fold zeros at the origin,so that we can write

$$G(z) = z^t H(z), \tag{3.2}$$

where $H(z)$ is a polynomial of degree $n-t$ having all its zeros in $|z| \le K,\ K \le 1$.

From (3.2), we get

$$\frac{zG'(z)}{G(z)} = t + \frac{zH'(z)}{H(z)}. \tag{3.3}$$

If $z_1, z_2, \dots, z_{n-t}$ are the zeros of $H(z)$, then $|z_j| \le K \le 1$ and from (3.3), we have

$$\begin{aligned} \operatorname{Re}\left\{\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})}\right\} &= t + \operatorname{Re}\left\{\frac{e^{i\theta}H'(e^{i\theta})}{H(e^{i\theta})}\right\} \\ &= t + \operatorname{Re}\sum_{j=1}^{n-t} \frac{e^{i\theta}}{e^{i\theta} - z_j} \\ &= t + \sum_{j=1}^{n-t} \operatorname{Re}\left(\frac{1}{1 - z_j e^{-i\theta}}\right) \end{aligned} \tag{3.4}$$

for points $e^{i\theta}$, $0 \le \theta < 2\pi$ which are not the zeros of $H(z)$.

Now,if $|w| \le K \le 1$, then it can be easily verified that

$$\operatorname{Re}\left(\frac{1}{1-w}\right) \ge \frac{1}{1+K}.$$

Using this fact in (3.4), we see that

$$\begin{aligned} \left|\frac{G'(e^{i\theta})}{G(e^{i\theta})}\right| &\ge \operatorname{Re}\left(\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})}\right) \\ &= t + \sum_{j=1}^{n-t} \operatorname{Re}\left(\frac{1}{1 - z_j e^{-i\theta}}\right) \ge t + \frac{n-t}{1+K}, \end{aligned}$$

which gives,

$$|G'(e^{i\theta})| \ge \frac{n+tK}{1+K} |G(e^{i\theta})| \tag{3.5}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$ which are not the zeros of $G(z)$. Since inequality (3.5) is trivially true for points $e^{i\theta}$, $0 \leq \theta < 2\pi$ which are the zeros of $P(z)$, it follows that

$$|G^{'}(z)| \geq \frac{n+tK}{1+K}|G(z)| \quad \text{for} \quad |z|=1. \tag{3.6}$$

Replacing $G(z)$ by $P(z) + \frac{\alpha m}{K^t} z^t$ in (3.6), then we get

$$\left|P^{'}(z) + \frac{\alpha t m}{K^t} z^{t-1}\right| \geq \frac{n+tK}{1+K}\left|P(z) + \frac{\alpha m}{K^t} z^t\right| \quad \text{for} \quad |z|=1 \tag{3.7}$$

and for every α with $|\alpha| < 1$. Choosing the argument of α such that

$$\left|P(z) + \frac{\alpha m}{K^t} z^t\right| = |P(z)| + |\alpha|\frac{m}{K^t} \quad \text{for} \quad |z|=1,$$

it follows from (3.7) that

$$|P^{'}(z)| + \frac{t|\alpha|m}{K^t} \geq \frac{n+tK}{1+K}\left[|P(z)| + |\alpha|\frac{m}{K^t}\right] \quad \text{for} \quad |z|=1.$$

Letting $|\alpha| \to 1$, we obtain

$$\begin{aligned} \left|P^{'}(z)\right| &\geq \frac{n+tK}{1+K}|P(z)| + \left[\frac{n+tK}{1+K} - t\right]\frac{m}{K^t} \\ &= \frac{n+tK}{1+K}|P(z)| + \frac{n-t}{1+K}\frac{m}{K^t} \quad \text{for} \quad |z|=1. \end{aligned}$$

This implies

$$\underset{|z|=1}{\text{Max}}|P^{'}(z)| \geq \frac{n+tK}{1+K}\underset{|z|=1}{\text{Max}}|P(z)| + \frac{n-t}{(1+K)K^t}\underset{|z|=K}{\text{Min}}|P(z)|$$

which is the desired result. □

REFERENCES

[1] A. AZIZ AND Q.M. DAWOOD, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **54** (1988), 306–313.

[2] A. AZIZ AND W.M. SHAH, Inequalities for a polynomial and its derivative, *Math. Inequal. and Applics.*, **7**(3) (2004), 379–391.

[3] A. AZIZ AND B.A. ZARGAR, Inequalities for a polynomial and its derivative, *Math. Inequal. and Applics.*, **1**(4) (1998), 543–550.

[4] S. BERNSTEIN, Sur l' ordre de la meilleure approximation des fonctions continues pardes polynômes de degré donné, *Mémoires de l'Académie Royale de Belgique*, **4** (1912), 1–103.

[5] S. BERNSTEIN, Leçons sur les propriétés extrémales et la meilleure d'une fonctions réella, Paris 1926.

[6] K.K. DEWAN AND A. MIR, On the maximum modulus of a polynomial and its derivatives, *International Journal of Mathematics and Mathematical Sciences*, **16** (2005), 2641–2645.

[7] N.K. GOVIL, Q.I. RAHMAN AND G. SCHMEISSER, On the derivative of a polynomial, *Illinois J. Math.*, **23** (1979), 319–329.

[8] P.D. LAX, Proof of a conjecture of P. Erdős on the derivative of a polynomial, *Bull. Amer. Math. Soc. (N.S)*, **50** (1944), 509–513.

[9] M.A. MALIK, On the derivative of a polynomial, *J. London Math. Soc.*, **1** (1969), 57–60.

[10] A.C. SCHAFFER, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, *Bull. Amer. Math. Soc.*, **47** (1941), 565–579.

[11] G.V. MILOVANOVIĆ, D.S. MITRINOVIĆ AND Th.M. RASSIAS, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*,World Scientific, Singapore, 1994.

[12] P. TURÁN, Über die Ableitung fon Polynomen, *Compositio Math.*, **7** (1939-40), 89–95.