# ON SOME FENG QI TYPE $q$-INTEGRAL INEQUALITIES

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*Received:* 03 May, 2008

*Accepted:* 30 May, 2008

*Communicated by:* F. Qi

*2000 AMS Sub. Class.:* 26D10, 26D15, 33D05, 33D15.

*Key words:* $q$-series, $q$-integral, Inequalities.

*Abstract:* In this paper, we provide some Feng Qi type $q$-Integral Inequalities, by using analytic and elementary methods in Quantum Calculus.



journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

## 1. Introduction

In [9], F. Qi studied an interesting integral inequality and proved the following result:

**Theorem 1.1.** *For a positive integer $n$ and an $n^{th}$ order continuous derivative function $f$ on an interval $[a,b]$ such that $f^{(i)}(a) \geq 0$, $0 \leq i \leq n-1$ and $f^{(n)}(a) \geq n!$, we have*

$$\int_a^b [f(t)]^{n+2} dt \geq \left[\int_a^b f(t)dt\right]^{n+1}. \tag{1.1}$$

Then, he proposed the following open problem:

*Under what condition is the inequality (1.1) still true if $n$ is replaced by any positive real number $p$?*

In view of the interest in this type of inequality, much attention has been paid to the problem and many authors have extended the inequality to more general cases (see [1, 8]). In this paper, we shall discuss a $q$-analogue of the Feng Qi problem and we will generalize the inequalities given in [1], [7] and [8].

This paper is organized as follows: In Section 2, we present definitions and facts from $q$-calculus necessary for understanding this paper. In Section 3, we discuss some generalizations of the so-called Feng Qi inequality.


vol. 9, iss. 2, art. 43, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

## 2. Notations and Preliminaries

Throughout this paper, we will fix $q \in (0,1)$. For the convenience of the reader, we provide a summary of the mathematical notations and definitions used in this paper (see [3] and [5]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1-q^a}{1-q}, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2,\dots,\infty,$$

$$[0]_q! = 1, \qquad [n]_q! = [1]_q[2]_q...[n]_q, \quad n = 1,2,\dots$$

and

$$(x-a)_q^n = \begin{cases} 1 & \text{if } \ n = 0 \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } \ n \neq 0 \end{cases} \qquad x \in \mathbb{C}, \quad n \in \mathbb{N}.$$

The $q$-derivative $D_q f$ of a function $f$ is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } \ x \neq 0, \tag{2.1}$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

The $q$-Jackson integral from $0$ to $a$ is defined by (see [4])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.2}$$

provided the sum converges absolutely.



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issn: 1443-5756

The $q$-Jackson integral in a generic interval $[a, b]$ is given by (see [4])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{2.3}$$

We recall that for any function $f$, we have (see [5])

$$D_q\left(\int_a^x f(t)d_qt\right) = f(x). \tag{2.4}$$

Finally, for $b > 0$ and $a = bq^n$, $n$ a positive integer, we write

$$[a, b]_q = \{bq^k : \ 0 \leq k \leq n\} \quad \text{and} \quad (a, b]_q = [q^{-1}a, b]_q.$$



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issn: 1443-5756

## 3. $q$-Integral Inequalities of Feng Qi type

Let us begin with the following useful result:

**Lemma 3.1.** *Let $p \geq 1$ be a real number and $g$ be a nonnegative and monotone function on $[a,b]_q$. Then*

$$pg^{p-1}(qx)D_q g(x) \leq D_q[g(x)]^p \leq pg^{p-1}(x)D_q g(x), \quad x \in (a,b]_q.$$

*Proof.* We have

$$D_q[g^p](x) = \frac{g^p(x) - g^p(qx)}{(1-q)x} = \frac{1}{(1-q)x} p \int_{g(qx)}^{g(x)} t^{p-1} dt. \tag{3.1}$$

Since $g$ is a nonnegative and monotone function, we have

$$g^{p-1}(qx)\left[g(x) - g(qx)\right] \leq \int_{g(qx)}^{g(x)} t^{p-1} dt \leq g^{p-1}(x)\left[g(x) - g(qx)\right].$$

Therefore, according to the relation (3.1), we obtain

$$pg^{p-1}(qx)D_q g(x) \leq D_q[g^p](x) \leq pg^{p-1}(x)D_q g(x).$$

□

**Proposition 3.2.** *Let $f$ be a function defined on $[a,b]_q$ satisfying*

$$f(a) \geq 0 \quad \text{and} \quad D_q f(x) \geq (t-2)(x-a)^{t-3} \quad \text{for} \quad x \in (a,b]_q \quad \text{and} \quad t \geq 3.$$

*Then*

$$\int_a^b [f(x)]^t d_q x \geq \left(\int_a^b f(qx) d_q x\right)^{t-1}.$$


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*Proof.* Put $g(x) = \int_a^x f(qu)d_qu$ and

$$F(x) = \int_a^x [f(u)]^t d_qu - \left(\int_a^x f(qu)d_qu\right)^{t-1}.$$

We have

$$D_qF(x) = f^t(x) - D_q[g^{t-1}](x).$$

Since $f$ and $g$ increase on $[a,b]_q$, we obtain from Lemma 3.1,

$$\begin{aligned} D_qF(x) &\geq f^t(x) - (t-1)g^{t-2}(x)f(qx) \\ &\geq f^t(x) - (t-1)g^{t-2}(x)f(x) = f(x)h(x), \end{aligned}$$

where $h(x) = f^{t-1}(x) - (t-1)g^{t-2}(x)$.

On the other hand, we have

$$D_qh(x) = D_q[f^{t-1}](x) - (t-1)D_q[g^{t-2}](x).$$

By using Lemma 3.1, we obtain

$$\begin{aligned} D_qh(x) &\geq (t-1)f^{t-2}(qx)D_qf(x) - (t-1)(t-2)g^{t-3}(x)D_qg(x) && (3.2) \\ &\geq (t-1)f(qx)\left[f^{t-3}(qx)D_qf(x) - (t-2)g^{t-3}(x)\right]. && (3.3) \end{aligned}$$

Since the function $f$ increases, we have

$$\int_a^x f(qu)d_qu \leq f(qx)(x-a).$$

Then, from the conditions of the proposition and inequalities (3.2) and (3.3), we get

$$D_qh(x) \geq (t-1)f^{t-2}(qx)\left[D_qf(x) - (t-2)(x-a)^{t-3}\right] \geq 0,$$


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and from the fact $h(a) = f^{t-1}(a) \geq 0$, we get $h(x) \geq 0, \quad x \in [a,b]_q$.

From $F(a) = 0$ and $D_qF(x) = f(x)h(x) \geq 0$, it follows that $F(x) \geq 0$ for all $x \in [a,b]_q$, in particular

$$F(b) = \int_a^b [f(u)]^t d_qu - \left(\int_a^b f(qu)d_qu\right)^{t-1} \geq 0.$$

□

**Corollary 3.3.** *Let $n$ be a positive integer and $f$ be a function defined on $[a,b]_q$ satisfying*

$$f(a) \geq 0 \quad \textit{and} \quad D_qf(x) \geq n(x-a)^{n-1}, \quad x \in (a,b]_q.$$

*Then,*

$$\int_a^b (f(x))^{n+2} d_qx \geq \left(\int_a^b f(qx)d_qx\right)^{n+1}.$$

*Proof.* It suffices to take $t = n+2$ in Proposition 3.2 and the result follows. □

**Corollary 3.4.** *Let $n$ be a positive integer and $f$ be a function defined on $[a,b]_q$ satisfying*

$$D_q^i f(a) \geq 0, \quad 0 \leq i \leq n-1 \quad \textit{and} \quad D_q^n f(x) \geq n[n-1]_q! \quad x \in (a,b]_q.$$

*Then,*

$$\int_a^b (f(x))^{n+2} d_qx \geq \left(\int_a^b f(qx)d_qx\right)^{n+1}.$$

*Proof.* Since $D_q^n f(x) \geq n[n-1]_q!$, then by $q$-integrating $n-1$ times over $[a,x]$, we get

$$D_qf(x) \geq n(x-a)_q^{n-1} \geq n(x-a)^{n-1}.$$

The result follows from Corollary 3.3. □



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**Proposition 3.5.** *Let $p \geq 1$ be a real number and $f$ be a function defined on $[a,b]_q$ satisfying*

$$f(a) \geq 0, \qquad D_q f(x) \geq p, \qquad \forall\, x \in (a,b]_q. \tag{3.4}$$

*Then we have*

$$\int_a^b [f(x)]^{p+2} d_q x \geq \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(qx) d_q x\right]^{p+1}. \tag{3.5}$$

*Proof.* Put $g(t) = \int_a^t f(qx) d_q x$ and

$$H(t) = \int_a^t [f(x)]^{p+2} d_q x - \frac{1}{(b-a)^{p-1}} \left[\int_a^t f(qx) d_q x\right]^{p+1}, \quad t \in [a,b]_q. \tag{3.6}$$

We have

$$D_q H(t) = [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}} D_q[g^{p+1}](t), \quad t \in (a,b]_q.$$

Since $f$ and $g$ increase on $[a,b]_q$, we obtain, according to Lemma 3.1, for $t \in (a,b]_q$,

$$\begin{aligned}
D_q H(t) &\geq [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}}(p+1) g^p(t) f(qt) \\
&\geq [f(t)]^{p+2} - \frac{1}{(b-a)^{p-1}}(p+1) g^p(t) f(t) \\
&\geq \left([f(t)]^{p+1} - \frac{1}{(b-a)^{p-1}}(p+1) g^p(t)\right) f(t) = h(t) f(t),
\end{aligned}$$


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where

$$h(t) = [f(t)]^{p+1} - \frac{1}{(b-a)^{p-1}}(p+1)g^p(t).$$

On the other hand, we have

$$D_q h(t) = D_q[f^{p+1}](t) - \frac{1}{(b-a)^{p-1}}(p+1)D_q[g^p](t).$$

By using Lemma 3.1, we obtain

$$\begin{aligned} D_q h(t) &\geq (p+1)f^p(qt)D_q f(t) - \frac{(p+1)p}{(b-a)^{p-1}} g^{p-1}(t)f(qt) \\ &\geq (p+1)f(qt)\left[f^{p-1}(qt)D_q f(t) - \frac{p}{(b-a)^{p-1}} g^{p-1}(t)\right]. \end{aligned}$$

Since $f$ increases, then for $t \in [a,b]_q$,

$$\int_a^t f(qx)d_q x \leq (b-a)f(qt), \tag{3.7}$$

therefore,

$$D_q h(t) \geq (p+1)f^p(qt)[D_q f(t) - p]. \tag{3.8}$$

We deduce, from the relation (3.4), that $h$ increases on $[a,b]_q$.

Finally, since $h(a) = f^{p+1}(a) \geq 0$, then $H$ increases and $H(b) \geq H(a) \geq 0$, which completes the proof. □

**Corollary 3.6.** *Let $p \geq 1$ be a real number and $f$ be a nonnegative function on $[0,1]$ such that $D_q f(x) \geq 1$. Then*

$$\int_0^1 [f(x)]^{p+2} d_q x \geq \frac{1}{p}\left[\int_0^1 f(qx)d_q x\right]^{p+1}. \tag{3.9}$$



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*Proof.* Replacing, in the previous proposition, $f(x)$ by $pf(x)$, $b$ by 1 and $a$ by $q^N$ ($N = 1, 2, \ldots$), we obtain then the result by tending $N$ to $\infty$. $\square$

In what follows, we will adopt the terminology of the following definition.

**Definition 3.7.** *Let $b > 0$ and $a = bq^n$, where $n$ is a positive integer. For each real number $r$, we denote by $E_{q,r}([a,b])$ the set of functions defined on $[a,b]_q$ such that*

$$f(a) \geq 0 \qquad \text{and} \qquad D_q f(x) \geq [r]_q, \quad \forall x \in (a,b]_q.$$

**Proposition 3.8.** *Let $f \in E_{q,2}([a,b])$. Then for all $p > 0$, we have*

$$\int_a^b [f(x)]^{2p+1} d_q x > \left[ \int_a^b (f(x))^p d_q x \right]^2 . \tag{3.10}$$

*Proof.* For $t \in [a,b]_q$, we put

$$F(t) = \int_a^t [f(x)]^{2p+1} d_q x - \left[ \int_a^t (f(x))^p d_q x \right]^2 \quad \text{and} \quad g(t) = \int_a^t [f(x)]^p d_q x.$$

Then, we have for $t \in [a,b]_q$,

$$\begin{aligned} D_q F(t) &= [f(t)]^{2p+1} - [f(t)]^p (g(t) + g(qt)) \\ &= [f(t)]^p \left( [f(t)]^{p+1} - [g(t) + g(qt)] \right) \\ &= [f(t)]^p G(t), \end{aligned}$$

where $G(t) = [f(t)]^{p+1} - [g(t) + g(qt)]$.


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On the other hand, we have

$$D_qG(t) = \frac{f^{p+1}(t) - f^{p+1}(qt)}{(1-q)t} - f^p(t) - qf^p(qt)$$
$$= f^p(t)\frac{f(t) - (1-q)t}{(1-q)t} - f^p(qt)\frac{f(qt) + q(1-q)t}{(1-q)t}.$$

By using the relation $D_q f(t) \geq [2]_q$, we obtain $f(t) \geq f(qt) + (1-q^2)t$, therefore

$$D_qG(t) \geq \frac{f^p(t) - f^p(qt)}{(1-q)t}[f(qt) + q(1-q)t] > 0, \quad t \in (a,b]_q. \tag{3.11}$$

Hence, $G$ is strictly increasing on $[a,b]_q$. Moreover, we have

$$G(a) = [f(a)]^{p+1} + (1-q)af(a) \geq 0,$$

for all $t \in (a,b]_q$, $G(t) > G(a) \geq 0$, which proves that $D_qF(t) > 0$, for all $t \in (a,b]_q$. Thus, $F$ is strictly increasing on $[a,b]_q$. In particular, $F(b) > F(a) = 0$. $\square$

**Corollary 3.9.** *Let $\alpha > 0$ and $f \in E_{q,2}([a,b])$. Then for all positive integers $m$, we have*

$$\int_a^b [f(x)]^{(\alpha+1)2^m - 1} d_qx > \left[\int_a^b [f(x)]^\alpha d_qx\right]^{2^m}. \tag{3.12}$$

*Proof.* We suggest here a proof by induction. For this purpose, we put:

$$p_m(\alpha) = (\alpha+1)2^m - 1.$$

We have

$$p_m(\alpha) > 0 \quad \text{and} \quad p_{m+1}(\alpha) = 2p_m(\alpha) + 1. \tag{3.13}$$

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From Proposition 3.8, we deduce that the inequality (3.12) is true for $m = 1$. Suppose that (3.12) holds for an integer $m$ and let us prove it for $m+1$. By using the relation (3.13) and Proposition 3.8, we obtain

$$\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^2. \tag{3.14}$$

And, by assumption, we have

$$\int_a^b [f(x)]^{(\alpha+1)2^m-1} > \left[\int_a^b [f(x)]^{\alpha} d_q x\right]^{2^m}. \tag{3.15}$$

Finally, the relations (3.14) and (3.15) imply that the inequality (3.12) is true for $m+1$. This completes the proof. □

**Corollary 3.10.** *Let $f \in E_{q,2}([a,b])$ and $\alpha > 0$. For $m \in \mathbb{N}$, we have*

$$\left[\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x\right]^{\frac{1}{2^{m+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^{\frac{1}{2^m}}. \tag{3.16}$$

*Proof.* Since, from Proposition 3.8,

$$\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^2, \tag{3.17}$$

then

$$\left[\int_a^b [f(x)]^{(\alpha+1)2^{m+1}-1} d_q x\right]^{\frac{1}{2^{m+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^m-1} d_q x\right]^{\frac{1}{2^m}}. \tag{3.18}$$

□

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**Corollary 3.11.** *Let $f \in E_{q,2}([a,b])$. For all integers $m \geq 2$, we have*

$$\int_a^b [f(x)]^{2^{m+1}-1} d_q x > \left[\int_a^b [f(x)]^3 d_q x\right]^{2^{m-1}} \tag{3.19}$$

$$> \left[\int_a^b f(x) d_q x\right]^{2^m}. \tag{3.20}$$

*Proof.* By using Proposition 3.8 and the two previous corollaries for $\alpha = 1$, we obtain the required result. □

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issn: 1443-5756