

# ON CERTAIN INEQUALITIES IMPROVING THE HERMITE-HADAMARD INEQUALITY

SABIR HUSSAIN AND MATLOOB ANWAR

School of Mathematical Sciences

GC University, Lahore

Pakistan

E-Mail: [sabirhus@gmail.com](mailto:sabirhus@gmail.com) and [matloob\\_t@yahoo.com](mailto:matloob_t@yahoo.com)

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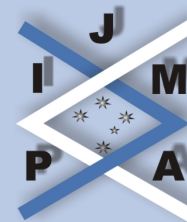
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*Key words:* Convex function, Hermite-Hadamard inequality, Mean value.

*Abstract:* A generalized form of the Hermite-Hadamard inequality for convex Lebesgue integrable functions are obtained.

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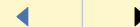
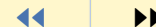
Improving the Hermite-Hadamard  
Inequality

Sabir Hussain and Matloob Anwar

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The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function  $f : [a, b] \rightarrow \mathbb{R}$  :

$$(HH) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

See [2, pp. 50-51], for details. This result can be easily improved by applying (HH) on each of the subintervals  $[a, (a+b)/2]$  and  $[(a+b)/2, b]$ ; summing up side by side we get

$$(SLHH) \quad \frac{1}{2} \left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \leq \frac{1}{b-a} \int_a^b f(x) dx$$

$$(SRHH) \quad \leq \frac{1}{2} \left[ f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right].$$

Usually, the precision in the (HH) inequalities is estimated via Ostrowski's and Iyengar's inequalities. See [2], p. 63 and respectively p. 191, for details. Based on previous work done by S.S. Dragomir and A.McAndrew [1], we shall prove here several better results, that apply to a slightly larger class of functions.

We start by estimating the deviation of the support line of a convex function from the mean value. The main ingredient is the existence of the subdifferential.

**Theorem 1.** *Assume that  $f$  is Lebesgue integrable and convex on  $(a, b)$ . Then*

$$\frac{1}{b-a} \int_a^b f(y) dy + \varphi(x) \left( x - \frac{a+b}{2} \right) - f(x)$$

$$\geq \left| \frac{1}{b-a} \int_a^b |f(y) - f(x)| dy - |\varphi(x)| \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right|$$

for all  $x \in (a, b)$ .



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Here  $\varphi : (a, b) \rightarrow \mathbb{R}$  is any function such that  $\varphi(x) \in [f'_-(x), f'_+(x)]$  for all  $x \in (a, b)$ .

*Proof.* In fact,

$$f(y) \geq f(x) + (y - x)\varphi(x)$$

for all  $x, y \in (a, b)$ , which yields

$$\begin{aligned} \text{(Sd)} \quad f(y) - f(x) - (y - x)\varphi(x) &= |f(y) - f(x) - (y - x)\varphi(x)| \\ &\geq ||f(y) - f(x)| - |y - x||\varphi(x)||. \end{aligned}$$

By integrating side by side we get

$$\begin{aligned} &\int_a^b f(y)dy - (b - a)f(x) + (b - a) \left( x - \frac{a + b}{2} \right) \varphi(x) \\ &\geq \int_a^b ||f(y) - f(x)| - |y - x||\varphi(x)|| dy \\ &\geq \left| \int_a^b |f(y) - f(x)| dy - |\varphi(x)| \int_a^b |y - x| dy \right| \\ &= \left| \int_a^b |f(y) - f(x)| dy - |\varphi(x)| \frac{(x - a)^2 + (b - x)^2}{2} \right| \end{aligned}$$

and it remains to simplify both sides by  $b - a$ . □

Theorem 1 applies for example to convex functions not necessarily defined on compact intervals, for example, to  $f(x) = (1 - x^2)^{-\alpha}$ ,  $x \in (-1, 1)$ , for  $\alpha \geq 0$ .



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**Theorem 2.** Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is a convex function. Then

$$\begin{aligned} & \frac{1}{2} \left[ f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(y) dy \\ & \geq \frac{1}{2} \left| \frac{1}{b-a} \int_a^b |f(x) - f(y)| dy - \frac{1}{b-a} \int_a^b |x-y| |f'(y)| dy \right| \end{aligned}$$

for all  $x \in (a, b)$ .

*Proof.* Without loss of generality we may assume that  $f$  is also continuous. See [2, p. 22] (where it is proved that  $f$  admits finite limits at the endpoints).

In this case  $f$  is absolutely continuous and thus it can be recovered from its derivative. The function  $f$  is differentiable except for countably many points, and letting  $\mathcal{E}$  denote this exceptional set, we have

$$f(x) \geq f(y) + (x-y)f'(y)$$

for all  $x \in [a, b]$  and all  $y \in [a, b] \setminus \mathcal{E}$ . This yields

$$\begin{aligned} f(x) - f(y) - (x-y)f'(y) &= |f(x) - f(y) - (x-y)f'(y)| \\ &\geq ||f(x) - f(y)| - |x-y| \cdot |f'(y)||, \end{aligned}$$

so that by integrating side by side with respect to  $y$  we get

$$\begin{aligned} (b-a)f(x) - 2 \int_a^b f(y) dy + f(b)(b-x) + f(a)(x-a) \\ \geq \left| \int_a^b |f(x) - f(y)| dy - \int_a^b |x-y| |f'(y)| dy \right| \end{aligned}$$



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equivalently,

$$f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a} - \frac{2}{b-a} \int_a^b f(y) dy$$

$$\geq \frac{1}{b-a} \left| \int_a^b |f(x) - f(y)| dy - \int_a^b |x-y| |f'(y)| dy \right|$$

and the result follows.  $\square$

A variant of Theorem 2, in the case where  $f$  is convex only on  $(a, b)$ , is as follows:

**Theorem 3.** Assume that  $f : [a, b] \rightarrow \mathbb{R}$  is monotone on  $[a, b]$  and convex on  $(a, b)$ . Then

$$\frac{1}{2} \left[ f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(y) dy$$

$$\geq \left| \frac{1}{b-a} \int_a^b \operatorname{sgn}(x-y) f(y) dy \right.$$

$$\left. + \frac{1}{2(b-a)} [f(x)(a+b-2x) + (x-a)f(a) + (b-x)f(b)] \right|$$

for all  $x \in (a, b)$ .

*Proof.* Consider for example the case where  $f$  is nondecreasing on  $[a, b]$ . Then

$$\int_a^b |f(x) - f(y)| dy = \int_a^x |f(x) - f(y)| dy + \int_x^b |f(x) - f(y)| dy$$

$$= (x-a)f(x) - \int_a^x f(y) dy + \int_x^b f(y) dy - (b-x)f(x)$$



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$$= (2x - a - b)f(x) - \int_a^x f(y)dy + \int_x^b f(y)dy.$$

As in the proof of Theorem 2, we may restrict ourselves to the case where  $f$  is absolutely continuous, which yields

$$\begin{aligned} \int_a^b |x - y| |f'(y)| dy &= \int_a^x (x - y)f'(y)dy + \int_x^b (y - x)f'(y)dy \\ &= (a - x)f(a) + (b - x)f(b) + \int_a^x f(y)dy - \int_x^b f(y)dy. \end{aligned}$$

By Theorem 2, we conclude that

$$\begin{aligned} \frac{1}{2} \left[ f(y) + \frac{f(b)(b - y) + f(a)(y - a)}{b - a} \right] - \frac{1}{b - a} \int_a^b f(x)dx \\ \geq \frac{1}{2} \left| \frac{2}{b - a} \left[ \int_x^b f(y)dy - \int_a^x f(y)dy \right] \right. \\ \left. + \frac{f(x)(2x - a - b)}{b - a} - \frac{(x - a)f(a) + (b - x)f(b)}{b - a} \right|. \end{aligned}$$

The case where  $f$  is nonincreasing can be treated in a similar way.  $\square$

For  $x = (a + b)/2$ , Theorem 3 gives us

$$\begin{aligned} \text{(UE)} \quad \frac{1}{2} \left[ f\left(\frac{a + b}{2}\right) + \frac{f(a) + f(b)}{2} \right] - \frac{1}{b - a} \int_a^b f(y)dy \\ \geq \left| \frac{1}{b - a} \int_a^b \operatorname{sgn}\left(\frac{a + b}{2} - y\right) f(y)dy + \frac{f(a) + f(b)}{4} \right|, \end{aligned}$$



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which in the case of the exponential function means

$$\frac{1}{2} \left[ \exp \frac{a+b}{2} + \frac{\exp a + \exp b}{2} \right] - \frac{\exp b - \exp a}{b-a} \geq \left| \frac{1}{b-a} \int_a^b \operatorname{sgn} \left( \frac{a+b}{2} - y \right) \exp y \, dy + \frac{\exp a + \exp b}{4} \right|$$

for all  $a, b \in \mathbb{R}$ ,  $a < b$ , equivalently,

$$\frac{1}{2} \left[ \sqrt{ab} + \frac{a+b}{2} \right] - \frac{b-a}{\ln b - \ln a} \geq \left| \frac{a+b}{4} - \frac{a+b-2\sqrt{ab}}{\ln b - \ln a} \right|$$

for all  $0 < a < b$ .

This represents an improvement on *Polya's inequality*,

$$(Po) \quad \frac{2}{3} \cdot \sqrt{ab} + \frac{1}{3} \cdot \frac{a+b}{2} > \frac{b-a}{\ln b - \ln a}$$

since

$$\frac{2}{3} \cdot \sqrt{ab} + \frac{1}{3} \cdot \frac{a+b}{2} > \frac{1}{2} \sqrt{ab} + \frac{a+b-2\sqrt{ab}}{\ln b - \ln a}.$$

In fact, the last inequality can be restated as

$$(x+1)^2 \ln x > 3(x-1)^2$$

for all  $x > 1$ , a fact that can be easily checked using calculus.

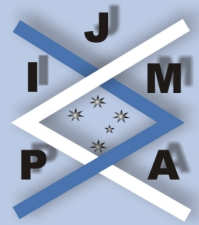
As Professor Niculescu has informed us, we can embed Polya's inequality into a long sequence of interpolating inequalities involving the geometric, the arithmetic,

the logarithmic and the identric means:

$$\begin{aligned}
 \sqrt{ab} &< \left(\sqrt{ab}\right)^{2/3} \left(\frac{a+b}{2}\right)^{1/3} \\
 &< \frac{b-a}{\ln b - \ln a} < \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{1/(b-a)} \\
 &< \frac{2}{3} \cdot \sqrt{ab} + \frac{1}{3} \cdot \frac{a+b}{2} \\
 &< \sqrt{\frac{a+b}{2} \sqrt{ab}} \\
 &< \frac{1}{2} \left(\frac{a+b}{2} + \sqrt{ab}\right) < \frac{a+b}{2}
 \end{aligned}$$

for all  $0 < a < b$ .

*Remark 1.* The extension of Theorems 1 – 3 above to the context of weighted measures is straightforward and we shall omit the details. However, the problem of estimating the Hermite-Hadamard inequality in the case of several variables is left open.



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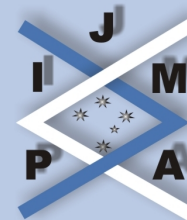
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