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LOCALIZATION OF FACTORED FOURIER SERIES

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ABSTRACT. In this paper we deal with a main theorem on the local property of $\mid \bar{N}, p_n \mid_k$ summability of factored Fourier series, which generalizes some known results.

Key words and phrases: Absolute summability, Fourier series, Local property.

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1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.3}$$

In the special case when $p_n = 1/(n+1)$ for all values of n (resp. $k = 1$), $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C, 1|$ (resp. $\left|\bar{N}, p_n\right|$) summability. Also if we take $k = 1$ and $p_n = 1/(n+1)$,

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summability $\left|\bar{N}, p_n\right|_k$ is equivalent to the summability $|R, \log n, 1|$. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{1.4}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t). \tag{1.5}$$

It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function f.

2. KNOWN RESULTS

Mohanty [4] has demonstrated that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum \frac{A_n(t)}{\log(n+1)} \tag{2.1}$$

at $t = x$, is a local property of the generating function of f, whereas the $|C, 1|$ summability of this series is not. Matsumoto [3] improved this result by replacing the series (2.1) by

$$\sum \frac{A_n(t)}{\{\log\log(n+1)\}^{\delta}}, \quad \delta > 1. \tag{2.2}$$

Generalizing the above result Bhatt [1] proved the following theorem.

Theorem A. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.*

3. THE MAIN RESULT

The aim of the present paper is to prove a more general theorem which includes of the above results as special cases. Also it should be noted that the conditions on the sequence (λ_n) in our theorem, are somewhat more general than in the above theorem.

Now we shall prove the following theorem.

Theorem 3.1. *Let $k \geq 1$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $\left|\bar{N}, p_n\right|_k$ of the series $\sum A_n(t)\lambda_n P_n$ at a point is a local property of the generating function f.*

We need the following lemmas for the proof of our theorem.

Lemma 3.2. *If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then $P_n\lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n\Delta\lambda_n < \infty$.*

Proof. Since (λ_n) is non-increasing, we have that

$$P_m\lambda_m = \lambda_m \sum_{n=0}^{m} p_n = O(1)\sum_{n=0}^{m} p_n\lambda_n = O(1) \quad \text{as} \quad m\to\infty.$$

Applying the Abel transform to the sum $\sum_{n=0}^{m} p_n\lambda_n$, we get that

$$\sum_{n=0}^{m} P_n\Delta\lambda_n = \sum_{n=0}^{m} p_n\lambda_n - P_m\lambda_{m+1}.$$

Since $\lambda_n \ge \lambda_{n+1}$, we obtain

$$\begin{aligned}\sum_{n=0}^{m} P_n\Delta\lambda_n &\le P_m\lambda_m + \sum_{n=0}^{m} p_n\lambda_n \\ &= O(1)+O(1)=O(1) \quad \text{as} \quad m\to\infty.\end{aligned}$$

□

Lemma 3.3. *Let $k\ge 1$ and $s_n = O(1)$. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n\to\infty$ as $n\to\infty$, then the series $\sum a_n\lambda_nP_n$ is summable $\left|\bar{N},p_n\right|_k$.*

Proof. Let (T_n) be the sequence of $(\bar{N},p_n)$ means of the series $\sum a_n\lambda_nP_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_rP_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_vP_v.$$

Then, for $n\ge 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}P_va_v\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned}T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vP_vs_v\Delta\lambda_v - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vs_vp_v\lambda_v \\ &\quad - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vp_{v+1}s_v\lambda_{v+1} + s_np_n\lambda_n \\ &= T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}, \quad \text{say.}\end{aligned}$$

By Minkowski's inequality for $k>1$, to complete the proof of Lemma 3.3, it is sufficient to show that

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}\left|T_{n,r}\right|^k < \infty, \quad \text{for} \quad r=1,2,3,4. \tag{3.1}$$

Now, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$ and $k>1$, we get that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,1}\right|^k \le \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}\left|s_v\right|^k P_vP_v\Delta\lambda_v\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right\}^{k-1}.$$

Since

$$\sum_{v=1}^{n-1} P_vP_v\Delta\lambda_v \le P_{n-1}\sum_{v=1}^{n-1}P_v\Delta\lambda_v,$$

it follows by Lemma 3.2 that

$$\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v \leq \sum_{v=1}^{n-1} P_v \Delta\lambda_v = O(1) \quad \text{as} \quad n \to \infty.$$

Therefore

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|s_v|^k P_vP_v\Delta\lambda_v\\ &= O(1)\sum_{v=1}^{m}|s_v|^k P_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v = O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. Again

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}|s_v|^k (P_v\lambda_v)^k p_v\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}\\ &= O(1)\sum_{v=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|s_v|^k (P_v\lambda_v)^k p_v\\ &= O(1)\sum_{v=1}^{m}|s_v|^k (P_v\lambda_v)^k p_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}|s_v|^k (P_v\lambda_v)^k \frac{p_v}{P_v}\\ &= O(1)\sum_{v=1}^{m}|s_v|^k (P_v\lambda_v)^{k-1} p_v\lambda_v\\ &= O(1)\sum_{v=1}^{m}p_v\lambda_v = O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

in view of the hypotheses of Theorem 3.1 and Lemma 3.2. Using the fact that $P_v < P_{v+1}$, similarly we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k = O(1)\sum_{v=1}^{m}p_{v+1}\lambda_{v+1} = O(1) \quad as \quad m\to\infty.$$

Finally, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &= \sum_{n=1}^{m}|s_n|^k (P_n\lambda_n)^{k-1}p_n\lambda_n\\ &= O(1)\sum_{n=1}^{m}p_n\lambda_n = O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 3.2. Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = O(1) \quad \text{as} \quad m\to\infty, \quad \text{for} \quad r=1,2,3,4.$$

This completes the proof of Lemma 3.3. □

In the particular case if we take $p_n = 1$ for all values of n in Lemma 3.3, then we get the following corollary.

Corollary 3.4. *Let* $k \geq 1$ *and and* $s_n = O(1)$. *If* (λ_n) *is a non-negative and non-increasing sequence such that* $\sum \lambda_n$ *is convergent, then the series* $\sum n a_n \lambda_n$ *is summable* $|C, 1|_k$.

Proof of Theorem 3.1. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 3.1 is a consequence of Lemma 3.3. If we take $p_n = 1$ for all values of n in this theorem, then we get a new local property result concerning the $|C, 1|_k$ summability. □

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