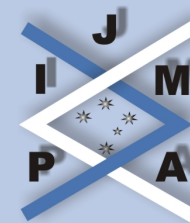


DRAGOMIR-AGARWAL TYPE INEQUALITIES FOR SEVERAL FAMILIES OF QUADRATURES



I. FRANJIĆ

Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6,
10000 Zagreb, Croatia
EMail: ifranjic@pbf.hr

J. PEČARIĆ

Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a
10000 Zagreb, Croatia
EMail: pecaric@hazu.hr

Received: 05 May, 2009

Accepted: 15 September, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 26D15, 65D30, 65D32.

Key words: Dragomir-Agarwal-type inequalities, k -convex functions, General 3-point, 4-point and 5-point quadrature formulae, Corrected quadrature formulae.

Abstract: Inequalities estimating the absolute value of the difference between the integral and the quadrature, i.e. the Dragomir-Agarwal-type inequalities, are given for the general 3, 4 and 5-point quadrature formulae, both classical and corrected. Beside values of the function in the chosen nodes, "corrected" quadrature formula includes values of the first derivative at the end points of the interval and has a higher accuracy than the adjoint classical quadrature formula.

Acknowledgements: The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grants 058-1170889-1050 (first author) and 117-1170889-0888 (second author).

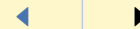
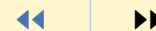
Dragomir-Agarwal Type Inequalities

I. Franjić and J. Pečarić

vol. 10, iss. 3, art. 65, 2009

Title Page

Contents



Page 1 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

Contents

1	Introduction	3
2	Preliminaries	5
3	Main Result	10
3.1	CASE $\alpha = Q3$ and $n = 3, 4$	12
3.2	CASE $\alpha = CQ3$ and $n = 5, 6$	13
3.3	CASE $\alpha = Q4$ and $n = 3, 4$	15
3.4	CASE $\alpha = CQ4$ and $n = 5, 6$	16
3.5	CASE $\alpha = Q5$ and $n = 5, 6$	17
3.6	CASE $\alpha = CQ5$ and $n = 7, 8$	17



Title Page

Contents



Page 2 of 23

Go Back

Full Screen

Close



1. Introduction

The well-known Hermite-Hadamard inequality states that if $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

This pair of inequalities has been improved and extended in a number of ways. One of the directions estimated the difference between the middle and rightmost term in (1.1). For example, Dragomir and Agarwal presented the following result in [2]: suppose $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is differentiable on I and $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, where I is an open interval in \mathbb{R} and $a, b \in I$ ($a < b$). Then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

Generalizations to higher-order convexity for this type of inequality were given in [3]. Related results for Euler-midpoint, Euler-twopoint, Euler-Simpson, dual Euler-Simpson, Euler-Maclaurin, Euler-Simpson 3/8 and Euler-Boole formulae were given in [11]. Furthermore, related results for the general Euler 2-point formulae were given in [10], unifying the cases of Euler trapezoid, Euler midpoint and Euler-twopoint formulae.

The aim of this paper is to give related results for the general 3, 4 and 5-point quadrature formulae, as well as for the corrected general 3, 4 and 5-point quadrature formulae. In addition to values of the function at the chosen nodes, "corrected" quadrature formulae include values of the first derivative at the end points of the interval and also have higher accuracy than adjoint classical quadrature formulae. They are sometimes called "quadratures with end corrections".

Title Page

Contents



Page 3 of 23

Go Back

Full Screen

Close



Our first course of action was to obtain the quadrature formulae. This was done using the extended Euler formulae, in which Bernoulli polynomials play an important role. For the reader's convenience, let us recall some basic properties of Bernoulli polynomials. Bernoulli polynomials $B_k(t)$ are uniquely determined by

$$B'_k(x) = kB_{k-1}(x), \quad B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0, \quad B_0(t) = 1.$$

For the k th Bernoulli polynomial we have $B_k(1-x) = (-1)^k B_k(x)$, $x \in \mathbb{R}$, $k \geq 1$.

The k th Bernoulli number B_k is defined by $B_k = B_k(0)$. For $k \geq 2$, we have $B_k(1) = B_k(0) = B_k$. Note that $B_{2k-1} = 0$, $k \geq 2$ and $B_1(1) = -B_1(0) = 1/2$.

$B_k^*(x)$ are periodic functions of period 1 defined by $B_k^*(x+1) = B_k^*(x)$, $x \in \mathbb{R}$, and related to Bernoulli polynomials as $B_k^*(x) = B_k(x)$, $0 \leq x < 1$. For $k \geq 2$, $B_k^*(t)$ is a continuous function, while $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer. For further details on Bernoulli polynomials, see [1] and [9].

Title Page

Contents



Page 4 of 23

Go Back

Full Screen

Close

2. Preliminaries

General 3-point quadrature formulas were obtained in [5] and general corrected 3-point quadrature formulas in [6]; general closed 4-point quadrature formulas were considered in [7] and finally, general closed 5-point quadrature formulas were derived in [8]. Namely, if $f : [0, 1] \rightarrow \mathbb{R}$ is such that $f^{(n-1)}$ is continuous and of bounded variation on $[0, 1]$ for some $n \geq 1$, then we have

$$(2.1) \quad \int_0^1 f(t)dt - Q_\alpha(x) + T_{n-1}^\alpha(x) = \frac{1}{n!} \int_0^1 F_n^\alpha(x, t)df^{(n-1)}(t),$$

for $\alpha = Q3$, $CQ3$ and $x \in [0, 1/2)$, for $\alpha = Q4$, $CQ4$ and $x \in (0, 1/2]$, and for $\alpha = Q5$, $CQ5$ and $x \in (0, 1/2)$, where

$$Q_{Q3}(x) := Q\left(x, \frac{1}{2}, 1-x\right) = \frac{f(x) + 24B_2(x)f\left(\frac{1}{2}\right) + f(1-x)}{6(1-2x)^2},$$

$$Q_{CQ3}(x) := Q_C\left(x, \frac{1}{2}, 1-x\right) = \frac{7f(x) - 480B_4(x)f\left(\frac{1}{2}\right) + 7f(1-x)}{30(1-2x)^2(1+4x-4x^2)},$$

$$\begin{aligned} Q_{Q4}(x) &:= Q(0, x, 1-x, 1) \\ &= \frac{-6B_2(x)f(0) + f(x) + f(1-x) - 6B_2(x)f(1)}{12x(1-x)}, \end{aligned}$$

$$\begin{aligned} Q_{CQ4}(x) &:= Q_C(0, x, 1-x, 1) \\ &= \frac{30B_4(x)f(0) + f(x) + f(1-x) + 30B_4(x)f(1)}{60x^2(1-x)^2}, \end{aligned}$$



Title Page

Contents



Page 5 of 23

Go Back

Full Screen

Close

$$\begin{aligned}
 Q_{Q_5}(x) &:= Q\left(0, x, \frac{1}{2}, 1-x, 1\right) \\
 &= \frac{1}{60x(1-x)(1-2x)^2} \left[f(x) + f(1-x) \right. \\
 &\quad \left. - (10x^2 - 10x + 1)(1-2x)^2(f(0) + f(1)) \right. \\
 &\quad \left. + 32x(1-x)(5x^2 - 5x + 1)f\left(\frac{1}{2}\right) \right],
 \end{aligned}$$

$$\begin{aligned}
 Q_{CQ_5}(x) &:= Q_C\left(0, x, \frac{1}{2}, 1-x, 1\right) \\
 &= \frac{1}{420x^2(1-x)^2(1-2x)^2} \left[f(x) + f(1-x) \right. \\
 &\quad \left. + (98x^4 - 196x^3 + 102x^2 - 4x - 1)(1-2x)^2(f(0) + f(1)) \right. \\
 &\quad \left. + 64x^2(1-x)^2(14x^2 - 14x + 3)f\left(\frac{1}{2}\right) \right]
 \end{aligned}$$

and

$$T_{n-1}^\alpha(x) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{1}{(2k)!} G_{2k}^\alpha(x, 0) [f^{(2k-1)}(1) - f^{(2k-1)}(0)],$$

$$(2.2) \quad F_n^\alpha(x, t) = G_n^\alpha(x, t) - G_n^\alpha(x, 0),$$

and finally,

$$(2.3) \quad G_n^{Q_3}(x, t) = \frac{B_n^*(x-t) + 24B_2(x) \cdot B_n^*\left(\frac{1}{2}-t\right) + B_n^*(1-x-t)}{6(1-2x)^2},$$

$$(2.4) \quad G_n^{CQ_3}(x, t) = \frac{7B_n^*(x-t) - 480B_4(x) \cdot B_n^*\left(\frac{1}{2}-t\right) + 7B_n^*(1-x-t)}{30(1-2x)^2(1+4x-4x^2)},$$



Title Page

Contents



Page 6 of 23

Go Back

Full Screen

Close



$$(2.5) \quad G_n^{Q4}(x, t) = \frac{B_n^*(x-t) - 12B_2(x) \cdot B_n^*(1-t) + B_n^*(1-x-t)}{12x(1-x)},$$

$$(2.6) \quad G_n^{CQ4}(x, t) = \frac{60B_4(x) \cdot B_n^*(1-t) + B_n^*(x-t) + B_n^*(1-x-t)}{60x^2(1-x)^2},$$

$$(2.7) \quad G_n^{Q5}(x, t) = \frac{10x^2 - 10x + 1}{30x(x-1)} B_n^*(1-t) \\ + \frac{B_n^*(x-t) + B_n^*(1-x-t)}{60x(1-x)(1-2x)^2} + \frac{8(5x^2 - 5x + 1)}{15(1-2x)^2} B_n^*\left(\frac{1}{2} - t\right),$$

$$(2.8) \quad G_n^{CQ5}(x, t) = \frac{98x^4 - 196x^3 + 102x^2 - 4x - 1}{210x^2(1-x)^2} B_n^*(1-t) \\ + \frac{B_n^*(x-t) + B_n^*(1-x-t)}{420x^2(1-x)^2(1-2x)^2} + \frac{16(14x^2 - 14x + 3)}{105(1-2x)^2} B_n^*\left(\frac{1}{2} - t\right).$$

The following lemma was the key result for obtaining the results in [5], [6], [7] and [8], and we shall need it here as well.

Lemma 2.1. For $x \in \{0\} \cup [1/6, 1/2)$ and $n \geq 2$, $G_{2n-1}^{Q3}(x, t)$ has no zeros in variable t on $(0, 1/2)$. The sign of the function is determined by:

$$(-1)^{n+1}G_{2n-1}^{Q3}(x, t) > 0 \quad \text{for } x \in [1/6, 1/2) \quad \text{and} \quad (-1)^n G_{2n-1}^{Q3}(0, t) > 0.$$

For $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$ and $n \geq 3$, $G_{2n-1}^{CQ3}(x, t)$ has no zeros in variable t on $(0, 1/2)$. The sign of the function is determined by:

$$(-1)^n G_{2n-1}^{CQ3}(x, t) > 0 \quad \text{for } x \in [0, 1/2 - \sqrt{15}/10],$$

$$(-1)^{n+1} G_{2n-1}^{CQ3}(x, t) > 0 \quad \text{for } x \in [1/6, 1/2).$$

[Title Page](#)

[Contents](#)



Page 7 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)



Title Page

Contents



Page 8 of 23

Go Back

Full Screen

Close

For $x \in (0, 1/2 - \sqrt{3}/6) \cup [1/3, 1/2]$ and $n \geq 2$, $G_{2n-1}^{Q4}(x, t)$ has no zeros in variable t on $(0, 1/2)$. The sign of the function is determined by:

$$(-1)^{n+1} G_{2n-1}^{Q4}(x, t) > 0 \quad \text{for } x \in (0, 1/2 - \sqrt{3}/6],$$

$$(-1)^n G_{2n-1}^{Q4}(x, t) > 0 \quad \text{for } x \in [1/3, 1/2].$$

For $x \in (0, 1/2 - \sqrt{5}/10) \cup [1/3, 1/2]$ and $n \geq 3$, $G_{2n-1}^{CQ4}(x, t)$ has no zeros in variable t on $(0, 1/2)$. The sign of the function is determined by:

$$(-1)^{n+1} G_{2n-1}^{CQ4}(x, t) > 0 \quad \text{for } x \in (0, 1/2 - \sqrt{5}/10],$$

$$(-1)^n G_{2n-1}^{CQ4}(x, t) > 0 \quad \text{for } x \in [1/3, 1/2].$$

For $x \in (0, 1/2 - \sqrt{15}/10) \cup [1/5, 1/2]$ and $n \geq 3$, $G_{2n-1}^{Q5}(x, t)$ has no zeros in variable t on $(0, 1/2)$. The sign of the function is determined by:

$$(-1)^n G_{2n-1}^{Q5}(x, t) > 0 \quad \text{for } x \in (0, 1/2 - \sqrt{15}/10],$$

$$(-1)^{n+1} G_{2n-1}^{Q5}(x, t) > 0 \quad \text{for } x \in [1/5, 1/2).$$

For $x \in (0, 1/2 - \sqrt{21}/14) \cup [3/7 - \sqrt{2}/7, 1/2]$ and $n \geq 4$, $G_{2n-1}^{CQ5}(x, t)$ has no zeros in variable t on $(0, 1/2)$. The sign of the function is determined by:

$$(-1)^n G_{2n-1}^{CQ5}(x, t) > 0 \quad \text{for } x \in (0, 1/2 - \sqrt{21}/14],$$

$$(-1)^{n+1} G_{2n-1}^{CQ5}(x, t) > 0 \quad \text{for } x \in [3/7 - \sqrt{2}/7, 1/2),$$

where G_{2n-1}^{Q3} is as in (2.3), G_{2n-1}^{CQ3} as in (2.4), G_{2n-1}^{Q4} as in (2.5), G_{2n-1}^{CQ4} as in (2.6), G_{2n-1}^{Q5} as in (2.7) and G_{2n-1}^{CQ5} as in (2.8).



Applying properties of Bernoulli polynomials, it easily follows that functions G_n^α for $\alpha = Q3, CQ3, Q4, CQ4, Q5, CQ5$ and $n \geq 1$, have the following properties:

$$(2.9) \quad G_n^\alpha(x, 1-t) = (-1)^n G_n^\alpha(x, t), \quad t \in [0, 1],$$

$$(2.10) \quad \frac{\partial^j G_n^\alpha(x, t)}{\partial t^j} = (-1)^j \frac{n!}{(n-j)!} G_{n-j}^\alpha(x, t), \quad j = 1, 2, \dots, n,$$

also, that $G_{2n-1}^\alpha(x, 0) = 0$ for $n \geq 1$, and so $F_{2n-1}^\alpha(x, t) = G_{2n-1}^\alpha(x, t)$.

These properties and Lemma 2.1 yield that functions F_{2n}^α , defined by (2.2), are monotonous on $(0, 1/2)$ and $(1/2, 1)$, have constant sign on $(0, 1)$, so the functions $|F_{2n}^\alpha(t)|$ attain their maximal value at $t = 1/2$. Finally, using (2.9) and (2.10), it is not hard to establish that under the assumptions of Lemma 2.1 we have:

$$(2.11) \quad \int_0^1 |F_{2n}^\alpha(x, t)| dt = 2 \int_0^1 t |F_{2n}^\alpha(x, t)| dt = |G_{2n}^\alpha(x, 0)|,$$

$$(2.12) \quad \int_0^1 |G_{2n-1}^\alpha(x, t)| dt = 2 \int_0^1 t |G_{2n-1}^\alpha(x, t)| dt = \frac{1}{n} |F_{2n}^\alpha(x, 1/2)|.$$

Now that we have stated all the previously obtained results which form a basis for the results of this paper, we proceed to the main result.

Title Page

Contents

◀▶

◀▶

Page 9 of 23

Go Back

Full Screen

Close



3. Main Result

To shorten notation, we denote the left-hand side of (2.1) by $\int_0^1 f(t)dt - \Delta_n^\alpha(x)$, i.e.

$$\Delta_n^\alpha(x) := Q_\alpha(x) - T_{n-1}^\alpha(x)$$

for $\alpha = Q3, CQ3, Q4, CQ4, Q5, CQ5$ and $n \geq 1$.

Theorem 3.1. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be n -times differentiable. If $|f^{(n)}|^p$ is convex for some $p \geq 1$ and*

• $n \geq 3$ and

1. $\alpha = Q3$ and $x \in \{0\} \cup [1/6, 1/2)$,
2. $\alpha = Q4$ and $x \in (0, 1/2 - \sqrt{3}/6] \cup [1/3, 1/2]$,

• $n \geq 5$ and

1. $\alpha = CQ3$ and $x \in [0, 1/2 - \sqrt{15}/10] \cup [1/6, 1/2)$,
2. $\alpha = CQ4$ and $x \in (0, 1/2 - \sqrt{5}/10] \cup [1/3, 1/2]$,
3. $\alpha = Q5$ and $x \in (0, 1/2 - \sqrt{15}/10] \cup [1/5, 1/2)$,

• $n \geq 7$ and

1. $\alpha = CQ5$ and $x \in (0, 1/2 - \sqrt{21}/14] \cup [3/7 - \sqrt{2}/7, 1/2)$,

then we have

$$(3.1) \quad \left| \int_0^1 f(t)dt - \Delta_n^\alpha(x) \right| \leq C_\alpha(n, x) \cdot \left(\frac{|f^{(n)}(0)|^p + |f^{(n)}(1)|^p}{2} \right)^{\frac{1}{p}}$$

[Title Page](#)

[Contents](#)



Page 10 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)



Title Page

Contents



Page 11 of 23

Go Back

Full Screen

Close

while if, under same conditions, $|f^{(n)}|$ is concave, then

$$(3.2) \quad \left| \int_0^1 f(t)dt - \Delta_n^\alpha(x) \right| \leq C_\alpha(n, x) \cdot \left| f^{(n)}\left(\frac{1}{2}\right) \right|,$$

where

$$C_\alpha(2k-1, x) = \frac{2}{(2k)!} \left| F_{2k}^\alpha\left(x, \frac{1}{2}\right) \right| \quad \text{and} \quad C_\alpha(2k, x) = \frac{1}{(2k)!} |G_{2k}^\alpha(x, 0)|$$

with functions F_{2k}^α defined as in (2.2) and G_{2k}^α as in (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8).

Proof. First, recall that $F_{2k-1}^\alpha(x, t) = G_{2k-1}^\alpha(x, t)$. Now, starting from (2.1), we apply Hölder's and then Jensen's inequality for the convex function $|f^{(n)}|^p$, to obtain

$$\begin{aligned} & n! \left| \int_0^1 f(t)dt - \Delta_n^\alpha(x) \right| \\ & \leq \int_0^1 |F_n^\alpha(x, t)| \cdot |f^{(n)}(t)| dt \\ & \leq \left(\int_0^1 |F_n^\alpha(x, t)| dt \right)^{1-\frac{1}{p}} \left(\int_0^1 |f^{(n)}((1-t) \cdot 0 + t \cdot 1)|^p \cdot |F_n^\alpha(x, t)| dt \right)^{\frac{1}{p}} \\ & \leq \left(\int_0^1 |F_n^\alpha(x, t)| dt \right)^{1-\frac{1}{p}} \\ & \quad \times \left(|f^{(n)}(0)|^p \int_0^1 (1-t) |F_n^\alpha(x, t)| dt + |f^{(n)}(1)|^p \int_0^1 t |F_n^\alpha(x, t)| dt \right)^{\frac{1}{p}}. \end{aligned}$$

Inequality (3.1) now follows from (2.11) and (2.12).



Title Page

Contents



Page 12 of 23

Go Back

Full Screen

Close

To prove (3.2), apply Jensen's integral inequality to (2.1) to obtain

$$\begin{aligned} & n! \left| \int_0^1 f(t)dt - \Delta_n^\alpha(x) \right| \\ & \leq \int_0^1 |F_n^\alpha(x, t)| \cdot |f^{(n)}((1-t) \cdot 0 + t \cdot 1)| dt \\ & \leq \int_0^1 |F_n^\alpha(x, t)| dt \cdot \left| f^{(n)} \left(\frac{\int_0^1 ((1-t) \cdot 0 + t \cdot 1) |F_n^\alpha(x, t)| dt}{\int_0^1 |F_n^\alpha(x, t)| dt} \right) \right|. \end{aligned}$$

□

Theorem 3.1 provides numerous interesting special cases. Particular choices of node will procure the Dragomir-Agarwal-type estimates for many classical quadrature formulas, as well as the adjoint corrected ones.

3.1. CASE $\alpha = Q3$ and $n = 3, 4$

For $x = 0$, Theorem 3.1 gives Dragomir-Agarwal-type estimates for Simpson's formula; for $x = 1/4$ it provides the estimates for the dual Simpson formula and for $x = 1/6$ for Maclaurin's formula. These were already obtained in [11]; Simpson's formula was also considered in [4].

For $x = 1/2 - \sqrt{3}/6$ ($\Leftrightarrow B_2(x) = 0$), the following estimates are obtained for the Gauss 2-point formula:

$$\Delta_n^{Q3} \left(\frac{3 - \sqrt{3}}{6} \right) = \frac{1}{2} f \left(\frac{3 - \sqrt{3}}{6} \right) + \frac{1}{2} f \left(\frac{3 + \sqrt{3}}{6} \right)$$

and

$$C_{Q3} \left(3, \frac{3 - \sqrt{3}}{6} \right) = \frac{9 - 4\sqrt{3}}{1728} \approx 1.2 \cdot 10^{-3},$$

$$C_{Q3} \left(4, \frac{3 - \sqrt{3}}{6} \right) = \frac{1}{4320} \approx 2.3 \cdot 10^{-4}.$$

3.2. CASE $\alpha = CQ3$ and $n = 5, 6$

For $x = 0$, the following estimates for the corrected Simpson's formula are produced:

$$\Delta_n^{CQ3}(0) = \frac{1}{30} \left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1) \right] - \frac{1}{60} [f'(1) - f'(0)]$$

and

$$C_{CQ3}(5, 0) = \frac{1}{115200} \approx 8.68 \cdot 10^{-6},$$

$$C_{CQ3}(6, 0) = \frac{1}{604800} \approx 1.65 \cdot 10^{-6}.$$

For $x = 1/6$, the following estimates for the corrected Maclaurin's formula are produced:

$$\Delta_n^{CQ3} \left(\frac{1}{6} \right) = \frac{1}{80} \left[27f\left(\frac{1}{6}\right) + 26f\left(\frac{1}{2}\right) + 27f\left(\frac{5}{6}\right) \right] + \frac{1}{240} [f'(1) - f'(0)]$$



Title Page

Contents



Page 13 of 23

Go Back

Full Screen

Close

and

$$C_{CQ3} \left(5, \frac{1}{6} \right) = \frac{1}{691200} \approx 1.45 \cdot 10^{-6},$$
$$C_{CQ3} \left(6, \frac{1}{6} \right) = \frac{31}{87091200} \approx 3.56 \cdot 10^{-7}.$$

For $x = 1/4$, the following estimates for the corrected dual Simpson's formula are produced:

$$\Delta_n^{CQ3} \left(\frac{1}{4} \right) = \frac{1}{15} \left[8f \left(\frac{1}{4} \right) - f \left(\frac{1}{2} \right) + 8f \left(\frac{3}{4} \right) \right] + \frac{1}{120} [f'(1) - f'(0)]$$

and

$$C_{CQ3} \left(5, \frac{1}{4} \right) = \frac{1}{115200} \approx 8.68 \cdot 10^{-6},$$
$$C_{CQ3} \left(6, \frac{1}{4} \right) = \frac{31}{19353600} \approx 1.6 \cdot 10^{-6}.$$

For $x = 1/2 - \sqrt{15}/10$ ($\Leftrightarrow G_2^{CQ3}(x, 0) = 0$), the following estimates for the Gauss 3-point formula are produced:

$$\Delta_n^{CQ3} \left(\frac{5 - \sqrt{15}}{10} \right) = \frac{1}{18} \left[5f \left(\frac{5 - \sqrt{15}}{10} \right) + 8f \left(\frac{1}{2} \right) + 5f \left(\frac{5 + \sqrt{15}}{10} \right) \right]$$



Title Page

Contents

◀▶

◀▶

Page 14 of 23

Go Back

Full Screen

Close

and

$$C_{CQ3} \left(5, \frac{5 - \sqrt{15}}{10} \right) = \frac{25 - 6\sqrt{15}}{576000} \approx 3.06 \cdot 10^{-6},$$

$$C_{CQ3} \left(6, \frac{5 - \sqrt{15}}{10} \right) = \frac{1}{2016000} \approx 4.96 \cdot 10^{-7}.$$

For $x = x_0 := 1/2 - \sqrt{225 - 30\sqrt{30}}/30$ ($\Leftrightarrow B_4(x) = 0$) (the case when the weight next to $f(1/2)$ is annihilated), the following estimates for the corrected Gauss 2-point formula are produced:

$$\Delta_n^{CQ3}(x_0) = \frac{1}{2}f(x_0) + \frac{1}{2}f(1-x_0) - \frac{5 - \sqrt{30}}{60}[f'(1) - f'(0)]$$

and

$$C_{CQ3} \left(5, \frac{15 - \sqrt{225 - 30\sqrt{30}}}{30} \right) = \frac{46\sqrt{225 - 30\sqrt{30}} - 120\sqrt{30} - 4\sqrt{30} + 150\sqrt{30} - 825}{1728000} \approx 7.86 \cdot 10^{-6},$$

$$C_{CQ3} \left(6, \frac{15 - \sqrt{225 - 30\sqrt{30}}}{30} \right) = \frac{45 - 7\sqrt{30}}{4536000} \approx 1.47 \cdot 10^{-6}.$$

3.3. CASE $\alpha = Q4$ and $n = 3, 4$

For $x = 1/3$, the estimates for the Simpson 3/8 formula from [11] are recaptured.



Title Page

Contents



Page 15 of 23

Go Back

Full Screen

Close



3.4. CASE $\alpha = CQ4$ and $n = 5, 6$

For $x = 1/3$, the following estimates for the corrected Simpson 3/8 formula are produced:

$$\Delta_n^{CQ4} \left(\frac{1}{3} \right) = \frac{1}{80} \left[13f(0) + 27f \left(\frac{1}{3} \right) + 27f \left(\frac{2}{3} \right) + 13f(1) \right] - \frac{1}{120} [f'(1) - f'(0)]$$

and

$$C_{CQ4} \left(5, \frac{1}{3} \right) = \frac{1}{691200} \approx 1.45 \cdot 10^{-6},$$
$$C_{CQ4} \left(6, \frac{1}{3} \right) = \frac{1}{2721600} \approx 3.67 \cdot 10^{-7}.$$

For $x = 1/2 - \sqrt{5}/10$ ($\Leftrightarrow G_2^{CQ4}(x, 0) = 0$), the following estimates for the Lobatto 4-point formula are produced:

$$\Delta_n^{CQ4} \left(\frac{1}{3} \right) = \frac{1}{12} \left[f(0) + 5f \left(\frac{5 - \sqrt{5}}{10} \right) + 5f \left(\frac{5 + \sqrt{5}}{10} \right) + f(1) \right]$$

and

$$C_{CQ4} \left(5, \frac{5 - \sqrt{5}}{10} \right) = \frac{\sqrt{5}}{576000} \approx 3.88 \cdot 10^{-6},$$
$$C_{CQ4} \left(6, \frac{5 - \sqrt{5}}{10} \right) = \frac{1}{1512000} \approx 6.61 \cdot 10^{-7}.$$

Title Page

Contents



Page 16 of 23

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)



Page 17 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

3.5. CASE $\alpha = Q5$ and $n = 5, 6$

For $x = 1/4$, the following estimates for Boole's formula from [11] are recaptured.

3.6. CASE $\alpha = CQ5$ and $n = 7, 8$

For $x = 1/2 - \sqrt{21}/14$ ($\Leftrightarrow G_2^{CQ5}(x, 0) = 0$), the following estimates for the Lobatto 5-point formula are produced:

$$\begin{aligned} & \Delta_n^{CQ5} \left(\frac{1}{4} \right) \\ &= \frac{1}{180} \left[9f(0) + 49f \left(\frac{7 - \sqrt{21}}{14} \right) + 64f \left(\frac{1}{2} \right) + 49f \left(\frac{7 + \sqrt{21}}{14} \right) + 9f(1) \right] \end{aligned}$$

and

$$\begin{aligned} C_{CQ5} \left(7, \frac{7 - \sqrt{21}}{14} \right) &= \frac{12\sqrt{21} - 49}{1264435200} \approx 4.74 \cdot 10^{-9}, \\ C_{CQ5} \left(8, \frac{7 - \sqrt{21}}{14} \right) &= \frac{1}{1422489600} \approx 7.03 \cdot 10^{-10}. \end{aligned}$$

For $x = 1/4$, the following estimates for the corrected Boole's formula are produced:

$$\begin{aligned} \Delta_n^{CQ5} \left(\frac{1}{4} \right) &= \frac{1}{1890} \left[217f(0) + 512f \left(\frac{1}{4} \right) + 432f \left(\frac{1}{2} \right) \right. \\ &\quad \left. + 512f \left(\frac{3}{4} \right) + 217f(1) \right] - \frac{1}{252} [f'(1) - f'(0)] \end{aligned}$$

and

$$C_{CQ5} \left(7, \frac{1}{4} \right) = \frac{17}{4877107200} \approx 3.49 \cdot 10^{-9},$$

$$C_{CQ5} \left(8, \frac{1}{4} \right) = \frac{1}{1625702400} \approx 6.15 \cdot 10^{-10}.$$

Further, for $x = 1/2 - \sqrt{7}/14$ ($\Leftrightarrow 14x^2 - 14x + 3 = 0$), which is the case when the weight next to $f(1/2)$ is annihilated, the following estimates for the corrected Lobatto 4-point formula are produced:

$$\Delta_n^{CQ5} \left(\frac{7 - \sqrt{7}}{14} \right) = \frac{1}{270} \left[37f(0) + 98f \left(\frac{7 - \sqrt{7}}{14} \right) + 98f \left(\frac{7 + \sqrt{7}}{14} \right) + 37f(1) \right] - \frac{1}{180} [f'(1) - f'(0)]$$

and

$$C_{CQ5} \left(7, \frac{7 - \sqrt{7}}{14} \right) = \frac{343 - 16\sqrt{7}}{34139750400} \approx 8.81 \cdot 10^{-9},$$

$$C_{CQ5} \left(8, \frac{7 - \sqrt{7}}{14} \right) = \frac{1}{711244800} \approx 1.41 \cdot 10^{-9}.$$



Title Page

Contents



Page 18 of 23

Go Back

Full Screen

Close



Title Page

Contents



Page 19 of 23

Go Back

Full Screen

Close

Finally, for

$$x = x_0 := \frac{1}{2} - \frac{\sqrt{45 - 2\sqrt{102}}}{14}$$

$$\left(\Leftrightarrow 98x^4 - 196x^3 + 102x^2 - 4x - 1 \right.$$

$$\left. = 98 \left(x^2 - x + \frac{1}{49} - \frac{\sqrt{102}}{98} \right) \left(x^2 - x + \frac{1}{49} + \frac{\sqrt{102}}{98} \right) = 0 \right),$$

which is the case when the weight next to $f(0)$ and $f(1)$ is annihilated, the estimates for the corrected Gauss 3-point formula are produced:

$$\Delta_n^{CQ5}(x_0) = \frac{1977 + 16\sqrt{102}}{6930} [f(x_0) + f(1 - x_0)]$$

$$+ \frac{1488 - 16\sqrt{102}}{3465} f\left(\frac{1}{2}\right) - \frac{9 - \sqrt{102}}{420} [f'(1) - f'(0)]$$

and

$$C_{CQ5}(7, x_0) = \frac{24\sqrt{60933 - 6014\sqrt{102}} - 49(87 - 8\sqrt{102})}{3793305600} \approx 8.12 \cdot 10^{-9},$$

$$C_{CQ5}(8, x_0) = \frac{43 - 3\sqrt{102}}{9957427200} \approx 1.28 \cdot 10^{-9}.$$

Remark 1. An interesting fact to point out is that out of all the 3-point quadrature formulae, Maclaurin's formula gives the least estimate of error in Theorem 3.1; among



[Title Page](#)

[Contents](#)



Page 20 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-575b

the corrected 3-point quadrature formulae, the corrected Maclaurin's formula has the same property.

Further, among the closed 4-point quadrature formulas, the Simpson 3/8 formula gives the best estimate and the corrected Simpson 3/8 formula is the optimal corrected closed 4-point quadrature formula.

Finally, the node $x = 1/5$ produces the closed 5-point quadrature formula with the best error estimate, while the node $x = 3/7 - \sqrt{2}/7$ produces the corrected closed 5-point quadrature formula with the same property.

The proofs are similar to those in [5], [6], [7] and [8], respectively.

In view of the previous remark, let us consider the case $\alpha = Q5$, $n = 5, 6$ and $x = 1/5$. We have:

$$\Delta_n^{Q5} \left(\frac{1}{5} \right) = \frac{1}{432} \left[27f(0) + 125f \left(\frac{1}{5} \right) + 128f \left(\frac{1}{2} \right) + 125f \left(\frac{4}{5} \right) + 27f(1) \right]$$

and

$$C_{Q5} \left(5, \frac{1}{5} \right) = \frac{1}{1152000} \approx 8.68 \cdot 10^{-7},$$

$$C_{Q5} \left(6, \frac{1}{5} \right) = \frac{1}{5040000} \approx 1.98 \cdot 10^{-7}.$$

Further, for $\alpha = CQ5$, $n = 7, 8$ and $x = x_0 := 3/7 - \sqrt{2}/7$ we obtain:

$$\begin{aligned} \Delta_n^{CQ5}(x_0) &= 0.10143 [f(0) + f(1)] + 0.259261 [f(x_0) + f(1 - x_0)] \\ &\quad + 0.278617 f \left(\frac{1}{2} \right) + 3.07832 \cdot 10^{-3} [f'(1) - f'(0)] \end{aligned}$$

and

$$C_{CQ_5}(7, x_0) = \frac{27 - 16\sqrt{2}}{3793305600} \approx 1.15 \cdot 10^{-9},$$
$$C_{CQ_5}(8, x_0) = \frac{11 - 6\sqrt{2}}{9957427200} \approx 2.53 \cdot 10^{-10}.$$



Dragomir-Agarwal Type Inequalities

I. Franjić and J. Pečarić

vol. 10, iss. 3, art. 65, 2009

Title Page

Contents



Page 21 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

References

- [1] M. ABRAMOWITZ AND I.A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulae, Graphs and Mathematical Tables*, National Bureau of Standards, Applied Math. Series 55, 4th printing, Washington, 1965.
- [2] S.S. DRAGOMIR AND R.P. AGARWAL, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoid formula, *Appl. Mat. Lett.*, **11**(5) (1998), 91–95.
- [3] Lj. DEDIĆ, C.E.M. PEARCE AND J. PEČARIĆ, Hadamard and Dragomir-Agarwal inequalities, higher-order convexity and the Euler formula, *J. Korean Math. Soc.*, **38** (2001), 1235–1243.
- [4] Lj. DEDIĆ, C.E.M. PEARCE AND J. PEČARIĆ, The Euler formulae and convex functions, *Math. Inequal. Appl.*, **3**(2) (2000), 211–221.
- [5] I. FRANJIĆ, J. PEČARIĆ AND I. PERIĆ, Quadrature formulae of Gauss type based on Euler identities, *Math. Comput. Modelling*, **45**(3-4) (2007), 355–370.
- [6] I. FRANJIĆ, J. PEČARIĆ AND I. PERIĆ, General 3-point quadrature formulas of Euler type, (submitted for publication).
- [7] I. FRANJIĆ, J. PEČARIĆ AND I. PERIĆ, General closed 4-point quadrature formulae of Euler type, *Math. Inequal. Appl.*, **12** (2009), 573–586.
- [8] I. FRANJIĆ, J. PEČARIĆ AND I. PERIĆ, On families of quadrature formulas based on Euler identities, (submitted for publication).
- [9] V.I. KRYLOV, *Approximate Calculation of Integrals*, Macmillan, New York-London, 1962.



Dragomir-Agarwal Type Inequalities

I. Franjić and J. Pečarić

vol. 10, iss. 3, art. 65, 2009

Title Page

Contents



Page 22 of 23

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



- [10] J. PEČARIĆ AND A. VUKELIĆ, Hadamard and Dragomir-Agarwal inequalities, the general Euler two point formulae and convex functions, *Rad Hrvat. akad. znan. umjet.*, 491. *Matematičke znanosti*, **15** (2005), 139–152.
- [11] J. PEČARIĆ AND A. VUKELIĆ, On generalizations of Dragomir-Agarwal inequality via some Euler-type identities, *Bulletin de la Société des Mathématiciens de R. Macèdonie*, **26** (LII) (2002), 463–483.

Dragomir-Agarwal Type Inequalities

I. Franjić and J. Pečarić

vol. 10, iss. 3, art. 65, 2009

[Title Page](#)

[Contents](#)



Page 23 of 23

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756