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# ON THE UNIVALENCY OF CERTAIN ANALYTIC FUNCTIONS

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## Abstract

Let $Q(\alpha, \beta, \gamma)$ denote the class of functions of the form $f(z) = z + a_2 z^2 + \cdots$, which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the condition

$$\Re\{\alpha(f(z)/z) + \beta f'(z)\} > \gamma \ \ (\alpha, \beta > 0;\ 0 \leq \gamma < \alpha + \beta \leq 1;\ z \in \mathcal{U}).$$

The extreme points for this class are provided, the coefficient bounds and radius of univalency for functions belonging to this class are also provided. The results presented here include a number of known results as their special cases.



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## Contents

# 1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. Also let $\mathcal{S}$ denote the familiar subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathcal{U}$.

In the present paper, we consider the following subclass of $\mathcal{A}$:

$$(1.1) \qquad Q(\alpha,\beta,\gamma) = \left\{ f(z) \in \mathcal{A} : \Re\left\{ \alpha \frac{f(z)}{z} + \beta f'(z) \right\} > \gamma \ \ (z \in \mathcal{U}) \right\},$$

where $\alpha, \beta > 0$ and $0 \le \gamma < \alpha + \beta \le 1$.

In some recent papers, Saitoh [2] and Owa [3, 4] discussed the related properties of the class $Q(1-\beta,\beta,\gamma)$. In the present paper, first we determine the extreme points of the class $Q(\alpha,\beta,\gamma)$, then we find the coefficient bounds and radius of univalency for functions belonging to this class. The results presented here include a number of known results as their special cases.

## 2. Extreme Points of the Class $Q(\alpha, \beta, \gamma)$

First we give the following theorem.

**Theorem 2.1.** *A function $f(z) \in Q(\alpha, \beta, \gamma)$ if and only if $f(z)$ can be expressed as*

$$f(z) = \frac{1}{\alpha+\beta} \int_{|x|=1} \left[ (2\gamma - \alpha - \beta) z + 2(\alpha+\beta-\gamma) \sum_{n=0}^{\infty} \frac{(\alpha+\beta) x^n z^{n+1}}{(n+1)\beta + \alpha} \right] d\mu(x), \tag{2.1}$$

*where $\mu(x)$ is the probability measure defined on $X = \{x : |x| = 1\}$. For fixed $\alpha$, $\beta$ and $\gamma$, $Q(\alpha, \beta, \gamma)$ and the probability measures $\{\mu\}$ defined on $X$ are one-to-one by the expression (2.1).*

*Proof.* By the definition of $Q(\alpha, \beta, \gamma)$, we know $f(z) \in Q(\alpha, \beta, \gamma)$ if and only if

$$\frac{\alpha(f(z)/z) + \beta f'(z) - \gamma}{\alpha + \beta - \gamma} \in \mathcal{P},$$

where $\mathcal{P}$ denotes the normalized well-known class of analytic functions which have positive real part. By the aid of Herglotz expressions of functions in $\mathcal{P}$, we have

$$\frac{\alpha(f(z)/z) + \beta f'(z) - \gamma}{\alpha + \beta - \gamma} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x),$$

or equivalently,

$$\frac{\alpha}{\beta}\frac{f(z)}{z}+f'(z)=\frac{1}{\beta}\int_{|x|=1}\frac{\alpha+\beta+(\alpha+\beta-2\gamma)xz}{1-xz}d\mu(x).$$

Thus we have

$$\begin{aligned} z^{-\frac{\alpha}{\beta}}&\int_0^z\left[\frac{\alpha}{\beta}\frac{f(\zeta)}{\zeta}+f'(\zeta)\right]\zeta^{\frac{\alpha}{\beta}}d\zeta \\ &=\frac{1}{\beta}\int_{|x|=1}\left[z^{-\frac{\alpha}{\beta}}\int_0^z\frac{\alpha+\beta+(\alpha+\beta-2\gamma)x\zeta}{1-x\zeta}\zeta^{\frac{\alpha}{\beta}}d\zeta\right]d\mu(x), \end{aligned}$$

that is,

$$\begin{aligned} f(z)=\frac{1}{\alpha+\beta}\int_{|x|=1}&\Bigg[(2\gamma-\alpha-\beta)z \\ &+2(\alpha+\beta-\gamma)\sum_{n=0}^{\infty}\frac{(\alpha+\beta)x^nz^{n+1}}{(n+1)\beta+\alpha}\Bigg]d\mu(x). \end{aligned}$$

This deductive process can be converse, so we have proved the first part of the theorem. We know that both probability measures $\{\mu\}$ and class $\mathcal{P}$, class $\mathcal{P}$ and $Q(\alpha,\beta,\gamma)$ are one-to-one, so the second part of the theorem is true. This completes the proof of Theorem 2.1. $\square$

**Corollary 2.2.** *The extreme points of the class $Q(\alpha,\beta,\gamma)$ are*

$$f_x(z) = \frac{1}{\alpha+\beta}\Bigg[(2\gamma-\alpha-\beta)z + 2(\alpha+\beta-\gamma)\sum_{n=0}^{\infty}\frac{(\alpha+\beta)x^n z^{n+1}}{(n+1)\beta+\alpha}\Bigg] \quad (|x|=1). \tag{2.2}$$

*Proof.* Using the notation $f_x(z)$, (2.1) can be written as

$$f_\mu(z) = \int_{|x|=1} f_x(z)\,d\mu(x).$$

By Theorem 2.1, the map $\mu \to f_\mu$ is one-to-one, so the assertion follows (see [1]). □

**Corollary 2.3.** *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha,\beta,\gamma)$, then for $n \geq 2$, we have*

$$|a_n| \leq \frac{2(\alpha+\beta-\gamma)}{n\beta+\alpha}.$$

*The results are sharp.*

*Proof.* The coefficient bounds are maximized at an extreme point. Now from (2.2), $f_x(z)$ can be expressed as

$$f_x(z) = z + 2(\alpha+\beta-\gamma)\sum_{n=2}^{\infty}\frac{x^{n-1}z^n}{n\beta+\alpha} \quad (|x|=1), \tag{2.3}$$

and the result follows. □

**Corollary 2.4.** *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, then for $|z| = r < 1$, we have*

$$|f(z)| \leq r + 2(\alpha + \beta - \gamma) \sum_{n=2}^{\infty} \frac{r^n}{n\beta + \alpha}.$$

*This result follows from (2.3).*

## 3. Radius of Univalency

In this section, we shall provide the radius of univalency for functions belonging to the class $Q(\alpha,\beta,\gamma)$.

**Theorem 3.1.** *Let $f(z) \in Q(\alpha,\beta,\gamma)$, then $f(z)$ is univalent in $|z| < R(\alpha,\beta,\gamma)$, where*

$$R(\alpha,\beta,\gamma) = \inf_n \left\{ \frac{n\beta+\alpha}{2n(\alpha+\beta-\gamma)} \right\}^{\frac{1}{n-1}}.$$

*This result is sharp.*

*Proof.* It suffices to show that

$$|f'(z) - 1| < 1. \tag{3.1}$$

For the left hand side of (3.1) we have

$$\left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1}.$$

This last expression is less than 1 if

$$|z|^{n-1} < \frac{n\beta+\alpha}{2n(\alpha+\beta-\gamma)}.$$

To show that the bound $R(\alpha,\beta,\gamma)$ is best possible, we consider the function $f(z) \in \mathcal{A}$ defined by

$$f(z) = z - \frac{2(\alpha+\beta-\gamma)}{n\beta+\alpha} z^n.$$

If $\delta > R(\alpha,\beta,\gamma)$, then there exists $n \geq 2$ such that

$$\left\{\frac{n\beta+\alpha}{2n(\alpha+\beta-\gamma)}\right\}^{\frac{1}{n-1}} < \delta.$$

Since $f'(0) = 1 > 0$ and

$$f'(\delta) = 1 - \frac{2n(\alpha+\beta-\gamma)}{n\beta+\alpha}\delta^{n-1} < 0.$$

Thus, there exists $\delta_0 \in (0,\delta)$ such that $f'(\delta_0) = 0$, which implies that $f(z)$ is not univalent in $|z| < \delta$. This completes the proof of Theorem 3.1. $\square$

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