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# ON A GEOMETRIC INEQUALITY BY J. SÁNDOR

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ABSTRACT. In this short note, we sharpen and generalize a geometric inequality by J. Sándor. As applications of our results, we give an alternative proof of Sándor's inequality and solve two conjectures posed by Liu.

*Key words and phrases:* Triangle, Hayashi's inequality, Hölder's inequality, Gerretsen's inequality, Euler's inequality.

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## 1. INTRODUCTION AND MAIN RESULTS

Let $P$ be an arbitrary point $P$ in the plane of triangle $ABC$. Let $a$, $b$, $c$ be the lengths of these sides, $\triangle$ the area, $s$ the semi-perimeter, $R$ the circumradius and $r$ the inradius, respectively. Denote by $R_1$, $R_2$, $R_3$ the distances from $P$ to the vertices $A$, $B$, $C$, respectively.

The following interesting geometric inequality from 1986 is due to J. Sándor [8], a proof of this inequality can be found in the monograph [9].

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**Theorem 1.1.** *For triangle $ABC$ and an arbitrary point $P$, we have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq \frac{16}{9}\triangle^2 . \tag{1.1}$$

Recently, J. Liu [6] also independently proved inequality (1.1).

In this short note, we sharpen and generalize inequality (1.1) and obtain the following results.

**Theorem 1.2.** *We have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq \frac{a^2b^2c^2}{a^2+b^2+c^2}. \tag{1.2}$$

**Theorem 1.3.** *If*

$$k \geq k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2} \approx 1.549800462,$$

*then*

$$(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \geq 3\left(\frac{4}{9}\sqrt{3}\triangle\right)^k . \tag{1.3}$$

## 2. PRELIMINARY RESULTS

**Lemma 2.1** (Hayashi's inequality, see [7, pp. 297, 311])**.** *For any $\triangle ABC$ and an arbitrary point $P$, we have*

$$aR_2R_3 + bR_3R_1 + cR_1R_2 \geq abc, \tag{2.1}$$

*with equality holding if and only if $P$ is the orthocenter of the acute triangle $ABC$ or one of the vertices of the triangle $ABC$.*

**Lemma 2.2** (see [2] and [4])**.** *For $\triangle ABC$, if*

$$0 \leq t \leq t_0 = \frac{\ln 9 - \ln 4}{\ln 4 - \ln 3},$$

*then we have*

$$a^t + b^t + c^t \leq 3\left(\sqrt{3}R\right)^t . \tag{2.2}$$

**Lemma 2.3.** *Let*

$$k \geq k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2} \approx 1.549800462.$$

*Then*

$$\frac{(abc)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}} \geq 3\left(\frac{4}{9}\sqrt{3}\triangle\right)^k . \tag{2.3}$$

*Proof.* From the well known identities $abc = 4Rrs$ and $\triangle = rs$, inequality (2.3) is equivalent to

$$\frac{(4Rrs)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}} \geq 3\left(\frac{4}{9}\sqrt{3}rs\right)^k ,$$

or

$$a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}} \le 3\left(\sqrt{3}R\right)^{\frac{k}{k-1}}. \tag{2.4}$$

It is easy to see that the function

$$f(x) = \frac{x}{x-1}$$

is strictly monotone decreasing on $(1, +\infty)$. If we let

$$t = \frac{k}{k-1} = f(k) \qquad \left(k \ge k_0 = \frac{2(\ln 3 - \ln 2)}{3\ln 3 - 4\ln 2}\right),$$

then

$$0 < f(k) = t \le \frac{\ln 9 - \ln 4}{\ln 4 - \ln 3} = f(k_0),$$

and inequality (2.4) is equivalent to (2.2).

The proof of Lemma 2.3 is thus complete from Lemma 2.2. □

**Lemma 2.4** ([3]). *For any $\lambda \ge 1$, we have*

$$[R - \lambda(\lambda+1)r]s^2 + r[4(\lambda^2-4)R^2 + (5\lambda^2+12\lambda+4)Rr + (\lambda^2+3\lambda+2)r^2] \ge 0. \tag{2.5}$$

**Lemma 2.5.** *In triangle $ABC$, we have*

$$\begin{aligned} a^9+b^9+c^9 = 2s[&s^8 - 18r(R+2r)s^6 + 18r^2(21Rr+7r^2+12R^2)s^4 \\ &- 6r^3(105r^2R+240rR^2+14r^3+160R^3)s^2 + 9r^4(r+2R)(r+4R)^3]. \end{aligned}$$

*Proof.* The identity directly follows from the known identities $a+b+c=2s$, $ab+bc+ca = s^2+4Rr+r^2$, $abc = 4Rrs$ and the following identity:

$$\begin{aligned} &a^9+b^9+c^9 \\ &= 3a^3b^3c^3 - 45abc(ab+bc+ca)(a+b+c)^4 + 54abc(ab+bc+ca)^2(a+b+c)^2 \\ &\quad - 27a^2b^2c^2(ab+bc+ca)(a+b+c) + (a+b+c)^9 \\ &\quad - 9(ab+bc+ca)(a+b+c)^7 + 9(ab+bc+ca)^4(a+b+c) \\ &\quad - 30(ab+bc+ca)^3(a+b+c)^3 + 18a^2b^2c^2(a+b+c)^3 \\ &\quad + 27(ab+bc+ca)^2(a+b+c)^5 + 9abc(a+b+c)^6 - 9abc(ab+bc+ca)^3. \end{aligned}$$

□

**Lemma 2.6** ([5]). *If $x, y, z \ge 0$, then*

$$x+y+z+3\sqrt[3]{xyz} \ge 2\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right).$$

## 3. PROOF OF THE MAIN RESULT

The proof of Theorem 1.2 is easy to find from the following inequality (3.1) for $k = 2$ of the proof of Theorem 1.3. Now, we prove Theorem 1.3.

***The proof of Theorem 1.3***. *Hölder's inequality* and Lemma 2.1 imply for $k > 1$ that

$$\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}} \left[(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k\right]^{\frac{1}{k}} \geq aR_2R_3 + bR_3R_1 + cR_1R_2 \geq abc,$$

or

$$(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \geq \frac{(abc)^k}{\left[a^{\frac{k}{k-1}} + b^{\frac{k}{k-1}} + c^{\frac{k}{k-1}}\right]^{k-1}}. \tag{3.1}$$

Combining inequality (3.1) and Lemma 2.3, we immediately see that Theorem 1.3 is true. □

## 4. APPLICATIONS

4.1. **Alternative Proof of Theorem 1.1.** From Theorem 1.2, in order to prove inequality (1.1), we only need to prove the following inequality:

$$\frac{a^2b^2c^2}{a^2+b^2+c^2} \geq \frac{16}{9} \triangle^2 . \tag{4.1}$$

With the known identities $abc = 4Rrs$ and $\triangle = rs$, inequality (4.1) is equivalent to

$$a^2 + b^2 + c^2 \leq 9R^2.$$

This is simply inequality (2.2) for $t = 2 < t_0$ in Lemma 2.2. This completes the proof of inequality (1.1).

**Remark 1.** The above proof of inequality (1.1) is simpler than Liu's proof [6].

4.2. **Solution of Two Conjectures.** In 2008, J. Liu [6] posed the following two geometric inequality conjectures, (4.2) and (4.3), involving $R_1, R_2, R_3, R$ and $r$.

**Conjecture 4.1.** *For $\triangle ABC$ and an arbitrary point $P$, we have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq 8(R^2 + 2r^2)r^2, \tag{4.2}$$

*and*

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 24r^3. \tag{4.3}$$

*Proof.* First of all, from *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

and *Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \geq 2r,$$

we have

$$2r^2(4R^2+4Rr+3r^2-s^2)+(R-2r)(4R^2+Rr+2r^2)r \geq 0$$
$$\iff \frac{16R^2r^2s^2}{2(s^2-4Rr-r^2)} \geq 8(R^2+2r^2)r^2.$$

Using Theorem 1.2 and the known identities [7, pp.52]

$$abc = 4Rrs \qquad \text{and} \qquad a^3+b^3+c^3 = 2s(s^2-6Rr-3r^2),$$

we see that inequality (4.2) holds true.

Secondly, from (3.1), in order to prove inequality (4.3), we only need to prove

$$\frac{(abc)^{\frac{3}{2}}}{[a^3+b^3+c^3]^{\frac{1}{2}}} \geq 24r^3. \tag{4.4}$$

With the known identities [7, pp. 52]

$$abc = 4Rrs \qquad \text{and} \qquad a^3+b^3+c^3 = 2s(s^2-6Rr-3r^2),$$

inequality (4.4) is equivalent to

$$\frac{(4Rrs)^{\frac{3}{2}}}{[2s(s^2-6Rr-3r^2)]^{\frac{1}{2}}} \geq 24r^3 \tag{4.5}$$
$$\iff 18r^3(4R^2+4Rr+3r^2-s^2)+R^3(s^2-16Rr+5r^2)$$
$$+Rr(R-2r)(16R^2+27Rr-18r^2) \geq 0.$$

From *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$16Rr-5r^2 \leq s^2 \leq 4R^2+4Rr+3r^2$$

and *Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \geq 2r,$$

we can conclude that inequality (4.5) holds, further, inequality (4.4) is true.

This completes the proof of Conjecture 4.1. □

**Corollary 4.2.** *For $\triangle ABC$ and an arbitrary point $P$, we have*

$$R_1^3+R_2^3+R_3^3+3R_1R_2R_3 \geq 48r^3. \tag{4.6}$$

*Proof.* Inequality (4.6) can directly be obtained from Lemma 2.6 and inequality (4.3). □

4.3. **Sharpened Form of Above Conjectures.** The inequalities (4.2) and (4.3) of Conjecture 4.1 can be sharpened as follows.

**Theorem 4.3.** *For $\triangle ABC$ and an arbitrary point $P$, we have*

$$(R_1R_2)^2+(R_2R_3)^2+(R_3R_1)^2 \geq 8(R+r)Rr^2, \tag{4.7}$$

*and*

$$(R_1R_2)^{\frac{3}{2}}+(R_2R_3)^{\frac{3}{2}}+(R_3R_1)^{\frac{3}{2}} \geq 12Rr^2. \tag{4.8}$$

*Proof.* The proof of inequality (4.7) is left to the readers. Now, we prove inequality (4.8).

From inequality (2.5) for $\lambda = 2$ in Lemma 2.4, the well-known *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2,$$

*Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \geq 2r$$

and the known identities [7, pp. 52]

$$abc = 4Rrs \text{ and } a^3 + b^3 + c^3 = 2s(s^2 - 6Rr - 3r^2),$$

we obtain that

$$\begin{aligned} &[(R-6r)s^2 + 12r^2(4R+r)] + 3r(4R^2 + 4Rr + 3r^2 - s^2) \\ &\qquad + R(s^2 - 16Rr + 5r^2) + r(R-2r)(4R-3r) \geq 0 \\ &\iff \frac{(4Rrs)^{\frac{3}{2}}}{[2s(s^2 - 6Rr - 3r^2)]^{\frac{1}{2}}} \geq 12Rr^2 \\ &\iff \frac{(abc)^{\frac{3}{2}}}{[a^3 + b^3 + c^3]^{\frac{1}{2}}} \geq 12Rr^2. \end{aligned} \tag{4.9}$$

Inequality (4.8) follows by Lemma 2.4.

Theorem 4.3 is thus proved. □

### 4.4. **Generalization of Inequality** (4.3).

**Theorem 4.4.** *If* $k \geq \frac{9}{8}$, *then*

$$(R_1R_2)^k + (R_2R_3)^k + (R_3R_1)^k \geq 3(4r^2)^k. \tag{4.10}$$

*Proof.* From the monotonicity of the power mean, we only need to prove that inequality (4.10) holds for $k = \frac{9}{8}$. By using inequality (3.1), we only need to prove the following inequality

$$\frac{(abc)^{\frac{9}{8}}}{(a^9 + b^9 + c^9)^{\frac{1}{8}}} \geq 3(4r^2)^{\frac{9}{8}}. \tag{4.11}$$

From *Gerretsen's inequality* [1, pp. 50, Theorem 5.8]

$$s^2 \geq 16Rr - 5r^2$$

and *Euler's inequality* [1, pp. 48, Theorem 5.1]

$$R \geq 2r,$$

it is obvious that

$$\begin{aligned} P = (R-2r)[&4096R^{10} + 12544R^9r + 34992R^8r^2 + 89667R^7r^3 + 218700R^6r^4 \\ &+ 516132R^5r^5 + 1189728R^4r^6 + 2493180R^3r^7 + 6018624(R-2r)Rr^8 \\ &+ 6753456r^{10} + 201204(R^2 - 4r^2)Rr^7] + 2799360r^{11} > 0, \end{aligned}$$

and

$$\begin{aligned}Q = (s^2-16Rr+5r^2)\{&R^9(s^2-16Rr+5r^2)+3R^4r(R-2r)(16R^5+27R^4r+54R^3r^2\\&+108R^2r^3+216Rr^4+432r^5)+324r^7[8(R^2-12r^2)^2+30r^2(R-2r)^2\\&+39Rr^3+267r^4]\}+17496r^7(R^2-3Rr+6r^2)(R^2-12Rr+24r^2)^2\\&+3r^2(R-2r)\{(R-2r)[256R^9+864R^8r+2457R^2r^2(R^5-32r^5)\\&+6372R^2r^3(R^4-16r^4)+15660R^2r^4(R^3-8r^3)+31320R^2r^5(R^2-4r^2)\\&+220104R^2r^6(R-2r)+2618784(R-2r)r^8+51840R^2r^7+501120Rr^8]\\&+687312r^{10}\}>0.\end{aligned}$$

Therefore, with the fundamental inequality [7, pp.1–3]

$$-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3\geq 0,$$

we have

$$\begin{aligned}W &= (R^9-13122r^9)s^8+236196r^{10}(2r+R)s^6-236196r^{11}(7r^2+12R^2+21Rr)s^4\\&\quad+78732r^{12}(105Rr^2+160R^3+240R^2r+14r^3)s^2-118098r^{13}(2R+r)(4R+r)^3\\&=13122r^9[s^4+9r^3(2R+r)][-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3]\\&\quad+r^3s^2(R-2r)P+s^2(s^2-16Rr+5r^2)Q\\&\geq 0.\end{aligned}$$

Hence, from Lemma 2.4, we get that

$$3\left(\frac{Rs}{3r}\right)^9-(a^9+b^9+c^9)=\frac{s}{6561r^9}W\geq 0, \tag{4.12}$$

or

$$3\left(\frac{Rs}{3r}\right)^9\geq a^9+b^9+c^9. \tag{4.13}$$

Inequality (4.13) is simply (4.11). Thus, we complete the proof of Theorem 4.4. □

## 5. TWO OPEN PROBLEMS

Finally, we pose two open problems as follows.

**Open Problem 1.** *For a triangle $ABC$ and an arbitrary point $P$, prove or disprove*

$$R_1^3+R_2^3+R_3^3+6R_1R_2R_3\geq 72r^3. \tag{5.1}$$

**Open Problem 2.** *For a triangle $ABC$ and an arbitrary point $P$, determine the best constant $k$ such that the following inequality holds:*

$$(R_1R_2)^{\frac{3}{2}}+(R_2R_3)^{\frac{3}{2}}+(R_3R_1)^{\frac{3}{2}}\geq 12[R+k(R-2r)]r^2. \tag{5.2}$$

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