Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 5, Article 180, 2006

ON A NEW STRENGTHENED VERSION OF A HARDY-HILBERT TYPE INEQUALITY AND APPLICATIONS

WEIHONG WANG AND DONGMEI XIN

DEPARTMENT OF MATHEMATICS,
GUANGDONG EDUCATION COLLEGE, GUANGZHOU,
GUANGDONG 510303, PEOPLE'S REPUBLIC OF CHINA.
xdm77108@gdei.edu.cn

Received 16 April, 2006; accepted 20 June, 2006
Communicated by B. Yang

ABSTRACT. By improving an inequality of the weight coefficient, we give a new strengthened version of Hardy-Hilbert's type inequality. As its applications, we build some strengthened versions of the equivalent form and some particular results.

Key words and phrases: Hardy-Hilbert's inequality, Weight coefficient, Hölder's inequality.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$, such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then we have the famous Hardy-Hilbert inequality as follows [1]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=0}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible.

Inequality (1.1) is important in analysis and its applications. In recent years, [2] – [5] considered the strengthened version, generalizations and improvements of inequality (1.1) and Pachpatte [6] built some inequalities similar to inequality (1.1).

Under the same condition with (1.1), we still have Mulholand's inequality (cf. [7]):

$$\sum_{n=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{mn\ln mn} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=2}^{\infty}\frac{1}{n}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty}\frac{1}{n}b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible.

ISSN (electronic): 1443-5756

116-06

For the double series:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^s n^t(\ln m+\ln n+\ln\alpha)}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^s n^t\ln\alpha m}\qquad (s,t\in\mathbb{R},\alpha\geq e^{7/6}),$$

in 2003, Yang [8] built an inequality of the weight coefficient as follows:

$$\sum_{m=1}^{\infty}\frac{1}{m\ln\alpha mn}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{1}{r}}<\frac{\pi}{\sin\pi(1-1/r)}\qquad (r>1,\alpha\geq e^{7/6}),$$

then he gave

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^s n^t(\ln\alpha mn)}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty}\left(n^{\frac{1}{q}-s}a_n\right)^p\right)^{\frac{1}{q}}\left(\sum_{n=1}^{\infty}\left(n^{\frac{1}{p}-t}b_n\right)^q\right)^{\frac{1}{p}}, \tag{1.3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^s\ln\alpha mn}\right)^p<\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p\sum_{n=1}^{\infty}\left(n^{\frac{1}{q}-s}a_n\right)^p, \tag{1.4}$$

where the constant factors $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p$ are the best possible.

In this paper, by using the refined Euler-Maclaurin formula, we have some strengthened versions of inequalities (1.3) and (1.4).

2. Some Lemmas

If $f^{(4)}\in C[1,\infty)$, $\int_1^{\infty}f(x)dx<\infty$, and $(-1)^n f^{(n)}(x)>0$, $f^{(n)}(\infty)=0$ $(n=0,1,2,3,4)$, then we have the following inequality (see [9]):

$$\sum_{m=1}^{\infty}f(m)<\int_1^{\infty}f(x)dx+\frac{1}{2}f(1)-\frac{1}{12}f'(1). \tag{2.1}$$

Lemma 2.1. *Setting $r>1, n\in\mathbb{N}$ and $\alpha\geq e^{7/6}$, define the function $\omega(r,n,\alpha)$ as:*

$$\omega(r,n,\alpha)=\sum_{m=1}^{\infty}\frac{1}{m\ln\alpha mn}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{1}{r}}.$$

The we have

$$\omega(r,n,\alpha)<\frac{\pi}{\sin\pi(1-1/r)}-\frac{3}{8(r-1)(2\ln n+1)^{1-1/r}}. \tag{2.2}$$

Proof. For fixed $x\in[1,\infty)$, setting $f(x)=\frac{1}{x\ln\alpha nx}\cdot\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}x}\right)^{\frac{1}{r}}$, we have

$$f(1)=\frac{1}{\ln\alpha n}\cdot\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}}\right)^{\frac{1}{r}}=\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}\cdot\ln\alpha n}\cdot\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}},$$

$$\begin{aligned}f'(1)&=-\left(\frac{1}{\ln\alpha n}+\frac{1}{\ln^2\alpha n}+\frac{1}{r\ln\sqrt{\alpha}\cdot\ln\alpha n}\right)\cdot\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}}\right)^{1-\frac{1}{r}}\\&=-\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}n\cdot\ln\alpha n}+\frac{\ln\sqrt{\alpha}n}{r\ln^2\sqrt{\alpha}\cdot\ln\alpha n}+\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}\cdot\ln^2\alpha n}\right)\cdot\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}},\end{aligned}$$

$$\begin{aligned}\int_1^\infty f(x)dx &= \int_1^\infty \frac{1}{x\ln\alpha nx}\cdot\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}x}\right)^{\frac{1}{r}}dx\\ &= \int_{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}^\infty \frac{1}{1+u}\cdot\left(\frac{1}{u}\right)^{\frac{1}{r}}du\\ &= \frac{\pi}{\sin(\pi/r)}-\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}\frac{1}{1+u}\cdot\left(\frac{1}{u}\right)^{\frac{1}{r}}du.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}\frac{1}{1+u}\cdot\left(\frac{1}{u}\right)^{\frac{1}{r}}du\\ &=\frac{r}{r-1}\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}\frac{1}{1+u}\cdot du^{1-\frac{1}{r}}\\ &=\frac{r\ln\sqrt{\alpha}n}{(r-1)\ln\alpha n}\cdot\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}+\frac{r}{r-1}\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}u^{1-\frac{1}{r}}\cdot\frac{1}{(1+u)^2}du\\ &=\frac{r\ln\sqrt{\alpha}n}{(r-1)\ln\alpha n}\cdot\left(\frac{ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}+\frac{r^2}{(r-1)(2r-1)}\int_0^{\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}}\frac{1}{(1+u)^2}du^{2-\frac{1}{r}}\\ &>\frac{r\ln\sqrt{\alpha}n}{(r-1)\ln\alpha n}\cdot\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}+\frac{r^2}{(r-1)(2r-1)}\cdot\left(\frac{ln\sqrt{\alpha}}{\ln\alpha}n\right)^2\cdot\left(\frac{ln\sqrt{\alpha}}{\ln\alpha}n\right)^{2-\frac{1}{r}}\\ &=\left(\frac{ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}\left[\frac{r\ln\sqrt{\alpha}n}{(r-1)\ln\alpha n}+\frac{r^2}{(r-1)(2r-1)}\cdot\frac{ln\sqrt{\alpha}\cdot\ln\sqrt{\alpha}n}{\ln^2\alpha n}\right],\end{aligned}$$

in view of (2.1) and the above result, we have

$$\begin{aligned}\omega(r,n,\alpha)&=\sum_{m=1}^\infty f(m)\\ &<\int_1^\infty f(x)dx+\frac{1}{2}f(1)-\frac{1}{12}f'(1)\\ &<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}\\ &\quad\cdot\left[\frac{r\ln\sqrt{\alpha}n}{(r-1)\ln\alpha n}+\frac{r^2}{(r-1)(2r-1)}\cdot\frac{\ln\sqrt{\alpha}\cdot ln\sqrt{\alpha}n}{\ln^2\alpha n}\right]\\ &\quad+\frac{\ln\sqrt{\alpha}n}{2\ln\sqrt{\alpha}\cdot\ln\alpha n}\cdot\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}\\ &\quad+\frac{1}{12}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}\cdot\ln\alpha n}\right)+\frac{\ln\sqrt{\alpha}n}{r\ln^2\sqrt{\alpha}\cdot\ln\alpha n}+\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}\cdot\ln^2\alpha n}\cdot\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}\\ &=\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}-\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}}\cdot\left\{\left(\frac{r}{r-1}-\frac{7}{6\ln\alpha}-\frac{1}{3r\ln^2\alpha}\right)\cdot\frac{\ln\sqrt{\alpha}n}{\ln\alpha n}\right.\\ &\quad\left.+\left[\frac{r^2\ln\alpha}{2(r-1)(2r-1)}-\frac{1}{6\ln\alpha}\right]\cdot\frac{\ln\sqrt{\alpha}n}{\ln^2\alpha n}\right\}.\end{aligned}$$

For $n \in \mathbb{N}$, $r > 1$, $\alpha \geq e^{7/6}$, since

$$\left(\frac{r}{1-r} - \frac{7}{6\ln\alpha} - \frac{1}{3r\ln^2\alpha}\right) \cdot \frac{\ln\sqrt{\alpha}n}{\ln\alpha n} \geq \left(1 + \frac{1}{r-1} - \frac{7}{6\cdot\frac{7}{6}} - \frac{1}{3r\cdot(\frac{7}{6})^2}\right) \cdot \frac{1}{2} = \frac{3}{8(r-1)},$$

$$\begin{aligned}\left[\frac{r^2\ln\alpha}{2(r-1)(2r-1)} - \frac{1}{6\ln\alpha}\right] &\cdot \frac{\ln\sqrt{\alpha}n}{\ln^2\alpha n} \\ &> \left[\frac{\ln\alpha}{4}\left(1+\frac{3}{2(r-1)}\right) - \frac{1}{6\ln\alpha}\right] \cdot \frac{\ln\sqrt{\alpha}n}{\ln^2\alpha n} \\ &> \left[\frac{7}{24}\left(1+\frac{3}{2(r-1)}\right) - \frac{1}{7}\right] \cdot \frac{\ln\sqrt{\alpha}n}{\ln^2\alpha n} > 0,\end{aligned}$$

and

$$\left(\frac{\ln\sqrt{\alpha}}{\ln\sqrt{\alpha}n}\right)^{1-\frac{1}{r}} > \frac{1}{(2\ln n+1)^{1-1/r}},$$

we have

$$\omega(r,n,\alpha) < \frac{\pi}{\sin\pi(1-1/r)} - \frac{3}{8(r-1)(2\ln n+1)^{1-1/r}}.$$

The lemma is proved. □

3. MAIN RESULTS AND APPLICATIONS

Theorem 3.1. *If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha \geq e^{7/6}$, $s, t \in \mathbb{R}$, $a_n b_n$ are two sequences of non-negative real numbers, such that $0 < \sum_{n=1}^{\infty}\left(n^{\frac{1}{q}-s}a_n\right)^p < \infty$ and $0 < \sum_{n=1}^{\infty}\left(n^{\frac{1}{p}-t}b_n\right)^q < \infty$, then we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^s n^t \ln\alpha mn} < \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right) - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{p}}}\left(n^{\frac{1}{q}-s}a_n\right)^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(q-1)}{8(2\ln n+1)^{\frac{1}{q}}}\right)\left(n^{\frac{1}{p}-t}b_n\right)^q\right]^{\frac{1}{q}}. \tag{3.1}$$

In particular,

(a) *for $s = \frac{1}{q}, t = \frac{1}{p}$, we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\frac{1}{p}} n^{\frac{1}{q}} ln\alpha mn} < \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{p}}}\right)a_n^p\right]^{\frac{1}{p}} \times \left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{q}}}\right)b_n^q\right]^{\frac{1}{q}}; \tag{3.2}$$

(b) *for* $s = t = 1$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{mnln\alpha mn} < \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{p}}}\right)\frac{a_n^p}{n}\right]^{\frac{1}{p}} \times \left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{q}}}\right)\frac{b_n^q}{n}\right]^{\frac{1}{q}} \tag{3.3}$$

(c) *for* $s = t = 0$, *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\ln\alpha mn} < \left[\sum_{n=1}^{\infty}\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{p}}}\right)n^{p-1}a_n^p\right]^{\frac{1}{p}} \times \left[\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} - \frac{3(p-1)}{8(2\ln n+1)^{\frac{1}{q}}}\right)n^{q-1}b_n^q\right]^{\frac{1}{q}}. \tag{3.4}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^s n^t\ln\alpha mn} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{(\ln\alpha mn)^{\frac{1}{p}}}\left(\frac{\ln\sqrt{\alpha}m}{\ln\sqrt{\alpha}n}\right)^{\frac{1}{pq}}\left(\frac{m^{\frac{1}{q}-s}}{n^{\frac{1}{p}}}\right)\right] \\ &\quad\times\left[\frac{b_n}{(\ln\alpha mn)^{\frac{1}{q}}}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{1}{pq}}\left(\frac{n^{\frac{1}{p}-t}}{m^{\frac{1}{q}}}\right)\right] \\ &\le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{a_m^p}{\ln\alpha mn}\left(\frac{\ln\sqrt{\alpha}m}{\ln\sqrt{\alpha}n}\right)^{\frac{1}{q}}\left(\frac{m^{p(\frac{1}{q}-s)}}{n}\right)\right]\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\ln\alpha mn}\left(\frac{\ln\sqrt{\alpha}n}{\ln\sqrt{\alpha}m}\right)^{\frac{1}{p}}\left(\frac{n^{q(\frac{1}{p}-t)}}{m}\right)\right]\right\}^{\frac{1}{q}} \\ &= \left[\sum_{m=1}^{\infty}\omega(q,m,\alpha)\left(m^{\frac{1}{q}-s}a_m\right)^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\omega(p,n,\alpha)\left(n^{\frac{1}{p}-t}b_n\right)^q\right]^{\frac{1}{q}}.\end{aligned}$$

In view of (2.2), we have (3.1). The theorem is proved. □

Theorem 3.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \ge e^{7/6}$, $s \in \mathbb{R}, a_n$ *is sequence of non-negative real numbers, such that* $0 < \sum_{n=1}^{\infty}\left(n^{\frac{1}{q}-s}a_n\right)^p < \infty$, *then we have*

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^s ln\alpha mn}\right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^{p-1}\sum_{m=1}^{\infty}\left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2\ln m+1)^{\frac{1}{p}}}\right)\left(m^{\frac{1}{q}-s}a_m\right)^p. \tag{3.5}$$

In particular,

(a) *for* $s = \frac{1}{q}$, *we have*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\frac{1}{q}} \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2\ln m + 1)^{\frac{1}{p}}} \right) (a_m)^p; \tag{3.6}$$

(b) *for* $s = 1$, *we have*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2\ln m + 1)^{\frac{1}{p}}} \right) \frac{a_m^p}{m}; \tag{3.7}$$

(c) *for* $s = 0$, *we have*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^{p-1} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2\ln m + 1)^{\frac{1}{p}}} \right) (m^{p-1} a_m)^p. \tag{3.8}$$

Proof. It is obvious that for any $m \in N_0, \omega(r, m, \alpha) < \frac{\pi}{\sin \pi(1-1/r)}$. By Cauchy's inequality, we obtain

$$\begin{aligned}
&\left(\sum_{m=1}^{\infty} \frac{a_m}{m^s \ln \alpha mn} \right)^p \\
&= \left[\sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)^{\frac{1}{p}}} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{pq}} \left(m^{\frac{1}{q}-s} \right) a_m \cdot \frac{1}{(\ln \alpha mn)^{\frac{1}{q}}} \left(\frac{\ln \sqrt{\alpha} n}{\ln \sqrt{\alpha} m} \right)^{\frac{1}{pq}} \left(\frac{1}{m^{\frac{1}{q}}} \right) \right]^p \\
&\le \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \cdot \sum_{m=1}^{\infty} \left[\frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} n}{\ln \sqrt{\alpha} m} \right)^{\frac{1}{p}} \frac{1}{m} \right]^{(p-1)} \\
&= \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \cdot [\omega(p, n, \alpha)]^{(p-1)} \\
&< \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p.
\end{aligned}$$

Hence, by (2.2) we find

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^s \ln \alpha mn} \right)^p &< \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{\infty} \frac{1}{(\ln \alpha mn)} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \\
&= \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \ln \alpha mn} \left(\frac{\ln \sqrt{\alpha} m}{\ln \sqrt{\alpha} n} \right)^{\frac{1}{q}} m^{p(\frac{1}{q}-s)} a_m^p \\
&= \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \omega(q, m, \alpha) \left(m^{\frac{1}{q}-s} a_m \right)^p \\
&< \left[\frac{\pi}{\sin \frac{\pi}{p}} \right]^{(p-1)} \sum_{m=1}^{\infty} \left(\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{3(p-1)}{8(2\ln m + 1)^{\frac{1}{p}}} \right) \left(m^{\frac{1}{q}-s} a_m \right)^p.
\end{aligned}$$

The theorem is proved. □

Remark 3.3. Obviously, inequalities (3.1) and (3.5) are separately strengthened versions of inequalities (1.3) and (1.4).

REFERENCES

[1] F.H. HARDY AND J.E. LITTLEWOOD, *Inequalities*, London, Cambridge Univ. Press, 1952.

[2] B.C. YANG AND L. DEBNATH, On a new generalization of Hardy-Hilbert's inequality and its applications., *J. Math. Anal. Appl.*, **233** (1999), 484–497.

[3] B.C. YANG, On a strengthened version of the more accurate Hardy-Hilbert's inequality, *Acta Mathematical Sinica*, **43**(6) (1999), 1003–1110 (in Chinese).

[4] B.C. YANG, On a strengthened Hardy-Hilbert's inequality, *J. Ineq. Pure and Appl. Math.*, **1**(2) (2000), Art. 22. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=116`].

[5] JICHANG KUANG AND L. DEBNATH, On new generalizations of Hilbert's inequality and their applications, *J. Math. Anal. Appl*., **245** (2000), 248–265.

[6] B.G. PACHPATTE, On some inequalities similar to Hilbert's inequality, *J. Math. Anal*., **226** (1998), 166–179.

[7] H.P. MULHOLAND, A further generalization of Hilbert's double series theorem, *J. London Math. Soc.*, **6** (1931), 100–106.

[8] B.C. YANG, A new Hardy-Hilbert's type inequality and Applications, *Acta Mathematica Sinica*, **46**(6) (2003), 1079–1086 (in Chinese).

[9] JICHANG KUANG AND L. DEBNATH, On new generalization of Hilbert's inequality and their applications, *J. Math. Anal. Appl*., **245** (2000), 248–265.