

Journal of Inequalities in Pure and Applied Mathematics

HILBERT-PACHPATTE TYPE INTEGRAL INEQUALITIES AND THEIR IMPROVEMENT

S.S. DRAGOMIR AND YOUNG-HO KIM

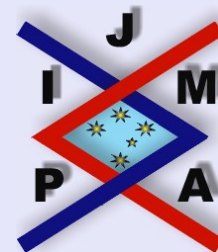
School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428 , Melbourne City MC
Victoria 8001, Australia.

*E*Mail: sever.dragomir@vu.edu.au

*U*RL: <http://rgmia.vu.edu.au/SSDragomirWeb.html>

Department of Applied Mathematics
Changwon National University
Changwon 641-773, Korea.

*E*Mail: yhkim@sarim.changwon.ac.kr



volume 4, issue 1, article 16,
2003.

Received 31 October, 2002;
accepted 8 January, 2003.

Communicated by: P.S. Bullen

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper, we obtain an extension of multivariable integral inequality of Hilbert-Pachpatte type. By specializing the upper estimate functions in the hypothesis and the parameters, we obtain many special cases.

2000 Mathematics Subject Classification: 26D15.

Key words: Hilbert's inequality, Hilbert-Pachpatte type inequality, Hölder's inequality, Jensen inequality.

The authors would like to thank Professor P.S. Bullen, University of British Columbia, Canada, for the careful reading of the manuscript which led to a considerable improvement in the presentation of this paper.

Contents

1	Introduction	3
2	Main Results	5
3	The Various Inequalities	13
	References	



Hilbert-Pachpatte Type Integral Inequalities and their Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 2 of 19

1. Introduction

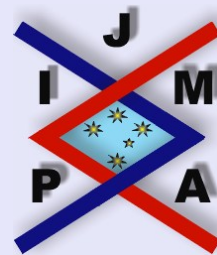
Hilbert's double series theorem [3, p. 226] was proved first by Hilbert in his lectures on integral equations. The determination of the constant, the integral analogue, the extension, other proofs of the whole or of parts of the theorems and generalizations in different directions have been given by several authors (cf. [3, Chap. 9]). Specifically, in [10] – [14] the author has established some new inequalities similar to Hilbert's double-series inequality and its integral analogue which we believe will serve as a model for further investigation. Recently, G.D. Handley, J.J. Koliha and J.E. Pečarić [2] established a new class of related integral inequalities from which the results of Pachpatte [12] – [14] are obtained by specializing the parameters and the functions Φ_i . A representative sample is the following.

Theorem 1.1 (Handley, Koliha and Pečarić [2, Theorem 3.1]). *Let $u_i \in C^{m_i}([0, x_i])$ for $i \in I$. If*

$$\left| u_i^{(k_i)}(s_i) \right| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i) d\tau_i, \quad s_i \in [0, x_i], \quad i \in I,$$

then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\sum_{i=1}^n \omega_i s_i^{(\alpha_i + 1)/(q_i \omega_i)}} ds_1 \cdots ds_n \leq U \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i} ds_i \right)^{\frac{1}{p_i}},$$



Hilbert-Pachpatte Type Integral Inequalities and their Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

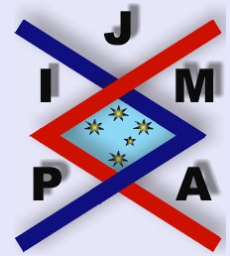
Close

Quit

Page 3 of 19

where $U = 1 / \prod_{i=1}^n [(\alpha_i + 1)^{\frac{1}{q_i}} (\beta_i + 1)^{\frac{1}{p_i}}]$.

The purpose of the present paper is to derive an extension of the inequality given in Theorem 1.1. In addition, we obtain some new inequalities as Hilbert-Pachpatte type inequalities, these inequalities improve the results obtained by Handley, Koliha and Pečarić [2].



**Hilbert-Pachpatte Type Integral
Inequalities and their
Improvement**

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 4 of 19

2. Main Results

In what follows we denote by \mathbb{R} the set of real numbers; \mathbb{R}_+ denotes the interval $[0, \infty)$. The symbols \mathbb{N}, \mathbb{Z} have their usual meaning. The following notation and hypotheses will be used throughout the paper:

$$I = \{1, \dots, n\} \quad n \in \mathbb{N}$$

$$m_i, i \in I \quad m_i \in \mathbb{N}$$

$$k_i, i \in I \quad k_i \in \{0, 1, \dots, m_i - 1\}$$

$$x_i, i \in I \quad x_i \in \mathbb{R}, x_i > 0$$

$$p_i, q_i, i \in I \quad p_i, q_i \in \mathbb{R}, p_i, q_i > 0, \frac{1}{p_i} + \frac{1}{q_i} = 1$$

$$p, q \quad \frac{1}{p} = \sum_{i=1}^n \left(\frac{1}{p_i}\right), \quad \frac{1}{q} = \sum_{i=1}^n \left(\frac{1}{q_i}\right)$$

$$a_i, b_i, i \in I \quad a_i, b_i \in \mathbb{R}_+, a_i + b_i = 1$$

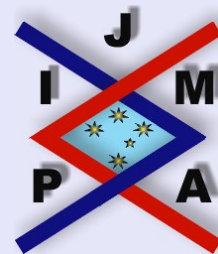
$$\omega_i, i \in I \quad \omega_i \in \mathbb{R}, \omega_i > 0, \sum_{i=1}^n \omega_i = \Omega_n$$

$$\alpha_i, i \in I \quad \alpha_i = (a_i + b_i q_i)(m_i - k_i - 1)$$

$$\beta_i, i \in I \quad \beta_i = a_i(m_i - k_i - 1)$$

$$u_i, i \in I \quad u_i \in C^{m'_i}([0, x_i]) \quad \text{for some } m'_i \geq m_i$$

$$\Phi_i, i \in I \quad \Phi_i \in C^1([0, x_i]), \Phi_i \geq m_i.$$



Hilbert-Pachpatte Type Integral Inequalities and their Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents

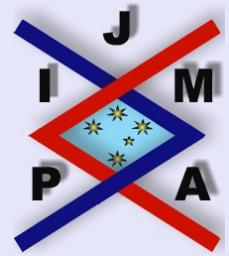


Go Back

Close

Quit

Page 5 of 19



Hilbert-Pachpatte Type Integral Inequalities and their Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 6 of 19

Here the u_i are given functions of sufficient smoothness, and the Φ_i are subject to choice. The coefficients p_i, q_i are conjugate Hölder exponents to be used in applications of Hölder's inequality, and the coefficients a_i, b_i will be used in exponents to factorize integrands. The coefficients ω_i will act as weights in applications of the geometric-arithmetic mean inequality. The coefficients α_i and β_i arise naturally in the derivation of the inequalities. Our main results are given in the following theorems.

Theorem 2.1. Let $u_i \in C^{m_i}([0, x_i])$ for $i \in I$. If

$$(2.1) \quad \left| u_i^{(k_i)}(s_i) \right| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i) d\tau_i, \quad s_i \in [0, x_i], \quad i \in I,$$

then

$$(2.2) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i + 1)/(q_i \omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n \leq V \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i} ds_i \right)^{\frac{1}{p_i}},$$

where

$$(2.3) \quad V = \frac{1}{\prod_{i=1}^n \left[(\alpha_i + 1)^{\frac{1}{q_i}} (\beta_i + 1)^{\frac{1}{p_i}} \right]}.$$

Proof. Factorize the integrand on the right side of (2.1) as

$$(s_i - \tau_i)^{(a_i/q_i + b_i)(m_i - k_i - 1)} \times (s_i - \tau_i)^{(a_i/p_i)(m_i - k_i - 1)} \Phi_i(\tau_i)$$

and apply Hölder's inequality [9, p.106]. Then

$$\begin{aligned} \left| u_i^{(k_i)}(s_i) \right| &\leq \left(\int_0^{s_i} (s_i - \tau_i)^{(a_i+b_iq_i)(m_i-k_i-1)} d\tau_i \right)^{\frac{1}{q_i}} \\ &\quad \times \left(\int_0^{s_i} (s_i - \tau_i)^{a_i(m_i-k_i-1)} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}} \\ &= \frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i + 1)^{\frac{1}{q_i}}} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}}. \end{aligned}$$

Using the inequality of means [9, p. 15]

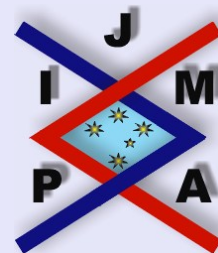
$$\left(\prod_{i=1}^n s_i^{w_i} \right)^{\frac{1}{\Omega_n}} \leq \left(\frac{1}{\Omega_n} \sum_{i=1}^n w_i s_i^r \right)^{\frac{1}{r}}$$

for $r > 0$, we deduce that

$$\prod_{i=1}^n s_i^{w_i r} \leq \left[\frac{1}{\Omega_n} \sum_{i=1}^n w_i s_i^r \right]^{\Omega_n}$$

for $r > 0$. According to above inequality, we have

$$\begin{aligned} \prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right| &\leq \frac{1}{\prod_{i=1}^n (\alpha_i + 1)^{\frac{1}{q_i}}} \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i \omega_i)} \right]^{\Omega_n} \\ &\quad \times \prod_{i=1}^n \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{\frac{1}{p_i}} \end{aligned}$$



**Hilbert-Pachpatte Type Integral
Inequalities and their
Improvement**

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 7 of 19

for $r = (\alpha_i + 1)/q_i\omega_i$. In the following estimate we apply Hölder's inequality and, at the end, change the order of integration:

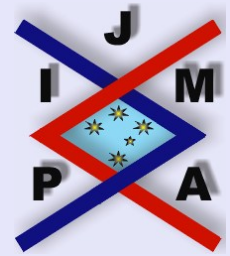
$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)} s_i|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n \\ & \leq \frac{1}{\prod_{i=1}^n (\alpha_i + 1)^{\frac{1}{q_i}}} \prod_{i=1}^n \left[\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i, \right)^{\frac{1}{p_i}} ds_i \right] \\ & \leq \frac{1}{\prod_{i=1}^n (\alpha_i + 1)^{\frac{1}{q_i}}} \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i, \right) ds_i \right]^{\frac{1}{p_i}} \\ & = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{\frac{1}{q_i}} (\beta_i + 1)^{\frac{1}{p_i}}]} \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} ds_i \right]^{\frac{1}{p_i}}. \end{aligned}$$

This proves the theorem. \square

Remark 2.1. In Theorem 2.1, setting $\Omega_n = 1$, we have Theorem 1.1.

Corollary 2.2. Under the assumptions of Theorem 2.1, if $r > 0$, we have

$$\begin{aligned} & \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)} s_i|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n \\ & \leq p^{\frac{1}{r-p}} V \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[\sum_{i=1}^n \frac{1}{p_i} \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i s_i^{p_i} ds_i \right)^r \right]^{\frac{1}{r-p}}, \end{aligned}$$



Hilbert-Pachpatte Type Integral Inequalities and their Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 8 of 19

where V is defined by (2.3).

Proof. By the inequality of means, for any $A_i \geq 0$ and $r > 0$, we obtain

$$\prod_{i=1}^n A_i^{\frac{1}{p_i}} \leq \left[p \sum_{i=1}^n \frac{1}{p_i} A_i^r \right]^{\frac{1}{r \cdot p}}.$$

The corollary then follows from the preceding theorem. □

Lemma 2.3. Let $\gamma_1 > 0$ and $\gamma_2 < -1$. Let $\omega_i > 0$, $\sum_{i=1}^n \omega_i = \Omega_n$ and let $s_i > 0$, $i = 1, \dots, n$ be real numbers. Then

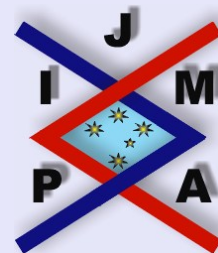
$$\prod_{i=1}^n s_i^{\omega_i \gamma_1 \gamma_2} \geq \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-\gamma_1 \Omega_n}.$$

Proof. By the inequality of means, for any $\gamma_1 > 0$ and $\gamma_2 < -1$, we have

$$\prod_{i=1}^n s_i^{\omega_i \gamma_1 \gamma_2} \geq \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i \right]^{\gamma_1 \gamma_2 \Omega_n}.$$

Using the fact that $x^{-\frac{1}{\gamma_2}}$ is concave and using the Jensen inequality, we have that

$$\begin{aligned} \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i \right]^{\gamma_1 \gamma_2 \Omega_n} &= \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i f(s_i^{-\gamma_2}) \right]^{\gamma_1 \gamma_2 \Omega_n} \\ &\geq \left[f \left(\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right) \right]^{\gamma_1 \gamma_2 \Omega_n} \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 9 of 19

$$\begin{aligned}
&= \left[\left(\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right)^{-\frac{1}{\gamma_2}} \right]^{\gamma_1 \gamma_2 \Omega_n} \\
&= \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-\gamma_1 \Omega_n}.
\end{aligned}$$

The proof of the lemma is complete. \square

Theorem 2.4. *Under the assumptions of Theorem 2.1, if $\gamma_2 < -1$, then*

$$\begin{aligned}
&\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n \\
&\leq V \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} ds_i \right]^{\frac{1}{p_i}},
\end{aligned}$$

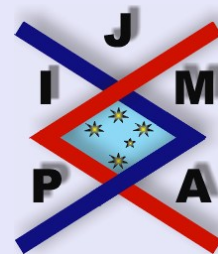
where V is given by (2.3).

Proof. Using the inequality of Lemma 2.3, for any $\gamma_1 > 0$ and $\gamma_2 < -1$, we get

$$\prod_{i=1}^n s_i^{\omega_i \gamma_1} \leq \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-\frac{\gamma_1 \Omega_n}{\gamma_2}}.$$

According to above inequality, we deduce that

$$\prod_{i=1}^n |u_i^{(k_i)}(s_i)| \leq \frac{1}{\prod_{i=1}^n (\alpha_i + 1)^{\frac{1}{q_i}}} \left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-W_1}$$



**Hilbert-Pachpatte Type Integral
Inequalities and their
Improvement**

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 10 of 19

$$\times \prod_{i=1}^n \left[\int_0^{(s_i)} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right]^{\frac{1}{p_i}},$$

where $W_1 = (\alpha_i + 1)\Omega_n/\gamma_2 q_i \omega_i$. The proof of the theorem then follows from the preceding Theorem 2.1. \square

Corollary 2.5. *Under the assumptions of Theorem 2.4, if $r > 0$, we have*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n$$

$$\leq p^{\frac{1}{r-p}} V \prod_{i=1}^n x_i^{\frac{q_i}{r}} \left[\sum_{i=1}^n \frac{1}{p_i} \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} ds_i \right)^r \right]^{\frac{1}{r-p}},$$

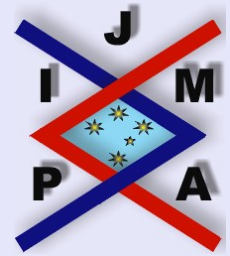
where V is given by (2.3).

Proof. By the inequality of means, for any $A_i \geq 0$ and $r > 0$, we obtain

$$\prod_{i=1}^n A_i^{\frac{1}{p_i}} \leq \left[p \sum_{i=1}^n \frac{1}{p_i} A_i^r \right]^{\frac{1}{r-p}}.$$

The corollary then follows from the preceding Theorem 2.4. \square

In the following section we discuss some choice of the functions Φ_i .



Title Page

Contents



Go Back

Close

Quit

Page 11 of 19

3. The Various Inequalities

Theorem 3.1. Let $u_i \in C^{m_i}([0, x_i])$ be such that $u_i^{(j)}(0) = 0$ for $j \in \{0, \dots, m_i - 1\}$, $i \in I$. Then

$$(3.1) \quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i \omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right]^{\frac{1}{p_i}},$$

where

$$(3.2) \quad V_1 = \frac{1}{\prod_{i=1}^n \left[(m_i - k_i - 1)! (\alpha_i + 1)^{\frac{1}{q_i}} (\beta_i + 1)^{\frac{1}{p_i}} \right]}.$$

Proof. Inequality (3.1) is proved when we set

$$\Phi_i(s_i) = \frac{|u_i^{(m_i)}(s_i)|}{(m_i - k_i - 1)!}$$

in Theorem 2.1. □



Hilbert-Pachpatte Type Integral
Inequalities and their
Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 12 of 19

Corollary 3.2. Under the assumptions of Theorem 3.1, if $r > 0$, we have

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i \omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq p^{\frac{1}{r \cdot p}} V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[\sum_{i=1}^n \frac{1}{p_i} \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right]^r \right]^{\frac{1}{r \cdot p}},$$

where V_1 is given by (3.2).

Theorem 3.3. Under the assumptions of Theorem 3.1, if $\gamma_2 < -1$, then

$$(3.3) \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n$$

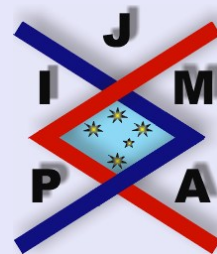
$$\leq V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right]^{\frac{1}{p_i}},$$

where V_1 is given by (3.2).

Proof. Inequality (3.3) is proved when we set

$$\Phi_i(s_i) = \frac{|u_i^{(m_i)}(s_i)|}{(m_i - k_i - 1)!}$$

in Theorem 2.4. □



Title Page

Contents



Go Back

Close

Quit

Page 13 of 19

Corollary 3.4. Under the assumptions of Theorem 3.3, if $r > 0$, we have

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{-\gamma_2} \right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n$$

$$\leq p^{\frac{1}{r-p}} V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[\sum_{i=1}^n \frac{1}{p_i} \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right]^r \right]^{\frac{1}{r-p}}.$$

We discuss a number of special cases of Theorem 3.1. Similar examples apply also to Corollary 3.2, Theorem 3.3 and Corollary 3.4.

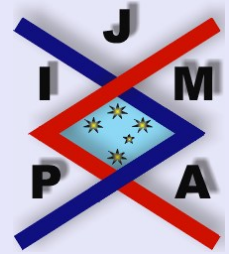
Example 3.1. If $a_i = 0$ and $b_i = 1$ for $i \in I$, then Theorem 3.1 becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(q_i m_i - q_i k_i - q_i + 1)/(q_i \omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq V_2 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i) |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right]^{\frac{1}{p_i}},$$

where

$$V_2 = \frac{1}{\prod_{i=1}^n \left[(m_i - k_i - 1)! (q_i m_i - q_i k_i - q_i + 1)^{\frac{1}{q_i}} \right]}.$$



Hilbert-Pachpatte Type Integral
Inequalities and their
Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents

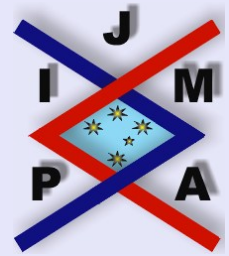


Go Back

Close

Quit

Page 14 of 19



Title Page

Contents



Go Back

Close

Quit

Page 15 of 19

Example 3.2. If $a_i = 0$, $b_i = 1$, $q_i = n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$, then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(nm-nk-n+1)/(n\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq \frac{\sqrt[n]{x_1 \cdots x_n}}{\left((m-k-1)! \right)^n (nm-nk-n+1)}$$

$$\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i) |u_i^{(m)}(s_i)|^{\frac{n}{n-1}} ds_i \right]^{\frac{n-1}{n}}.$$

For $q = p = n = 2$ and $\omega_i = \frac{1}{n}$ this is [12, Theorem 1]. Setting $q = p = 2$, $k = 0$, $n = 1$ and $\omega_i = \frac{1}{n}$, we recover the result of [14].

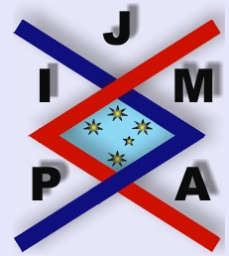
Example 3.3. If $a_i = 0$ and $b_i = 1$ for $i \in I$, then Theorem 3.1 becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(m_i-k_i)/(q_i\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq V_3 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m_i-k_i} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right]^{\frac{1}{p_i}},$$

where

$$V_3 = \frac{1}{\prod_{i=1}^n [(m_i - k_i)!]}.$$



Title Page

Contents



Go Back

Close

Quit

Page 16 of 19

Example 3.4. If $a_i = 1$, $b_i = 0$, $q_i = n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (3.1) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(m-k)/(n\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k)!]^n} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m-k} |u_i^{(m)}(s_i)|^{n/(n-1)} ds_i \right]^{\frac{(n-1)}{n}}.$$

Example 3.5. Let $p_1, p_2 \in \mathbb{R}_+$. If we set $n = 2$, $\omega_1 = \frac{1}{p_1}$, $\omega_2 = \frac{1}{p_2}$, $m_i = 1$ and $k_i = 0$ for $i = 1, 2$ in Theorem 3.1, then by our assumptions $q_1 = p_1/(p_1 - 1)$, $q_2 = p_2/(p_2 - 1)$, and we obtain

$$\int_0^{x_1} \int_0^{x_2} \frac{|u_1(s_1)| |u_2(s_2)|}{\left[\frac{1}{p_1 p_2 \Omega_2} \left(p_2 s_1^{(p_1-1)} + p_1 s_2^{(p_2-1)} \right) \right]^{\Omega_2}} ds_1 ds_2$$

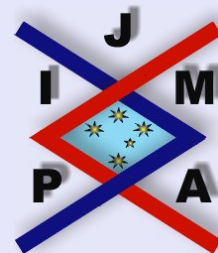
$$\leq x_1^{(p_1-1)/p_1} x_2^{(p_2-1)/p_2} \left(\int_0^{x_1} (x_1 - s_1) |u_1'(s_1)|^{p_1} ds_1 \right)^{\frac{1}{p_1}}$$

$$\times \left(\int_0^{x_2} (x_2 - s_2) |u_2'(s_2)|^{p_2} ds_2 \right)^{\frac{1}{p_2}}.$$

If we set $\omega_1 + \omega_2 = 1$ in Example 3.5, then we have [13, Theorem 2]. (The values of a_i and b_i are irrelevant.)

References

- [1] BICHENG YANG, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220** (1988), 778–785.
- [2] G.D. HANDLEY, J.J. KOLIHA AND PEČARIĆ, New Hilbert-Pachpatte type integral inequalities, *J. Math. Anal. Appl.*, **257** (2001), 238–250.
- [3] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, London, 1952.
- [4] YOUNG-HO KIM, Refinements and Extensions of an inequality, *J. Math. Anal. Appl.*, **245** (2000), 628–632.
- [5] V. LEVIN, On the two-parameter extension and analogue of Hilbert's inequality, *J. London Math. Soc.*, **11** (1936), 119–124.
- [6] G. MINGZE, On Hilbert's inequality and its applications, *J. Math. Anal. Appl.*, **212** (1997), 316–323.
- [7] D.S. MITRINOVIĆ, *Analytic inequalities*, Springer-Verlag, Berlin, New York, 1970.
- [8] D.S. MITRINOVIĆ AND J.E. PEČARIĆ, On inequalities of Hilbert and Widder, *Proc. Edinburgh Math. Soc.*, **34** (1991), 411–414.
- [9] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.
- [10] B.G. PACHPATTE, A note on Hilbert type inequality, *Tamkang J. Math.*, **29** (1998), 293–298.



Hilbert-Pachpatte Type Integral Inequalities and their Improvement

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



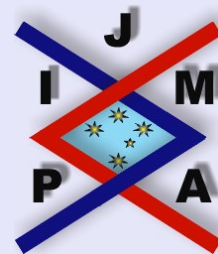
Go Back

Close

Quit

Page 17 of 19

- [11] B.G. PACHPATTE, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.*, **226** (1998), 166–179.
- [12] B.G. PACHPATTE, Inequalities similar to the integral analogue of Hilbert's Inequality, *Tamkang J. Math.*, **30** (1999), 139–146.
- [13] B.G. PACHPATTE, Inequalities similar to certain extensions of Hilbert's inequality, *J. Math. Anal. Appl.*, **243** (2000), 217–227.
- [14] B.G. PACHPATTE, A note on inequality of Hilbert type, *Demonstratio Math.*, in press.
- [15] D.V. WIDDER, An inequality related to one of Hilbert's, *J. London Math. Soc.*, **4** (1929), 194–198.



**Hilbert-Pachpatte Type Integral
Inequalities and their
Improvement**

S.S. Dragomir and Young-Ho Kim

Title Page

Contents



Go Back

Close

Quit

Page 18 of 19