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# HILBERT-PACHPATTE TYPE INTEGRAL INEQUALITIES AND THEIR IMPROVEMENT

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## Abstract

In this paper, we obtain an extension of multivariable integral inequality of Hilbert-Pachpatte type. By specializing the upper estimate functions in the hypothesis and the parameters, we obtain many special cases.

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*Key words:* Hilbert's inequality, Hilbert-Pachpatte type inequality, Hölder's inequality, Jensen inequality.

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# Contents

# 1. Intoduction

Hilbert's double series theorem [3, p. 226] was proved first by Hilbert in his lectures on integral equations. The determination of the constant, the integral analogue, the extension, other proofs of the whole or of parts of the theorems and generalizations in different directions have been given by several authors (cf. [3, Chap. 9]). Specifically, in [10] – [14] the author has established some new inequalities similar to Hilbert's double-series inequality and its integral analogue which we believe will serve as a model for further investigation. Recently, G.D. Handley, J.J. Koliha and J.E. Pečarić [2] established a new class of related integral inequalities from which the results of Pachpatte [12] – [14] are obtained by specializing the parameters and the functions $\Phi_i$. A representative sample is the following.

**Theorem 1.1 (Handley, Koliha and Pečarić [2, Theorem 3.1]).** *Let* $u_i \in C^{m_i}([0, x_i])$ *for* $i \in I$. *If*

$$\left| u_i^{(k_i)}(s_i) \right| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i)\, d\tau_i, \quad s_i \in [0, x_i],\ i \in I,$$

*then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)}}\, ds_1 \cdots ds_n$$

$$\leq U \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i}\, ds_i \right)^{\frac{1}{p_i}},$$

*where* $U = 1 \Big/ \prod_{i=1}^{n} [(\alpha_i + 1)^{\frac{1}{q_i}} (\beta_i + 1)^{\frac{1}{p_i}}]$ .

The purpose of the present paper is to derive an extension of the inequality given in Theorem 1.1. In addition, we obtain some new inequalities as Hilbert-Pachpatte type inequalities, these inequalities improve the results obtained by Handley, Koliha and Pečarić [2].

## 2. Main Results

In what follows we denote by $\mathbb{R}$ the set of real numbers; $\mathbb{R}_+$ denotes the interval $[0, \infty)$. The symbols $\mathbb{N}, \mathbb{Z}$ have their usual meaning. The following notation and hypotheses will be used throughout the paper:

| | |
|---|---|
| $I = \{1, ..., n\}$ | $n \in \mathbb{N}$ |
| $m_i,\ i \in I$ | $m_i \in \mathbb{N}$ |
| $k_i,\ i \in I$ | $k_i \in \{0, 1, \dots, m_i - 1\}$ |
| $x_i,\ i \in I$ | $x_i \in \mathbb{R},\ x_i > 0$ |
| $p_i,\ q_i,\ \ i \in I$ | $p_i, q_i \in \mathbb{R},\ p_i, q_i > 0,\ \frac{1}{p_i} + \frac{1}{q_i} = 1$ |
| $p, q$ | $\frac{1}{p} = \sum_{i=1}^{n} \left(\frac{1}{p_i}\right),\ \ \frac{1}{q} = \sum_{i=1}^{n} \left(\frac{1}{q_i}\right)$ |
| $a_i, b_i,\ i \in I$ | $a_i, b_i \in \mathbb{R}_+,\ a_i + b_i = 1$ |
| $\omega_i,\ i \in I$ | $\omega_i \in \mathbb{R},\ \omega_i > 0,\ \sum_{i=1}^{n} \omega_i = \Omega_n$ |
| $\alpha_i,\ i \in I$ | $\alpha_i = (a_i + b_i q_i)(m_i - k_i - 1)$ |
| $\beta_i,\ i \in I$ | $\beta_i = a_i(m_i - k_i - 1)$ |
| $u_i,\ i \in I$ | $u_i \in C^{m_i'}([0, x_i])$ for some $m_i' \geq m_i$ |
| $\Phi_i,\ i \in I$ | $\Phi_i \in C^1([0, x_i]),\ \Phi_i \geq m_i.$ |

Here the $u_i$ are given functions of sufficient smoothness, and the $\Phi_i$ are subject to choice. The coefficients $p_i, q_i$ are conjugate Hölder exponents to be used in applications of Hölder's inequality, and the coefficients $a_i, b_i$ will be used in exponents to factorize integrands. The coefficients $\omega_i$ will act as weights in applications of the geometric-arithmetic mean inequality. The coefficients $\alpha_i$ and $\beta_i$ arise naturally in the derivation of the inequalities. Our main results are given in the following theorems.

**Theorem 2.1.** *Let* $u_i \in C^{m_i}([0, x_i])$ *for* $i \in I$. *If*

$$\left|u_i^{(k_i)}(s_i)\right| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i)\, d\tau_i, \quad s_i \in [0, x_i],\ i \in I, \tag{2.1}$$

*then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left|u_i^{(k_i)}(s_i)\right|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i \omega_i)}\right]^{\Omega_n}} ds_1 \cdots ds_n \leq V \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i}\, ds_i\right)^{\frac{1}{p_i}}, \tag{2.2}$$

*where*

$$V = \frac{1}{\prod_{i=1}^n \left[(\alpha_i + 1)^{\frac{1}{q_i}} (\beta_i + 1)^{\frac{1}{p_i}}\right]}. \tag{2.3}$$

*Proof.* Factorize the integrand on the right side of (2.1) as

$$(s_i - \tau_i)^{(a_i/q_i + b_i)(m_i - k_i - 1)} \times (s_i - \tau_i)^{(a_i/p_i)(m_i - k_i - 1)} \Phi_i(\tau_i)$$

and apply Hölder's inequality [9, p.106]. Then

$$\left|u_i^{(k_i)}(s_i)\right| \le \left(\int_0^{s_i} (s_i-\tau_i)^{(a_i+b_iq_i)(m_i-k_i-1)}\,d\tau_i\right)^{\frac{1}{q_i}} \times \left(\int_0^{s_i} (s_i-\tau_i)^{a_i(m_i-k_i-1)}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}} = \frac{s_i^{(\alpha_i+1)/q_i}}{(\alpha_i+1)^{\frac{1}{q_i}}}\left(\int_0^{s_i} (s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}\,d\tau_i\right)^{\frac{1}{p_i}}.$$

Using the inequality of means [9, p. 15]

$$\left(\prod_{i=1}^{n} s_i^{w_i}\right)^{\frac{1}{\Omega_n}} \le \left(\frac{1}{\Omega_n}\sum_{i=1}^{n} w_i s_i^r\right)^{\frac{1}{r}}$$

for $r>0$, we deduce that

$$\prod_{i=1}^{n} s_i^{w_i r} \le \left[\frac{1}{\Omega_n}\sum_{i=1}^{n} w_i s_i^r\right]^{\Omega_n}$$

for $r>0$. According to above inequality, we have

$$\prod_{i=1}^{n}\left|u_i^{(k_i)}(s_i)\right| \le \frac{1}{\prod_{i=1}^{n}(\alpha_i+1)^{\frac{1}{q_i}}}\left[\frac{1}{\Omega_n}\sum_{i=1}^{n}\omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)}\right]^{\Omega_n} \times \prod_{i=1}^{n}\left(\int_0^{s_i}(s_i-\tau_i)^{\beta_i}\Phi_i(\tau_i)^{p_i}d\tau_i\right)^{\frac{1}{p_i}}$$

for $r = (\alpha_i + 1)/q_i\omega_i$. In the following estimate we apply Hölder's inequality and, at the end, change the order of integration:

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)} s_i \right|}{\left[ \frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq \frac{1}{\prod_{i=1}^n (\alpha_i+1)^{\frac{1}{q_i}}} \prod_{i=1}^n \left[ \int_0^{x_i} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i, \right)^{\frac{1}{p_i}} ds_i \right]$$

$$\leq \frac{1}{\prod_{i=1}^n (\alpha_i+1)^{\frac{1}{q_i}}} \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[ \int_0^{x_i} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i, \right) ds_i \right]^{\frac{1}{p_i}}$$

$$= \frac{1}{\prod_{i=1}^n [(\alpha_i+1)^{\frac{1}{q_i}} (\beta_i+1)^{\frac{1}{p_i}}]} \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[ \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} \, ds_i \right]^{\frac{1}{p_i}}.$$

This proves the theorem. □

**Remark 2.1.** *In Theorem 2.1, setting $\Omega_n = 1$, we have Theorem 1.1.*

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, if $r > 0$, we have*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)} s_i \right|}{\left[ \frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq p^{\frac{1}{r \cdot p}} V \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[ \sum_{i=1}^n \frac{1}{p_i} \left( \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i s_i^{p_i} \, ds_i \right)^r \right]^{\frac{1}{r \cdot p}},$$

*where $V$ is defined by (2.3).*

*Proof.* By the inequality of means, for any $A_i \geq 0$ and $r > 0$, we obtain

$$\prod_{i=1}^{n} A_i^{\frac{1}{p_i}} \leq \left[ p \sum_{i=1}^{n} \frac{1}{p_i} A_i^r \right]^{\frac{1}{r \cdot p}}.$$

The corollary then follows from the preceding theorem. □

**Lemma 2.3.** *Let $\gamma_1 > 0$ and $\gamma_2 < -1$. Let $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = \Omega_n$ and let $s_i > 0$, $i = 1, \ldots, n$ be real numbers. Then*

$$\prod_{i=1}^{n} s_i^{\omega_i \gamma_1 \gamma_2} \geq \left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right]^{-\gamma_1 \Omega_n}.$$

*Proof.* By the inequality of means, for any $\gamma_1 > 0$ and $\gamma_2 < -1$, we have

$$\prod_{i=1}^{n} s_i^{\omega_i \gamma_1 \gamma_2} \geq \left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i \right]^{\gamma_1 \gamma_2 \Omega_n}.$$

Using the fact that $x^{-\frac{1}{\gamma_2}}$ is concave and using the Jensen inequality, we have that

$$\left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i \right]^{\gamma_1 \gamma_2 \Omega_n} = \left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i f(s_i^{-\gamma_2}) \right]^{\gamma_1 \gamma_2 \Omega_n}$$
$$\geq \left[ f\left( \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right) \right]^{\gamma_1 \gamma_2 \Omega_n}$$

$$= \left[ \left( \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right)^{-\frac{1}{\gamma_2}} \right]^{\gamma_1 \gamma_2 \Omega_n}$$

$$= \left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right]^{-\gamma_1 \Omega_n}.$$

The proof of the lemma is complete. □

**Theorem 2.4.** *Under the assumptions of Theorem 2.1, if $\gamma_2 < -1$, then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} \left| u_i^{(k_i)}(s_i) \right|}{\left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n$$
$$\leq V \prod_{i=1}^{n} x_i^{\frac{1}{q_i}} \prod_{i=1}^{n} \left[ \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} \, ds_i \right]^{\frac{1}{p_i}},$$

*where $V$ is given by (2.3).*

*Proof.* Using the inequality of Lemma 2.3, for any $\gamma_1 > 0$ and $\gamma_2 < -1$, we get

$$\prod_{i=1}^{n} s_i^{\omega_i \gamma_1} \leq \left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right]^{-\frac{\gamma_1 \Omega_n}{\gamma_2}}.$$

According to above inequality, we deduce that

$$\prod_{i=1}^{n} \left| u_i^{(k_i)}(s_i) \right| \leq \frac{1}{\prod_{i=1}^{n} (\alpha_i + 1)^{\frac{1}{q_i}}} \left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right]^{-W_1}$$

$$\times \prod_{i=1}^{n} \left[ \int_0^{(s_i)} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right]^{\frac{1}{p_i}},$$

where $W_1 = (\alpha_i + 1)\Omega_n / \gamma_2 q_i \omega_i$. The proof of the theorem then follows from the preceding Theorem 2.1. □

**Corollary 2.5.** *Under the assumptions of Theorem 2.4, if $r > 0$, we have*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} \left| u_i^{(k_i)}(s_i) \right|}{\left[ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^{-\gamma_2} \right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n$$
$$\leq p^{\frac{1}{r \cdot p}} V \prod_{i=1}^{n} x_i^{\frac{1}{q_i}} \left[ \sum_{i=1}^{n} \frac{1}{p_i} \left( \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i} \, ds_i \right)^r \right]^{\frac{1}{r \cdot p}},$$

*where $V$ is given by (2.3).*

*Proof.* By the inequality of means, for any $A_i \geq 0$ and $r > 0$, we obtain

$$\prod_{i=1}^{n} A_i^{\frac{1}{p_i}} \leq \left[ p \sum_{i=1}^{n} \frac{1}{p_i} A_i^r \right]^{\frac{1}{r \cdot p}}.$$

The corollary then follows from the preceding Theorem 2.4. □

In the following section we discuss some choice of the functions $\Phi_i$.

## 3. The Various Inequalities

**Theorem 3.1.** *Let* $u_i \in C^{m_i}([0,x_i])$ *be such that* $u_i^{(j)}(0) = 0$ *for* $j \in \{0,\dots,m_i - 1\}$, $i \in I$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n \left|u_i^{(k_i)}(s_i)\right|}{\left[\frac{1}{\Omega_n}\sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)}\right]^{\Omega_n}} ds_1\cdots ds_n \le V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left|u_i^{(m_i)}(s_i)\right|^{p_i} ds_i\right]^{\frac{1}{p_i}}, \tag{3.1}$$

*where*

$$V_1 = \frac{1}{\prod_{i=1}^n \left[(m_i - k_i - 1)!(\alpha_i+1)^{\frac{1}{q_i}}(\beta_i+1)^{\frac{1}{p_i}}\right]}. \tag{3.2}$$

*Proof.* Inequality (3.1) is proved when we set

$$\Phi_i(s_i) = \frac{\left|u_i^{(m_i)}(s_i)\right|}{(m_i - k_i - 1)!}$$

in Theorem 2.1. □

**Corollary 3.2.** *Under the assumptions of Theorem 3.1, if $r > 0$, we have*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\left[\frac{1}{\Omega_n}\sum_{i=1}^n \omega_i s_i^{(\alpha_i+1)/(q_i\omega_i)}\right]^{\Omega_n}} ds_1 \cdots ds_n$$
$$\leq p^{\frac{1}{r\cdot p}} V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \left[ \sum_{i=1}^n \frac{1}{p_i} \left[ \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left| u_i^{(m_i)}(s_i) \right|^{p_i} ds_i \right]^r \right]^{\frac{1}{r\cdot p}},$$

*where $V_1$ is given by (3.2).*

**Theorem 3.3.** *Under the assumptions of Theorem 3.1, if $\gamma_2 < -1$, then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\left[\frac{1}{\Omega_n}\sum_{i=1}^n \omega_i s_i^{-\gamma_2}\right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i \omega_i}} ds_1 \cdots ds_n$$
$$\leq V_1 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[ \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \left| u_i^{(m_i)}(s_i) \right|^{p_i} ds_i \right]^{\frac{1}{p_i}}, \tag{3.3}$$

*where $V_1$ is given by (3.2).*

*Proof.* Inequality (3.3) is proved when we set

$$\Phi_i(s_i) = \frac{\left| u_i^{(m_i)}(s_i) \right|}{(m_i - k_i - 1)!}$$

in Theorem 2.4. □

**Corollary 3.4.** *Under the assumptions of Theorem 3.3, if $r > 0$, we have*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n\left|u_i^{(k_i)}(s_i)\right|}{\left[\frac{1}{\Omega_n}\sum_{i=1}^n\omega_i s_i^{-\gamma_2}\right]^{-(\alpha_i+1)\Omega_n/\gamma_2 q_i\omega_i}}\,ds_1\cdots ds_n$$
$$\le p^{\frac{1}{r\cdot p}}V_1\prod_{i=1}^n x_i^{\frac{1}{q_i}}\left[\sum_{i=1}^n\frac{1}{p_i}\left[\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\left|u_i^{(m_i)}(s_i)\right|^{p_i}\,ds_i\right]^r\right]^{\frac{1}{r\cdot p}}.$$

We discuss a number of special cases of Theorem 3.1. Similar examples apply also to Corollary 3.2, Theorem 3.3 and Corollary 3.4.

**Example 3.1.** *If $a_i = 0$ and $b_i = 1$ for $i \in I$, then Theorem 3.1 becomes*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n\left|u_i^{(k_i)}(s_i)\right|}{\left[\frac{1}{\Omega_n}\sum_{i=1}^n\omega_i s_i^{(q_im_i-q_ik_i-q_i+1)/(q_i\omega_i)}\right]^{\Omega_n}}\,ds_1\cdots ds_n$$
$$\le V_2\prod_{i=1}^n x_i^{\frac{1}{q_i}}\prod_{i=1}^n\left[\int_0^{x_i}(x_i-s_i)\left|u_i^{(m_i)}(s_i)\right|^{p_i}\,ds_i\right]^{\frac{1}{p_i}},$$

*where*

$$V_2=\frac{1}{\prod_{i=1}^n\left[(m_i-k_i-1)!(q_im_i-q_ik_i-q_i+1)^{\frac{1}{q_i}}\right]}.$$

**Example 3.2.** *If* $a_i = 0$, $b_i = 1$, $q_i = n$, $p_i = n/(n-1)$, $m_i = m$ *and* $k_i = k$ *for* $i \in I$, *then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(nm-nk-n+1)/(n\omega_i)}\right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq \frac{\sqrt[n]{x_1 \cdots x_n}}{\left((m-k-1)!\right)^n (nm-nk-n+1)}$$

$$\times \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i) \left| u_i^{(m)}(s_i) \right|^{\frac{n}{n-1}} ds_i\right]^{\frac{n-1}{n}}.$$

*For* $q = p = n = 2$ *and* $\omega_i = \frac{1}{n}$ *this is [12, Theorem 1]. Setting* $q = p = 2$, $k = 0$, $n = 1$ *and* $\omega_i = \frac{1}{n}$, *we recover the result of [14].*

**Example 3.3.** *If* $a_i = 0$ *and* $b_i = 1$ *for* $i \in I$, *then Theorem 3.1 becomes*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\left[\frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(m_i-k_i)/(q_i\omega_i)}\right]^{\Omega_n}} ds_1 \cdots ds_n$$

$$\leq V_3 \prod_{i=1}^n x_i^{\frac{1}{q_i}} \prod_{i=1}^n \left[\int_0^{x_i} (x_i - s_i)^{m_i-k_i} \left| u_i^{(m_i)}(s_i) \right|^{p_i} ds_i\right]^{\frac{1}{p_i}},$$

*where*

$$V_3 = \frac{1}{\prod_{i=1}^n \left[(m_i - k_i)!\right]}.$$

**Example 3.4.** *If $a_i = 1$, $b_i = 0$, $q_i = n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (3.1) becomes*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \left| u_i^{(k_i)}(s_i) \right|}{\left[ \frac{1}{\Omega_n} \sum_{i=1}^n \omega_i s_i^{(m-k)/(n\omega_i)} \right]^{\Omega_n}} ds_1 \cdots ds_n$$
$$\le \frac{\sqrt[n]{x_1 \cdots x_n}}{[(m-k)!]^n} \prod_{i=1}^n \left[ \int_0^{x_i} (x_i - s_i)^{m-k} \left| u_i^{(m)}(s_i) \right|^{n/(n-1)} ds_i \right]^{\frac{(n-1)}{n}}.$$

**Example 3.5.** *Let $p_1, p_2 \in \mathbb{R}_+$. If we set $n = 2$, $\omega_1 = \frac{1}{p_1}$, $\omega_2 = \frac{1}{p_2}$, $m_i = 1$ and $k_i = 0$ for $i = 1, 2$ in Theorem 3.1, then by our assumptions $q_1 = p_1/(p_1 - 1)$, $q_2 = p_2/(p_2 - 1)$, and we obtain*

$$\int_0^{x_1} \int_0^{x_2} \frac{|u_1(s_1)|\,|u_2(s_2)|}{\left[ \frac{1}{p_1 p_2 \Omega_2} \left( p_2 s_1^{(p_1-1)} + p_1 s_2^{(p_2-1)} \right) \right]^{\Omega_2}} ds_1\, ds_2$$
$$\le x_1^{(p_1-1)/p_1} x_2^{(p_2-1)/p_2} \left( \int_0^{x_1} (x_1 - s_1) \left| u_1'(s_1) \right|^{p_1} ds_1 \right)^{\frac{1}{p_1}}$$
$$\times \left( \int_0^{x_2} (x_2 - s_2) \left| u_2'(s_2) \right|^{p_2} ds_2 \right)^{\frac{1}{p_2}}.$$

*If we set $\omega_1 + \omega_2 = 1$ in Example 3.5, then we have [13, Theorem 2]. (The values of $a_i$ and $b_i$ are irrelevant.)*

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