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APPROXIMATION OF THE DILOGARITHM FUNCTION

MEHDI HASSANI
INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES
P.O. BOX 45195-1159
ZANJAN, IRAN.
mmhassany@srttu.edu

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Dedicated to Professor Yousef Sobouti on the occasion of his 75th birthday.

ABSTRACT. In this short note, we approximate Dilogarithm function, defined by $\operatorname{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt$. Letting

$$\mathcal{D}(x, N) = -\frac{1}{2}\log^2 x - \frac{\pi^2}{6} + \sum_{n=1}^{N} \frac{\frac{1}{n^2} + \frac{1}{n}\log x}{x^n},$$

we show that for every $x > 1$, the inequalities

$$\mathcal{D}(x, N) < \operatorname{dilog}(x) < \mathcal{D}(x, N) + \frac{1}{x^N}$$

hold true for all $N \in \mathbb{N}$.

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Definition. The Dilogarithm function $\operatorname{dilog}(x)$ is defined for every $x > 0$ as follows [5]:

$$\operatorname{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt.$$

Expansion. The following expansion holds true when x tends to infinity:

$$\operatorname{dilog}(x) = \mathcal{D}(x, N) + O\left(\frac{1}{x^{N+1}}\right),$$

where

$$\mathcal{D}(x, N) = -\frac{1}{2}\log^2 x - \frac{\pi^2}{6} + \sum_{n=1}^{N} \frac{\frac{1}{n^2} + \frac{1}{n}\log x}{x^n}.$$

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Aim of Present Work. The aim of this note is to prove that:

$$0 < \operatorname{dilog}(x) - \mathcal{D}(x,N) < \frac{1}{x^N} \qquad (x > 1,\ N \in \mathbb{N}).$$

Lower Bound. For every $x > 0$ and $N \in \mathbb{N}$, let:

$$\mathcal{L}(x,N) = \operatorname{dilog}(x) - \mathcal{D}(x,N).$$

A simple computation, yields that:

$$\frac{d}{dx}\mathcal{L}(x,N) = \log x \left(\frac{x}{1-x} + \sum_{n=0}^{N+1} \frac{1}{x^n} \right) < \log x \left(\frac{x}{1-x} + \sum_{n=0}^{\infty} \frac{1}{x^n} \right) = 0.$$

So, $\mathcal{L}(x,N)$ is a strictly decreasing function of the variable x, for every $N \in \mathbb{N}$. Considering $\mathcal{L}(x,N) = O\left(\frac{1}{x^{N+1}}\right)$, we obtain a desired lower bound for the Dilogarithm function, as follows:

$$\mathcal{L}(x,N) > \lim_{x\to+\infty} \mathcal{L}(x,N) = 0.$$

Upper Bound. For every $x > 0$ and $N \in \mathbb{N}$, let:

$$\mathcal{U}(x,N) = \operatorname{dilog}(x) - \mathcal{D}(x,N) - \frac{1}{x^N}.$$

First, we observe that

$$\mathcal{U}(1,N) = \frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2} - 1 = \Psi(1, N+1) - 1 \le \frac{\pi^2}{6} - 2 < 0,$$

in which $\Psi(m,x)$ is the m-th polygamma function, the m-th derivative of the digamma function, $\Psi(x) = \frac{d}{dx}\log\Gamma(x)$, with $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ (see [1, 2]). A simple computation, yields that:

$$\frac{d}{dx}\mathcal{U}(x,N) = \log x \left(\frac{x}{1-x} + \sum_{n=0}^{N+1} \frac{1}{x^n} \right) + \frac{N}{x^{N+1}}.$$

To determine the sign of $\frac{d}{dx}\mathcal{U}(x,N)$, we distinguish two cases:

(1) Suppose $x > 1$. Since, $\frac{\log x}{x-1}$ is strictly decreasing, we have

$$N \ge 1 = \lim_{x\to1} \frac{\log x}{x-1} > \frac{\log x}{x-1},$$

which is $\frac{N}{\log x} > \frac{1}{x-1}$ or equivalently $\frac{N}{x^{N+1}\log x} > \sum_{n=N+2}^{\infty} \frac{1}{x^n}$, and this yields that $\frac{d}{dx}\mathcal{U}(x,N) > 0$. So, $\mathcal{U}(x,N)$ is strictly increasing for every $N \in \mathbb{N}$. Thus, $\mathcal{U}(x,N) < \lim_{x\to+\infty}\mathcal{U}(x,N) = 0$; as desired in this case. Also, note that in this case we obtain

$$\mathcal{U}(x,N) > \mathcal{U}(1,N) = \Psi(1,N+1) - 1.$$

(2) Suppose $0 < x < 1$ and $N - \frac{\log x}{x-1} \ge 0$. We observe that $1 < \frac{\log x}{x-1} < +\infty$ and $\sum_{n=0}^{N+1} \frac{1}{x^n} = \frac{1-x^{N+2}}{x^{N+1}(1-x)}$. Considering these facts, we see that $\frac{d}{dx}\mathcal{U}(x,N)$ and $N - \frac{\log x}{x-1}$ have same sign; i.e.

$$\operatorname{sgn}\left(\frac{d}{dx}\mathcal{U}(x,N)\right) = \operatorname{sgn}\left(N - \frac{\log x}{x-1}\right).$$

Thus, $\mathcal{U}(x,N)$ is increasing and so,

$$\mathcal{U}(x,N) \leq \lim_{x\to 1^-} \mathcal{U}(x,N) = \Psi(1,N+1) - 1 \leq \frac{\pi^2}{6} - 2 < 0.$$

Connection with Other Functions. Using Maple, we have:

$$\mathcal{D}(x,N) = -\frac{1}{2}\log^2 x - \frac{\pi^2}{6} + \frac{1}{N^2x^N} + \frac{\log x}{Nx^N} - \log\left(\frac{x-1}{x}\right)\log x + \operatorname{polylog}\left(2,\frac{1}{x}\right) - \frac{\log x}{x^N}\Phi\left(\frac{1}{x},1,N\right) - \frac{1}{x^N}\Phi\left(\frac{1}{x},2,N\right),$$

in which

$$\operatorname{polylog}(a,z) = \sum_{n=1}^{\infty}\frac{z^n}{n^a},$$

is the polylogarithm function of index a at the point z and defined by the above series if $|z| < 1$, and by analytic continuation otherwise [4]. Also,

$$\Phi(z,a,v) = \sum_{n=1}^{\infty}\frac{z^n}{(v+n)^a},$$

is the Lerch zeta (or Lerch-Φ) function defined by the above series for $|z| < 1$, with $v \neq 0,-1,-2,\dots$, and by analytic continuation, it is extended to the whole complex z-plane for each value of a and v (see [3, 6]).

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