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SOME SUBORDINATION RESULTS ASSOCIATED WITH CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

H.M. SRIVASTAVA AND A.A. ATTIYA

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA V8W 3P4
CANADA
harimsri@math.uvic.ca

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF MANSOURA
MANSOURA 35516, EGYPT
aattiy@mans.edu.eg

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ABSTRACT. For functions belonging to each of the subclasses $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ of normalized analytic functions in the open unit disk $\mathbb{U}$, which are investigated in this paper when $\alpha > 1$, the authors derive several subordination results involving the Hadamard product (or convolution) of the associated functions. A number of interesting consequences of some of these subordination results are also discussed.

Key words and phrases: Analytic functions, Univalent functions, Convex functions, Subordination principle, Hadamard product (or convolution), Subordinating factor sequence.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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We denote by $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ two interesting subclasses of the class $\mathcal{A}$, which are defined (for $\alpha > 1$) as follows:

$$\mathcal{M}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \qquad (z \in \mathbb{U};\ \alpha > 1) \right\} \tag{1.2}$$

and

$$\mathcal{N}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \qquad (z \in \mathbb{U};\ \alpha > 1) \right\}. \tag{1.3}$$

The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced and studied by Owa *et al*. ([1] and [2]). In fact, for $1 < \alpha \leqq \frac{4}{3}$, these classes were investigated earlier by Uralegaddi *et al*. (*cf*. [5]; see also [3] and [4]).

It follows from the definitions (1.2) and (1.3) that

$$f(z) \in \mathcal{N}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{M}(\alpha). \tag{1.4}$$

We begin by recalling each of the following coefficient inequalities associated with the function classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$.

Theorem A (Nishiwaki and Owa [1, p. 2, Theorem 2.1])**.** *If $f \in \mathcal{A}$, given by (1.1), satisfies the coefficient inequality*:

$$\sum_{n=2}^{\infty} [(n-\lambda) + |n+\lambda-2\alpha|]\,|a_n| \leqq 2(\alpha-1) \tag{1.5}$$
$$(\alpha > 1;\ 0 \leqq \lambda \leqq 1),$$

then $f \in \mathcal{M}(\alpha)$.

Theorem B (Nishiwaki and Owa [1, p. 3, Theorem 2.3])**.** *If $f \in \mathcal{A}$, given by (1.1), satisfies the coefficient inequality*:

$$\sum_{n=2}^{\infty} n\,[(n-\lambda) + |n+\lambda-2\alpha|]\,|a_n| \leqq 2(\alpha-1) \tag{1.6}$$
$$(\alpha > 1;\ 0 \leqq \lambda \leqq 1),$$

then $f \in \mathcal{N}(\alpha)$.

In view of Theorem A and Theorem B, we now introduce the subclasses

$$\mathcal{M}^*(\alpha) \subset \mathcal{M}(\alpha) \qquad \text{and} \qquad \mathcal{N}^*(\alpha) \subset \mathcal{N}(\alpha) \qquad (\alpha > 1), \tag{1.7}$$

which consist of functions $f \in \mathcal{A}$ whose Taylor-Maclaurin coefficients a_n satisfy the inequalities (1.5) and (1.6), respectively. In our proposed investigation of functions in the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$, we shall also make use of the following definitions and results.

Defintition 1 (Hadamard Product or Convolution)**.** *Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product $($or convolution$)$ $f * g$ is defined $($as usual$)$ by

$$(f*g)(z) := z + \sum_{n=2}^{\infty} a_n\, b_n\, z^n =: (g*f)(z).$$

Defintition 2 (Subordination Principle)**.** *For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write*

$$f \prec g \qquad \textit{or} \qquad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \textit{and} \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \textit{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Defintition 3 (Subordinating Factor Sequence)**.** *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n\, b_n\, z^n \prec f(z) \qquad (z \in \mathbb{U};\ a_1 := 1). \tag{1.8}$$

Theorem C (*cf.* Wilf [6])**.** *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{n=1}^{\infty} b_n\, z^n\right) > 0 \qquad (z \in \mathbb{U}). \tag{1.9}$$

2. SUBORDINATION RESULTS FOR THE CLASSES $\mathcal{M}^*(\alpha)$ AND $\mathcal{M}(\alpha)$

Our first main result (Theorem 1 below) provides a sharp subordination result involving the function class $\mathcal{M}^*(\alpha)$.

Theorem 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{M}^*(\alpha)$. Also let $\mathcal{K}$ denote the familiar class of functions $f \in \mathcal{A}$ which are also univalent and convex in $\mathbb{U}$. Then*

$$\frac{(2-\lambda) + |2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda) + |2+\lambda-2\alpha|\right]}\,(f * g)(z) \prec g(z) \tag{2.1}$$
$$(z \in \mathbb{U};\ 0 \leqq \lambda \leqq 1;\ \alpha > 1;\ g \in \mathcal{K})$$

and

$$\Re\big(f(z)\big) > -\frac{(2\alpha-\lambda) + |2+\lambda-2\alpha|}{(2-\lambda) + |2+\lambda-2\alpha|} \qquad (z \in \mathbb{U}). \tag{2.2}$$

The following constant factor in the subordination result (2.1):

$$\frac{(2-\lambda) + |2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda) + |2+\lambda-2\alpha|\right]}$$

cannot be replaced by a larger one.

Proof. Let $f(z) \in \mathcal{M}^*(\alpha)$ and suppose that

$$g(z) = z + \sum_{n=2}^{\infty} c_n\, z^n \in \mathcal{K}.$$

Then we readily have

$$\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}\,(f*g)(z) = \frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}\left(z+\sum_{n=2}^{\infty} c_n\, a_n\, z^n\right). \tag{2.3}$$

Thus, by Definition 3, the subordination result (2.1) will hold true if

$$\left\{\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}\,a_n\right\}_{n=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence (with, of course, $a_1 = 1$). In view of Theorem C, this is equivalent to the following inequality:

$$\Re\left(1+\sum_{n=1}^{\infty}\frac{(2-\lambda)+|2+\lambda-2\alpha|}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\,a_n\, z^n\right)>0 \qquad (z\in\mathbb{U})\,. \tag{2.5}$$

Now, since

$$(n-\lambda)+|n+\lambda-2\alpha| \qquad (0\leqq\lambda\leqq 1;\ \alpha>1)$$

is an *increasing* function of n, we have

$$\begin{aligned}
&\Re\left(1+\sum_{n=1}^{\infty}\frac{(2-\lambda)+|2+\lambda-2\alpha|}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\,a_n\, z^n\right)\\
&\quad=\Re\left(1+\frac{(2-\lambda)+|2+\lambda-2\alpha|}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\,z\right.\\
&\qquad\left.+\frac{1}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\sum_{n=2}^{\infty}\left[(2-\lambda)+|2+k-2\alpha|\right]a_n\, z^n\right)\\
&\quad\geqq 1-\frac{(2-\lambda)+|2+\lambda-2\alpha|}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\,r\\
&\qquad-\frac{1}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\sum_{n=2}^{\infty}\left[(n-\lambda)+|n+\lambda-2\alpha|\right]|a_n|\ r^n\\
&\quad>1-\frac{(2-\lambda)+|2+\lambda-2\alpha|}{(2\alpha-\lambda)+|2+\lambda-2\alpha|}\,r-\frac{2(\alpha-1)}{\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}\,r\\
&\quad>0 \qquad (|z|=r<1)\,,
\end{aligned} \tag{2.6}$$

where we have also made use of the assertion (1.5) of Theorem A. This evidently proves the inequality (2.5), and hence also the subordination result (2.1) asserted by Theorem 1.

The inequality (2.2) follows from (2.1) upon setting

$$g\,(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in\mathcal{K}. \tag{2.7}$$

Next we consider the function:

$$q(z):=z-\frac{2(\alpha-1)}{(2-\lambda)+|2+\lambda-2\alpha|}\,z^2 \qquad (0\leqq\lambda\leqq 1;\ \alpha>1)\,, \tag{2.8}$$

which is a member of the class $\mathcal{M}^*(\alpha)$. Then, by using (2.1), we have

$$\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}\,q(z) \prec \frac{z}{1-z} \qquad (z\in\mathbb{U})\,. \tag{2.9}$$

It is also easily verified for the function $q\,(z)$ defined by (2.7) that

$$\min\left\{\Re\left(\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}\,q(z)\right)\right\} = -\frac{1}{2} \qquad (z\in\mathbb{U})\,, \tag{2.10}$$

which completes the proof of Theorem 1. □

Corollary 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{M}(\alpha)$*. Then the assertions* (2.1) *and* (2.2) *of Theorem* 1 *hold true. Furthermore, the following constant factor*:

$$\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(2\alpha-\lambda)+|2+\lambda-2\alpha|\right]}$$

cannot be replaced by a larger one.

By taking $\lambda = 1$ and $1 < \alpha \leqq \dfrac{3}{2}$ in Corollary 1, we obtain

Corollary 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{M}(\alpha)$*. Then*

$$\left(1-\frac{1}{2}\alpha\right)(f*g)(z) \prec g(z) \tag{2.11}$$

$$\left(z\in\mathbb{U};\ 1<\alpha\leqq\frac{3}{2};\ g\in\mathcal{K}\right)$$

and

$$\Re\big(f(z)\big) > -\frac{1}{2-\alpha} \qquad (z\in\mathbb{U})\,. \tag{2.12}$$

The constant factor $1-\frac{1}{2}\alpha$ *in the subordination result* (2.11) *cannot be replaced by a larger one.*

3. SUBORDINATION RESULTS FOR THE CLASSES $\mathcal{N}^*(\alpha)$ AND $\mathcal{N}(\alpha)$

Our proof of Theorem 2 below is much akin to that of Theorem 1. Here we make use of Theorem B in place of Theorem A.

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{N}^*(\alpha)$*. Then*

$$\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(\alpha+1-\lambda)+|2+\lambda-2\alpha|\right]}\,(f*g)(z) \prec g(z) \tag{3.1}$$

$$(z\in\mathbb{U};\ 0\leqq\lambda\leqq 1;\ \alpha>1;\ g\in\mathcal{K})$$

and

$$\Re\big(f(z)\big) > -\frac{(\alpha+1-\lambda)+|2+\lambda-2\alpha|}{(2-\lambda)+|2+\lambda-2\alpha|} \qquad (z\in\mathbb{U})\,. \tag{3.2}$$

The following constant factor in the subordination result (3.1):

$$\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(\alpha+1-\lambda)+|2+\lambda-2\alpha|\right]}$$

cannot be replaced by a larger one.

Corollary 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{N}(\alpha)$. *Then the assertions* (3.1) *and* (3.2) *of Theorem* 2 *hold true. Furthermore, the following constant factor*:

$$\frac{(2-\lambda)+|2+\lambda-2\alpha|}{2\left[(\alpha+1-\lambda)+|2+\lambda-2\alpha|\right]}$$

cannot be replaced by a larger one.

By letting $\lambda = 1$ and $1 < \alpha \leqq \frac{3}{2}$ in Corollary 3, we obtain the following *further* consequence of Theorem 2.

Corollary 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{N}(\alpha)$. *Then*

$$\frac{2-\alpha}{2(3-\alpha)}\;(f*g)(z)\;\prec g(z) \qquad \left(z \in \mathbb{U};\ 1 < \alpha \leqq \frac{3}{2};\ g \in \mathcal{K}\right). \tag{3.3}$$

and

$$\Re\big(f(z)\big) > -\,\frac{3-\alpha}{2-\alpha} \qquad (z \in \mathbb{U})\,. \tag{3.4}$$

The following constant factor in the subordination result (3.3):

$$\frac{2-\alpha}{2\,(3-\alpha)}$$

cannot be replaced by a larger one.

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