Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 4, Article 92, 2005

ON HARDY-HILBERT INTEGRAL INEQUALITIES WITH SOME PARAMETERS

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Received 07 April, 2005; accepted 06 June, 2005
Communicated by B. Yang

ABSTRACT. In this paper, we give a new Hardy-Hilbert's integral inequality with some parameters and a best constant factor. It includes an overwhelming majority of results of many papers.

Key words and phrases: Hardy-Hilbert's integral inequality, Weight, Parameter, Best constant factor, β-function, Γ-function.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION AND MAIN RESULT

Let $\frac{1}{p}+\frac{1}{q}=1(p>1)$, $f\geq 0$, $g\geq 0$, $0<\int_0^\infty f^p(x)dx<+\infty$, $0<\int_0^\infty g^q(x)dx<+\infty$, then we have the well known Hardy-Hilbert inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{x+y}dxdy<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}; \tag{1.1}$$

and an equivalent form as:

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy<\left[\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right]^p\int_0^\infty f^p(x)dx. \tag{1.2}$$

In recent years, many results have been obtained in the research of these two inequalities (see [1] – [13]). Yang [1] and [2] gave:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy<B\left(\frac{p+\lambda-2}{p},\frac{p+\lambda-2}{q}\right) \times\left(\int_0^\infty x^{1-\lambda}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda}g^q(x)dx\right)^{\frac{1}{q}}, \tag{1.3}$$

ISSN (electronic): 1443-5756

109-05

where $B(r,s)$ is the β-function; and Kuang [3] gave:

$$\text{(1.4)} \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin^{\frac{1}{p}}\left(\frac{\pi}{p\lambda}\right) \sin^{\frac{1}{q}}\left(\frac{\pi}{q\lambda}\right)} \left(\int_0^\infty x^{1-\lambda} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{1-\lambda} g^q(x)dx\right)^{\frac{1}{q}}.$$

Recently, Hong [4] gave:

$$\text{(1.5)} \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{\sqrt{x^2+y^2}} dxdy \le \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2q}\right) \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}.$$

And Yang [5] gave:

$$\text{(1.6)} \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty x^{(q-1)(1-\lambda)} g^q(x)dx\right)^{\frac{1}{q}};$$

$$\text{(1.7)} \quad \int_0^\infty y^{\lambda-1} \left(\int_0^\infty \frac{f(x)}{x^\lambda + y^\lambda} dx\right)^p dy < \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)}\right]^p \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)dx.$$

These results generalize and improve (1.1) and (1.2) in a certain degree.

In this paper, by introducing a few parameters, we obtain a new Hardy-Hilbert integral inequality with a best constant factor, which is a more extended inequality, and includes all the results above and the overwhelming majority of results of many recent papers.

Our main result is as follows:

Theorem 1.1. *If* $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$, $\alpha > 0$, $\lambda > 0$, $m, n \in \mathbb{R}$, *such that* $0 < 1 - mp < \alpha\lambda$, $0 < 1 - nq < \alpha\lambda$, *and* $f \ge 0$, $g \ge 0$, *satisfy*

$$\text{(1.8)} \quad 0 < \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x)dx < \infty,$$

$$\text{(1.9)} \quad 0 < \int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y)dy < \infty,$$

then

$$\text{(1.10)} \quad \int_0^\infty \int_0^\infty \frac{f(x)g(x)}{(x^\alpha + y^\alpha)^\lambda} dxdy < H_{\lambda,\alpha}(m,n,p,q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x)dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y)dy\right)^{\frac{1}{q}};$$

and

$$\int_0^\infty y^{\frac{(1-\alpha\lambda)+q(m-n)}{1-q}}\left[\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^\lambda}dx\right]^p dy < \widetilde{H}_{\lambda,\alpha}(m,n,p,q)\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)}f^p(x)dx, \tag{1.11}$$

where

$$H_{\lambda,\alpha}(m,n,p,q)=\frac{1}{\alpha}B^{\frac{1}{p}}\left(\frac{1-mp}{\alpha},\lambda-\frac{1-mp}{\alpha}\right)B^{\frac{1}{q}}\left(\frac{1-nq}{\alpha},\lambda-\frac{1-nq}{\alpha}\right)$$

and

$$\widetilde{H}_{\lambda,\alpha}(m,n,p,q)=\frac{1}{\alpha^p}B\left(\frac{1-mp}{\alpha},\lambda-\frac{1-mp}{\alpha}\right)B^{p-1}\left(\frac{1-nq}{\alpha},\lambda-\frac{1-nq}{\alpha}\right).$$

Theorem 1.2. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$, $\lambda>0$, $m,n\in\mathbb{R}$, *such that* $0<1-mp<\alpha\lambda$, $mp+nq=2-\alpha\lambda$, *and* $f(x)\geq 0$, $g(y)\geq 0$, *satisfy*

$$0<\int_0^\infty x^{n(p+q)-1}f^p(x)dx<\infty, \tag{1.12}$$

$$0<\int_0^\infty y^{m(p+q)-1}g^q(y)dy<\infty, \tag{1.13}$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{(x^\alpha+y^\alpha)^\lambda}dxdy < \frac{1}{\alpha}B\left(\frac{1-mp}{\alpha},\lambda-\frac{1-mp}{\alpha}\right) \times\left(\int_0^\infty x^{n(p+q)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{m(p+q)-1}g^q(y)dy\right)^{\frac{1}{q}}; \tag{1.14}$$

$$\int_0^\infty y^{\frac{m(p+q)-1}{1-q}}\left[\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^\lambda}dx\right]^p dy < \frac{1}{\alpha^p}B^p\left(\frac{1-mp}{\alpha},\lambda-\frac{1-mp}{\alpha}\right)\int_0^\infty x^{n(p+q)-1}f^p(x)dx, \tag{1.15}$$

where the constant factors $\frac{1}{\alpha}B\left(\frac{1-mp}{\alpha},\lambda-\frac{1-mp}{\alpha}\right)$ *in (1.14) and* $\frac{1}{\alpha^p}B^p\left(\frac{1-mp}{\alpha},\lambda-\frac{1-mp}{\alpha}\right)$ *in (1.15) are the best possible.*

2. WEIGHT FUNCTION AND LEMMAS

The weight function is defined as follows

$$\omega_{\lambda,\alpha}(m,n,y)=\int_0^\infty \frac{1}{(x^\alpha+y^\alpha)^\lambda}\cdot\frac{y^n}{x^m}dx,\quad y\in(0,+\infty).$$

Lemma 2.1. *If* $\alpha>0$, $\lambda>0$, $m\in\mathbb{R}$, $0<1-m<\alpha\lambda$, *then*

$$\omega_{\lambda,\alpha}(m,n,y)=\frac{1}{\alpha}y^{(1-\alpha\lambda)+(n-m)}B\left(\frac{1-m}{\alpha},\lambda-\frac{1-m}{\alpha}\right). \tag{2.1}$$

Proof. Setting $t = \frac{x^\alpha}{y^\alpha}$, then

$$\begin{aligned}\omega_{\lambda,\alpha}(m,n,y) &= \frac{1}{\alpha}\int_0^\infty \frac{1}{(1+t)^\lambda} y^{(1-\alpha\lambda)+(n-m)} t^{\frac{1-m}{\alpha}-1} dt \\ &= \frac{1}{\alpha} y^{(1-\alpha\lambda)+(n-m)} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-m}{\alpha}-1} dt \\ &= \frac{1}{\alpha} y^{(1-\alpha\lambda)+(n-m)} B\left(\frac{1-m}{\alpha}, \lambda - \frac{1-m}{\alpha}\right).\end{aligned}$$

Hence (2.1) is valid. The lemma is proved. □

Lemma 2.2. *If* $\alpha > 0$, $\lambda > 0$, $\beta < 1$, $a \in \mathbb{R}$, *then*

$$\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-\beta}{\alpha}-1-a\varepsilon} dt dx = O(1), \quad (\varepsilon \to 0^+). \tag{2.2}$$

Proof. Since $(1-\beta)/\alpha > 0$, for ε small enough, such that $\frac{1-\beta}{\alpha} - a\varepsilon > 0$, then

$$\begin{aligned}\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-\beta}{\alpha}-1-a\varepsilon} dt dx &< \int_1^\infty \frac{1}{x} \int_0^{\frac{1}{x^\alpha}} t^{\left(\frac{1-\beta}{\alpha}-a\varepsilon\right)-1} dt dx \\ &= \frac{1}{1-\beta-a\varepsilon\alpha} \int_1^\infty x^{\beta+a\varepsilon\alpha-2} dx \\ &= \frac{1}{(1-\beta-a\varepsilon\alpha)^2}.\end{aligned}$$

Hence (2.2) is valid. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}G &= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha+y^\alpha)^\lambda} dx dy \\ &= \int_0^\infty \int_0^\infty \left[\frac{f(x)}{(x^\alpha+y^\alpha)^{\lambda/p}} \frac{x^n}{y^m}\right] \left[\frac{g(y)}{(x^\alpha+y^\alpha)^{\lambda/q}} \frac{y^m}{x^n}\right] dx dy \\ &\le \left\{\int_0^\infty \int_0^\infty \frac{f^p(x)}{(x^\alpha+y^\alpha)^\lambda} \frac{x^{np}}{y^{mp}}\right\}^{\frac{1}{p}} \left\{\int_0^\infty \int_0^\infty \frac{g^q(x)}{(x^\alpha+y^\alpha)^\lambda} \frac{y^{mq}}{x^{nq}} dx dy\right\}^{\frac{1}{q}},\end{aligned} \tag{3.1}$$

according to the condition of taking equality in Hölder's inequality, if (3.1) takes equality, then there exists a constant C, such that

$$\left[\frac{f^p(x)}{(x^\alpha+y^\alpha)^\lambda} \frac{x^{np}}{y^{mp}}\right] \Big/ \left[\frac{g^q(y)}{(x^\alpha+y^\alpha)^\lambda} \frac{y^{mq}}{x^{nq}}\right] \equiv C, \quad \text{a.e. } (x,y) \in (0,+\infty)\times(0,+\infty)$$

it follows that

$$f^p(x) x^{n(p+q)} \equiv C g^q(y) y^{m(p+q)} \equiv C_1 \text{ (constant)}, \quad \text{a.e. } (x,y) \in (0,+\infty)\times(0,+\infty)$$

hence

$$\begin{aligned}\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x)dx &= \int_0^\infty x^{(1-\alpha\lambda)+n(p+q)-nq-mp} f^p(x)dx \\ &= C_1 \int_0^\infty x^{(1-\alpha\lambda)-np-mq} dx \\ &= C_1 \int_0^1 x^{(1-\alpha\lambda)-np-mq} dx + C_1 \int_1^\infty x^{(1-\alpha\lambda)-np-mq} dx \\ &= +\infty,\end{aligned}$$

which contradicts (1.8) and (1.9). Hence, by (3.1), we have

$$\begin{aligned} G &< \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} \frac{x^{np}}{y^{mp}} dy \right] f^p(x)dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty \left[\int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda} \frac{y^{mq}}{x^{nq}} dy \right] g^q(y)dy \right\}^{\frac{1}{q}} \\ &= \left\{ \frac{1}{\alpha} \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha} \right) f^p(x)dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \frac{1}{\alpha} \int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} B\left(\frac{1-nq}{\alpha}, \lambda - \frac{1-nq}{\alpha} \right) g^q(y)dy \right\}^{\frac{1}{q}} \\ &= H_{\lambda,\alpha}(m,n,p,q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} g^q(y)dy \right)^{\frac{1}{q}}. \end{aligned}$$

Hence (1.10) is vaild.

Let $\beta = [(1-\alpha\lambda) + q(m-n)]/(1-q)$, and

$$\widetilde{g}(y) = y^\beta \left(\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right)^{\frac{p}{q}}.$$

By (1.10), we have

$$\begin{aligned} &\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} \widetilde{g}^q(y)dy \\ &\quad = \int_0^\infty y^{\beta(1-q)} \widetilde{g}^q(y)dy \\ &\quad = \int_0^\infty y^\beta \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^p dy \\ &\quad = \int_0^\infty y^\beta \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right]^{\frac{p}{q}} \left[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \right] dy \\ &\quad = \int_0^\infty \int_0^\infty \frac{f(x)\widetilde{g}(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy \\ &\quad < H_{\lambda,\alpha}(m,n,p,q) \left(\int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x)dx \right)^{\frac{1}{p}} \\ &\qquad \times \left(\int_0^\infty y^{(1-\alpha\lambda)+q(m-n)} \widetilde{g}^q(y)dy| \right)^{\frac{1}{q}}, \end{aligned}$$

It follows that

$$\int_0^\infty y^{\frac{(1-\alpha\lambda)+q(m-n)}{1-q}} \left[\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^\lambda}dx\right]^p dy$$
$$< \widetilde{H}_{\lambda,\alpha}(m,n,p,q) \int_0^\infty x^{(1-\alpha\lambda)+p(n-m)} f^p(x)dx.$$

Hence, (1.11) is valid. □

Proof of Theorem 1.2. Since $0 < 1 - mp < \alpha\lambda$, $mp + nq = 2 - \alpha\lambda$, then

$$0 < 1 - nq < \alpha\lambda,$$
$$(1-\alpha\lambda) + p(n-m) = n(p+q) - 1,$$
$$(1-\alpha\lambda) + q(m-n) = m(p+q) - 1,$$
$$\frac{1-mp}{\alpha} = \lambda - \frac{1-nq}{\alpha}.$$

By Theorem 1.1, (1.14) and (1.15) are valid.

For $\varepsilon > 0$, setting

$$f_0(x) = \begin{cases} x^{[-n(p+q)-\varepsilon]/p}, & x \ge 1; \\ 0, & 0 \le x < 1, \end{cases}$$

and

$$g_0(y) = \begin{cases} y^{[-m(p+q)-\varepsilon]/q}, & y \ge 1; \\ 0, & 0 \le y < 1. \end{cases}$$

We have

$$0 < \int_0^\infty x^{n(p+q)-1} f_0^p(x)dx = \int_1^\infty x^{-1-\varepsilon}dx = \frac{1}{\varepsilon} < \infty, \tag{3.2}$$

$$0 < \int_0^\infty y^{m(p+q)-1} g_0^q(y)dy = \int_1^\infty y^{-1-\varepsilon}dy = \frac{1}{\varepsilon} < \infty. \tag{3.3}$$

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f_0(x)g_0(y)}{(x^\alpha+y^\alpha)^\lambda}dxdy \\
&= \int_1^\infty \int_1^\infty \frac{1}{(x^\alpha+y^\alpha)^\lambda} x^{-\frac{n(p+q)+\varepsilon}{p}} y^{-\frac{m(p+q)+\varepsilon}{q}} dxdy \\
&= \int_1^\infty x^{-\frac{n(p+q)+\varepsilon}{p}} \int_1^\infty \frac{1}{(x^\alpha+y^\alpha)^\lambda} y^{-\frac{m(p+q)+\varepsilon}{q}} dydx \\
&= \frac{1}{\alpha}\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_{\frac{1}{x^\alpha}}^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dtdx \\
&= \frac{1}{\alpha}\left[\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-mt}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dtdx \right. \\
&\qquad \left. - \int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-mt}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dtdx \right].
\end{aligned}$$

By Lemma 2.2, when $\varepsilon \to 0$, we have

$$\int_1^\infty \frac{1}{x^{1+\varepsilon}} \int_0^{\frac{1}{x^\alpha}} \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dtdx = O(1).$$

Since

$$\int_0^\infty \frac{1}{(1+t)^\lambda} t^{\frac{1-mp}{\alpha}-1-\frac{\varepsilon}{q\alpha}} dt = B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right) + o(1),$$

we have

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{f_0(x)g_0(y)}{(x^\alpha+y^\alpha)^\lambda} dxdy &= \frac{1}{\alpha}\left\{\frac{1}{\varepsilon}\left[B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right) + o(1)\right] - O(1)\right\} \\ &= \frac{1}{\varepsilon}\left[\frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right) - o(1)\right] \\ &= \frac{1}{\varepsilon\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)(1-o(1)). \end{aligned} \tag{3.4}$$

If the constant $\frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)$ in (1.14) is not the best possible, then there exists a $K < \frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)$, such that (1.14) still is valid when we replace $\frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)$ by K. By (3.2), (3.3) and (3.4), we find

$$\begin{aligned}&\frac{1}{\varepsilon\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)(1-o(1)) \\ &\qquad < K\left(\int_0^\infty x^{n(p+q)-1} f_0^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{m(p+q)-1} g_0^q(y)dy\right)^{\frac{1}{q}} = K\frac{1}{\varepsilon}.\end{aligned}$$

For $\varepsilon \to 0^+$, we have $\frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right) \le K$, which contradicts the fact that $K < \frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)$. It follows that $\frac{1}{\alpha}B\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)$ in (1.14) is the best possible.

Since (1.14) is equivalent to (1.15), then the constant $\frac{1}{\alpha^p}B^p\left(\frac{1-mp}{\alpha}, \lambda - \frac{1-mp}{\alpha}\right)$ in (1.15) is the best possible. The theorem is proved. □

4. SOME COROLLARIES

When we take the appropriate parameters, many new inequalities can be obtained as follows:

Corollary 4.1. *If* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\alpha>0$, $\lambda>0$, $f\ge 0$, $g \ge 0$, *and* $x^{(1-\alpha\lambda)(p-1)/p}f(x) \in L^p(0,+\infty)$, $x^{(1-\alpha\lambda)(q-1)/q}g(x) \in L^q(0,+\infty)$, *then*

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha+y^\alpha)^\lambda} dxdy \\ &\quad < \frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\alpha\Gamma(\lambda)}\left(\int_0^\infty x^{(1-\alpha\lambda)(p-1)} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{(1-\alpha\lambda)(q-1)} g^q(x)dx\right)^{\frac{1}{q}};\end{aligned} \tag{4.1}$$

$$\int_0^\infty y^{\alpha-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^\lambda}dx\right)^p dy < \left[\frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\alpha\Gamma(\lambda)}\right]^p \int_0^\infty x^{(1-\alpha\lambda)(p-1)} f^p(x)dx, \tag{4.2}$$

where the constants $\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)\Big/(\alpha\Gamma(\lambda))$ *in (4.1) and* $\left[\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)\Big/(\alpha\Gamma(\lambda))\right]^p$ *in (4.2) are the best possible.*

Proof. If we take $m = \frac{1}{p} - \frac{\alpha\lambda}{p^2}$, $n = \frac{1}{q} - \frac{\alpha\lambda}{q^2}$ in Theorem 1.2, (4.1) and (4.2) can be obtained. □

Corollary 4.2. *If* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\lambda>0$, $f\geq 0$, $g\geq 0$ *and* $x^{(1-\lambda)(p-1)/p}f(x)\in L^p(0,+\infty)$, $x^{(1-\lambda)(q-1)/q}g(x)\in L^q(0,+\infty)$*, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < \frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}\left(\int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{(1-\lambda)(q-1)}g^q(x)dx\right)^{\frac{1}{q}}, \tag{4.3}$$

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy < \left[\frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}\right]^p\int_0^\infty x^{(1-\lambda)(p-1)}f^p(x)dx, \tag{4.4}$$

where $\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)\Big/\Gamma(\lambda)$ *in (4.3) and* $\left[\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)\Big/\Gamma(\lambda)\right]^p$ *in (4.4) are the best possible.*

Proof. If we take $\alpha=1$ in Corollary 4.1, (4.3) and (4.4) can be obtained. ☐

Corollary 4.3. *If* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\lambda>0$, $p+\lambda-2>0$, $q+\lambda-2>0$, $f\geq 0, g\geq 0$*, and* $x^{(1-\lambda)/p}f(x)\in L^p(0,+\infty)$, $x^{(1-\lambda)/q}g(x)\in L^q(0,+\infty)$*, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left(\int_0^\infty x^{1-\lambda}f^p(x)\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda}g^q(x)dx\right)^{\frac{1}{q}}, \tag{4.5}$$

$$\int_0^\infty y^{\frac{1-\lambda}{1-q}}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy < B^p\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\int_0^\infty x^{1-\lambda}f^p(x)dx, \tag{4.6}$$

where $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *in (4.5) and* $B^p\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *in (4.6) are the best possible.*

Proof. If we take $\alpha=1$, $m=n=\frac{2-\lambda}{pq}$ in Theorem 1.2, (4.5) and (4.6) can be obtained. ☐

Corollary 4.4. *If* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\alpha>0$, $f\geq 0$, $g\geq 0$*, and* $x^{(1-\alpha)/p}f(x)\in L^p(0,+\infty)$, $x^{(1-\alpha)/q}g(x)\in L^q(0,+\infty)$*, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{x^\alpha+y^\alpha}dxdy < \frac{\pi}{\alpha\sin^{\frac{1}{p}}\left(\frac{\pi}{p\alpha}\right)\sin^{\frac{1}{q}}\left(\frac{\pi}{q\alpha}\right)}\left(\int_0^\infty x^{1-\alpha}f^p(x)dx\right)^p\left(\int_0^\infty x^{1-\alpha}g^q(x)dx\right)^q, \tag{4.7}$$

$$\int_0^\infty y^{\frac{1-\alpha}{1-q}}\left(\int_0^\infty \frac{f(x)}{x^\alpha+y^\alpha}dx\right)^p dy < \left(\frac{\pi}{\alpha\sin^{\frac{1}{p}}\left(\frac{\pi}{p\alpha}\right)\sin^{\frac{1}{q}}\left(\frac{\pi}{q\alpha}\right)}\right)^p\int_0^\infty x^{1-\alpha}f^p(x)dx. \tag{4.8}$$

Proof. If we take $\lambda=1$, $m=n=\frac{1}{pq}$, in Theorem 1.1, (4.7) and (4.8) can be obtained. ☐

Corollary 4.5. *If* $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\alpha>0$, $f\geq 0$, $g\geq 0$, *and* $f(x)\in L^p(0,+\infty)$, $g(x)\in L^q(0,+\infty)$, *then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{(x^\alpha+y^\alpha)^{\frac{1}{\alpha}}}dxdy < \frac{\Gamma\left(\frac{1}{p\alpha}\right)\Gamma\left(\frac{1}{q\alpha}\right)}{\alpha\Gamma\left(\frac{1}{\alpha}\right)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \tag{4.9}$$

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^{\frac{1}{\alpha}}}dx\right)^p dy < \left(\frac{\Gamma\left(\frac{1}{p\alpha}\right)\Gamma\left(\frac{1}{q\alpha}\right)}{\alpha\Gamma\left(\frac{1}{\alpha}\right)}\right)^p\int_0^\infty f^p(x)dx, \tag{4.10}$$

where $\Gamma\left(\frac{1}{p\alpha}\right)\Gamma\left(\frac{1}{q\alpha}\right)\Big/\left(\alpha\Gamma\left(\frac{1}{\alpha}\right)\right)$ *in (4.9) and* $\left[\Gamma\left(\frac{1}{p\alpha}\right)\Gamma\left(\frac{1}{q\alpha}\right)\Big/\left(\alpha\Gamma\left(\frac{1}{\alpha}\right)\right)\right]^p$ *in (4.10) are the best possible.*

Proof. If we take $\lambda=\frac{1}{\alpha}$, $m=n=\frac{1}{pq}$ in Theorem 1.2, (4.9) and (4.10) can be obtained. □

Remark 4.6. (4.1) and (4.2) are respectively generalizations of (1.6) and (1.7). For $\alpha=1$ in (4.1) and (4.2), (1.6) and (1.7) can be obtained.

Remark 4.7. (4.3) and (4.4) are respectively generalizations of (1.1) and (1.2) .

Remark 4.8. (4.5) is the result of [1] and [2]. (4.6) is a new inequality.

Remark 4.9. (4.7) is the result of [3]. (1.7) is a new inequality.

Remark 4.10. (4.9) is a generalization of (1.5). (4.10) is a new inequality.

For other appropriate values of parameters taken in Theorems 1.1 and 1.2, many new inequalities and the inequalities of [6] – [13] can yet be obtained.

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