INEQUALITIES ON THE LAMBERT W FUNCTION AND HYPERPOWER FUNCTION

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- Received:09 April, 2007Accepted:15 March, 2008Communicated by:S.S. Dragomir
- 2000 AMS Sub. Class.: 33E20, 26D07.

Abstract:

Key words: Lambert W function, hyperpower function, special function, inequality.

In this note, we obtain inequalities for the Lambert W function W(x), defined by $W(x)e^{W(x)} = x$ for $x \ge -e^{-1}$. Also, we get upper and lower bounds for the hyperpower function $h(x) = x^{x^{x^{-1}}}$.



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journal of inequalities in pure and applied mathematics issn: 1443-5756

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1. Introduction

The Lambert W function W(x), is defined by $W(x)e^{W(x)} = x$ for $x \ge -e^{-1}$. For $-e^{-1} \le x < 0$, there are two possible values of W(x), which we take values not less than -1. The history of the function goes back to J. H. Lambert (1728-1777). One can find in [2] a more detailed definition of W as a complex variable function, some historical background and various applications of it in Mathematics and Physics. The expansion

$$W(x) = \log x - \log \log x + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \frac{(\log \log x)^m}{(\log x)^{k+m}},$$

holds true for large values of x, with $c_{km} = \frac{(-1)^k}{m!}S[k+m, k+1]$, where S[k+m, k+1] is Stirling cycle number [2]. The series in the above expansion is absolutely convergent and it can be rearranged into the form

$$W(x) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} + O\left(\left(\frac{L_2}{L_1}\right)^4\right),$$

where $L_1 = \log x$ and $L_2 = \log \log x$. Note that by log we mean logarithm in the base e. Since the Lambert W function appears in some problems in Mathematics, Physics and Engineering, it is very useful to have some explicit bounds for it. In [5] it is shown that

(1.1)
$$\log x - \log \log x < W(x) < \log x$$

where the left hand side holds true for x > 41.19 and the right hand side holds true for x > e. The aim of the present paper is to obtain some sharper bounds.



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2. Some Sharp Bounds for the Lambert W Function

It is easy to see that $W(-e^{-1}) = -1$, W(0) = 0 and W(e) = 1. Also, for x > 0, since $W(x)e^{W(x)} = x > 0$ and $e^{W(x)} > 0$, we have W(x) > 0. An easy calculation yields that

$$\frac{d}{dx}W(x) = \frac{W(x)}{x(1+W(x))}$$

Thus, $x \frac{d}{dx} W(x) > 0$ holds true for x > 0 and consequently W(x) is strictly increasing for x > 0 (and also for $-e^{-1} \le x \le 0$, but not for this reason).

Theorem 2.1. For every $x \ge e$, we have

(2.1)
$$\log x - \log \log x \le W(x) \le \log x - \frac{1}{2} \log \log x,$$

with equality holding only for x = e. The coefficients -1 and $-\frac{1}{2}$ of $\log \log x$ both are best possible for the range $x \ge e$.

Proof. For the given constant 0 consider the function

$$f(x) = \log x - \frac{1}{p}\log\log x - W(x)$$

for $x \ge e$. Obviously,

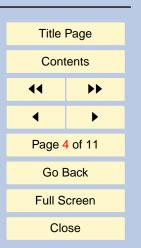
$$\frac{d}{dx}f(x) = \frac{p\log x - 1 - W(x)}{px(1 + W(x))\log x},$$

and if p = 2, then

$$\frac{d}{dx}f(x) = \frac{(\log x - W(x)) + (\log x - 1)}{2x(1 + W(x))\log x}$$



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Considering the right hand side of (1.1), we have $\frac{d}{dx}f(x) > 0$ for x > e and consequently f(x) > f(e) = 0, and this gives right hand side of (2.1). Trivially, equality only holds for x = e. If $0 , then <math>\frac{d}{dx}f(e) = \frac{p-2}{2ep} < 0$, and this yields that the coefficient $-\frac{1}{2}$ of $\log \log x$ in the right hand side of (2.1) is the best possible for the range $x \ge e$.

For the other side, note that $\log W(x) = \log x - W(x)$ and the inequality $\log W(x) \le \log \log x$ holds for $x \ge e$, because of the right hand side of (1.1). Thus, $\log x - W(x) \le \log \log x$ holds for $x \ge e$ with equality only for x = e. The sharpness of (2.1) with coefficient -1 for $\log \log x$ comes from the relation $\lim_{x\to\infty} (W(x) - \log x + \log \log x) = 0$. This completes the proof.

Now, we try to obtain some upper bounds for the function W(x) with the main term $\log x - \log \log x$. To do this, we need the following lemma.

Lemma 2.2. For every $t \in \mathbb{R}$ and y > 0, we have

$$(t - \log y)e^t + y \ge e^t,$$

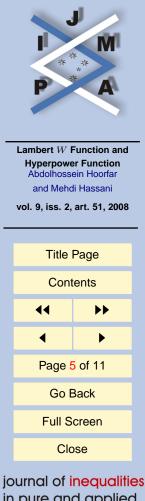
with equality for $t = \log y$.

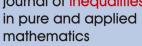
Proof. Letting $f(t) = (t - \log y)e^t + y - e^t$, we have $\frac{d}{dt}f(t) = (t - \log y)e^t$ and $\frac{d^2}{dt^2}f(t) = (t + 1 - \log y)e^t$. Now, we observe that $f(\log y) = \frac{d}{dt}f(\log y) = 0$ and $\frac{d^2}{dt^2}f(\log y) = y > 0$. These show that the function f(t) takes its only minimum value (equal to 0) at $t = \log y$, which yields the result of Lemma 2.2.

Theorem 2.3. For $y > \frac{1}{e}$ and $x > -\frac{1}{e}$ we have

(2.2)
$$W(x) \le \log\left(\frac{x+y}{1+\log y}\right)$$

with equality only for $x = y \log y$.





Proof. Using the result of Lemma 2.2 with t = W(x), we get

$$(W(x) - \log y)e^{W(x)} - (e^{W(x)} - y) \ge 0$$

which, considering $W(x)e^{W(x)} = x$, gives $(1 + \log y)e^{W(x)} \le x + y$ and this is desired inequality for $y > \frac{1}{e}$ and $x > -\frac{1}{e}$. The equality holds when $W(x) = \log y$, i.e., $x = y \log y$.

Corollary 2.4. For $x \ge e$ we have

(2.3) $\log x - \log \log x \le W(x) \le \log x - \log \log x + \log(1 + e^{-1}),$

where equality holds in the left hand side for x = e and in the right hand side for $x = e^{e+1}$.

Proof. Consider (2.2) with $y = \frac{x}{e}$, and the left hand side of (2.1).

Remark 1. Taking y = x in (2.2) we get $W(x) \le \log x - \log(\frac{1+\log x}{2})$, which is sharper than the right hand side of (2.1).

Theorem 2.5. For x > 1, we have

(2.4)
$$W(x) \ge \frac{\log x}{1 + \log x} (\log x - \log \log x + 1),$$

with equality only for x = e.

Proof. For t > 0 and x > 1, let

$$f(t) = \frac{t - \log x}{\log x} - (\log t - \log \log x).$$



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We have $\frac{d}{dt}f(t) = \frac{1}{\log x} - \frac{1}{t}$ and $\frac{d^2}{dt^2}f(t) = \frac{1}{t^2} > 0$. Now, we observe that $\frac{d}{dt}f(\log x) = 0$ and so

$$\min_{t>0} f(t) = f(\log x) = 0.$$

Thus, for t > 0 and x > 1 we have $f(t) \ge 0$, with equality at $t = \log x$. Putting t = W(x) and simplifying, we get the result, with equality at $W(x) = \log x$, or, equivalently, at x = e.

Corollary 2.6. For x > 1 we have

$$W(x) \le (\log x)^{\frac{\log x}{1 + \log x}}.$$

Proof. This refinement of the right hand side of (1.1) can be obtained by simplifying (2.4) with $W(x) = \log x - \log W(x)$.

The bounds we have obtained up to now have the form $W(x) = \log x - \log \log x + O(1)$. Now, we give bounds with the error term $O(\frac{\log \log x}{\log x})$ instead of O(1).

Theorem 2.7. For every $x \ge e$ we have

$$(2.5) \log x - \log \log x + \frac{1}{2} \frac{\log \log x}{\log x} \le W(x) \le \log x - \log \log x + \frac{e}{e-1} \frac{\log \log x}{\log x},$$

with equality only for x = e.

Proof. Taking the logarithm of the right hand side of (2.1), we have

$$\log W(x) \le \log \left(\log x - \frac{1}{2} \log \log x \right) = \log \log x + \log \left(1 - \frac{\log \log x}{2 \log x} \right)$$

Using $\log W(x) = \log x - W(x)$, we get

$$W(x) \ge \log x - \log \log x - \log \left(1 - \frac{\log \log x}{2 \log x}\right),$$



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which, considering $-\log(1-t) \ge t$ for $0 \le t < 1$ (see [1]) with $t = \frac{\log \log x}{2 \log x}$, implies the left hand side of (2.5). To prove the other side, we take the logarithm of the left hand side of (2.1) to get

$$\log W(x) \ge \log(\log x - \log\log x) = \log\log x + \log\left(1 - \frac{\log\log x}{\log x}\right).$$

Again, using $\log W(x) = \log x - W(x)$, we obtain

$$W(x) \le \log x - \log \log x - \log \left(1 - \frac{\log \log x}{\log x}\right).$$

Now we use the inequality $-\log(1-t) \le \frac{t}{1-t}$ for $0 \le t < 1$ (see [1]) with $t = \frac{\log \log x}{\log x}$, to get

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \le \frac{\log\log x}{\log x} \left(1 - \frac{\log\log x}{\log x}\right)^{-1} \le \frac{1}{m} \frac{\log\log x}{\log x},$$

where

$$m = \min_{x \ge e} \left(1 - \frac{\log \log x}{\log x} \right) = 1 - \frac{1}{e}$$

Thus, we have

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \le \frac{e}{e-1} \frac{\log\log x}{\log x},$$

which gives the desired bounds. This completes the proof.



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3. Studying the Hyperpower Function $h(x) = x^{x^x}$

Consider the hyperpower function $h(x) = x^{x^{x^{-1}}}$. One can define this function as the limit of the sequence $\{h_n(x)\}_{n \in \mathbb{N}}$ with $h_1(x) = x$ and $h_{n+1}(x) = x^{h_n(x)}$. It is proven that this sequence converges if and only if $e^{-e} \leq x \leq e^{\frac{1}{e}}$ (see [4] and references therein). This function satisfies the relation $h(x) = x^{h(x)}$, which, on taking the logarithm of both sides and a simple calculation yields

$$h(x) = \frac{W(\log(x^{-1}))}{\log(x^{-1})}$$

In this section we obtain some explicit upper and lower bounds for this function. Since the obtained bounds for W(x) hold for large values of x and since for such values of x the value of $\log(x^{-1})$ is negative, we cannot use these bounds to approximate h(x).

Theorem 3.1. Taking $\lambda = e - 1 - \log(e - 1) = 1.176956974..., for <math>e^{-e} \le x \le e^{\frac{1}{e}}$ we have

(3.1)
$$\frac{1 + \log(1 - \log x)}{1 - 2\log x} \le h(x) \le \frac{\lambda + \log(1 - \log x)}{1 - 2\log x},$$

where equality holds in the left hand side for x = 1 and in the right hand side for $x = e^{\frac{1}{e}}$.

Proof. For t > 0, we have $t \ge \log t + 1$, which taking $t = z - \log z$ with z > 0, implies

$$z\left(1-2\log(z^{\frac{1}{z}})\right) \ge \log\left(1-\log(z^{\frac{1}{z}})\right) + 1$$



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and putting $z^{\frac{1}{z}} = x$, or equivalently z = h(x), yields that $h(x)(1 - 2\log x) \ge \log(1 - \log x) + 1$; this is the left hand side (3.1), since $1 - 2\log x$ is positive for $e^{-e} \le x \le e^{\frac{1}{e}}$. Note that equality holds for t = z = x = 1.

For the right hand side, we define $f(z) = z - \log z$ with $\frac{1}{e} \le z \le e$. We immediately see that $1 \le f(z) \le e - 1$; in fact it takes its minimum value 1 at z = 1. Also, consider the function $g(t) = \log t - t + \lambda$ for $1 \le t \le e - 1$, with $\lambda = e - 1 - \log(e - 1)$. Since $\frac{d}{dt}g(t) = \frac{1}{t} - 1$ and g(e - 1) = 0, we obtain the inequality $\log t - t + \lambda \ge 0$ for $1 \le t \le e - 1$, and putting $t = z - \log z$ with $\frac{1}{e} \le z \le e$ in this inequality, we obtain

$$\log(1 - \log z) + \lambda \ge z \left(1 - 2\log(z^{\frac{1}{z}})\right).$$

Taking $z^{\frac{1}{z}} = x$, or equivalently z = h(x) yields the right hand side (3.1). Note that equality holds for $x = e^{\frac{1}{e}}$ (z = e, t = e - 1). This completes the proof.



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