# INEQUALITIES ON THE LAMBERT W FUNCTION AND HYPERPOWER FUNCTION

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Abstract:

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In this note, we obtain inequalities for the Lambert W function W(x), defined by  $W(x)e^{W(x)} = x$  for  $x \ge -e^{-1}$ . Also, we get upper and lower bounds for the hyperpower function  $h(x) = x^{x^{x^{-1}}}$ .



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## 1. Introduction

The Lambert W function W(x), is defined by  $W(x)e^{W(x)} = x$  for  $x \ge -e^{-1}$ . For  $-e^{-1} \le x < 0$ , there are two possible values of W(x), which we take values not less than -1. The history of the function goes back to J. H. Lambert (1728-1777). One can find in [2] a more detailed definition of W as a complex variable function, some historical background and various applications of it in Mathematics and Physics. The expansion

$$W(x) = \log x - \log \log x + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \frac{(\log \log x)^m}{(\log x)^{k+m}},$$

holds true for large values of x, with  $c_{km} = \frac{(-1)^k}{m!}S[k+m, k+1]$ , where S[k+m, k+1] is Stirling cycle number [2]. The series in the above expansion is absolutely convergent and it can be rearranged into the form

$$W(x) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2 - 2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} + O\left(\left(\frac{L_2}{L_1}\right)^4\right),$$

where  $L_1 = \log x$  and  $L_2 = \log \log x$ . Note that by log we mean logarithm in the base e. Since the Lambert W function appears in some problems in Mathematics, Physics and Engineering, it is very useful to have some explicit bounds for it. In [5] it is shown that

(1.1) 
$$\log x - \log \log x < W(x) < \log x$$

where the left hand side holds true for x > 41.19 and the right hand side holds true for x > e. The aim of the present paper is to obtain some sharper bounds.



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## 2. Some Sharp Bounds for the Lambert W Function

It is easy to see that  $W(-e^{-1}) = -1$ , W(0) = 0 and W(e) = 1. Also, for x > 0, since  $W(x)e^{W(x)} = x > 0$  and  $e^{W(x)} > 0$ , we have W(x) > 0. An easy calculation yields that

$$\frac{d}{dx}W(x) = \frac{W(x)}{x(1+W(x))}$$

Thus,  $x \frac{d}{dx} W(x) > 0$  holds true for x > 0 and consequently W(x) is strictly increasing for x > 0 (and also for  $-e^{-1} \le x \le 0$ , but not for this reason).

**Theorem 2.1.** For every  $x \ge e$ , we have

(2.1) 
$$\log x - \log \log x \le W(x) \le \log x - \frac{1}{2} \log \log x,$$

with equality holding only for x = e. The coefficients -1 and  $-\frac{1}{2}$  of  $\log \log x$  both are best possible for the range  $x \ge e$ .

*Proof.* For the given constant 0 consider the function

$$f(x) = \log x - \frac{1}{p}\log\log x - W(x)$$

for  $x \ge e$ . Obviously,

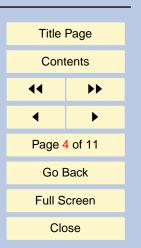
$$\frac{d}{dx}f(x) = \frac{p\log x - 1 - W(x)}{px(1 + W(x))\log x},$$

and if p = 2, then

$$\frac{d}{dx}f(x) = \frac{(\log x - W(x)) + (\log x - 1)}{2x(1 + W(x))\log x}$$



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Considering the right hand side of (1.1), we have  $\frac{d}{dx}f(x) > 0$  for x > e and consequently f(x) > f(e) = 0, and this gives right hand side of (2.1). Trivially, equality only holds for x = e. If  $0 , then <math>\frac{d}{dx}f(e) = \frac{p-2}{2ep} < 0$ , and this yields that the coefficient  $-\frac{1}{2}$  of  $\log \log x$  in the right hand side of (2.1) is the best possible for the range  $x \ge e$ .

For the other side, note that  $\log W(x) = \log x - W(x)$  and the inequality  $\log W(x) \le \log \log x$  holds for  $x \ge e$ , because of the right hand side of (1.1). Thus,  $\log x - W(x) \le \log \log x$  holds for  $x \ge e$  with equality only for x = e. The sharpness of (2.1) with coefficient -1 for  $\log \log x$  comes from the relation  $\lim_{x\to\infty} (W(x) - \log x + \log \log x) = 0$ . This completes the proof.

Now, we try to obtain some upper bounds for the function W(x) with the main term  $\log x - \log \log x$ . To do this, we need the following lemma.

**Lemma 2.2.** For every  $t \in \mathbb{R}$  and y > 0, we have

$$(t - \log y)e^t + y \ge e^t,$$

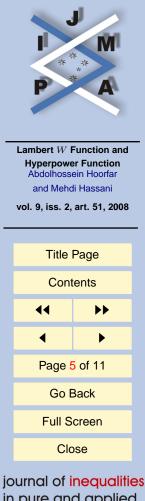
with equality for  $t = \log y$ .

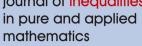
*Proof.* Letting  $f(t) = (t - \log y)e^t + y - e^t$ , we have  $\frac{d}{dt}f(t) = (t - \log y)e^t$  and  $\frac{d^2}{dt^2}f(t) = (t + 1 - \log y)e^t$ . Now, we observe that  $f(\log y) = \frac{d}{dt}f(\log y) = 0$  and  $\frac{d^2}{dt^2}f(\log y) = y > 0$ . These show that the function f(t) takes its only minimum value (equal to 0) at  $t = \log y$ , which yields the result of Lemma 2.2.

**Theorem 2.3.** For  $y > \frac{1}{e}$  and  $x > -\frac{1}{e}$  we have

(2.2) 
$$W(x) \le \log\left(\frac{x+y}{1+\log y}\right)$$

with equality only for  $x = y \log y$ .





*Proof.* Using the result of Lemma 2.2 with t = W(x), we get

$$(W(x) - \log y)e^{W(x)} - (e^{W(x)} - y) \ge 0$$

which, considering  $W(x)e^{W(x)} = x$ , gives  $(1 + \log y)e^{W(x)} \le x + y$  and this is desired inequality for  $y > \frac{1}{e}$  and  $x > -\frac{1}{e}$ . The equality holds when  $W(x) = \log y$ , i.e.,  $x = y \log y$ .

**Corollary 2.4.** For  $x \ge e$  we have

(2.3)  $\log x - \log \log x \le W(x) \le \log x - \log \log x + \log(1 + e^{-1}),$ 

where equality holds in the left hand side for x = e and in the right hand side for  $x = e^{e+1}$ .

*Proof.* Consider (2.2) with  $y = \frac{x}{e}$ , and the left hand side of (2.1).

*Remark* 1. Taking y = x in (2.2) we get  $W(x) \le \log x - \log(\frac{1+\log x}{2})$ , which is sharper than the right hand side of (2.1).

**Theorem 2.5.** For x > 1, we have

(2.4) 
$$W(x) \ge \frac{\log x}{1 + \log x} (\log x - \log \log x + 1),$$

with equality only for x = e.

*Proof.* For t > 0 and x > 1, let

$$f(t) = \frac{t - \log x}{\log x} - (\log t - \log \log x).$$



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We have  $\frac{d}{dt}f(t) = \frac{1}{\log x} - \frac{1}{t}$  and  $\frac{d^2}{dt^2}f(t) = \frac{1}{t^2} > 0$ . Now, we observe that  $\frac{d}{dt}f(\log x) = 0$  and so

$$\min_{t>0} f(t) = f(\log x) = 0.$$

Thus, for t > 0 and x > 1 we have  $f(t) \ge 0$ , with equality at  $t = \log x$ . Putting t = W(x) and simplifying, we get the result, with equality at  $W(x) = \log x$ , or, equivalently, at x = e.

**Corollary 2.6.** For x > 1 we have

$$W(x) \le (\log x)^{\frac{\log x}{1 + \log x}}.$$

*Proof.* This refinement of the right hand side of (1.1) can be obtained by simplifying (2.4) with  $W(x) = \log x - \log W(x)$ .

The bounds we have obtained up to now have the form  $W(x) = \log x - \log \log x + O(1)$ . Now, we give bounds with the error term  $O(\frac{\log \log x}{\log x})$  instead of O(1).

**Theorem 2.7.** For every  $x \ge e$  we have

$$(2.5) \log x - \log \log x + \frac{1}{2} \frac{\log \log x}{\log x} \le W(x) \le \log x - \log \log x + \frac{e}{e-1} \frac{\log \log x}{\log x},$$

with equality only for x = e.

*Proof.* Taking the logarithm of the right hand side of (2.1), we have

$$\log W(x) \le \log \left( \log x - \frac{1}{2} \log \log x \right) = \log \log x + \log \left( 1 - \frac{\log \log x}{2 \log x} \right)$$

Using  $\log W(x) = \log x - W(x)$ , we get

$$W(x) \ge \log x - \log \log x - \log \left(1 - \frac{\log \log x}{2 \log x}\right),$$



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which, considering  $-\log(1-t) \ge t$  for  $0 \le t < 1$  (see [1]) with  $t = \frac{\log \log x}{2 \log x}$ , implies the left hand side of (2.5). To prove the other side, we take the logarithm of the left hand side of (2.1) to get

$$\log W(x) \ge \log(\log x - \log\log x) = \log\log x + \log\left(1 - \frac{\log\log x}{\log x}\right).$$

Again, using  $\log W(x) = \log x - W(x)$ , we obtain

$$W(x) \le \log x - \log \log x - \log \left(1 - \frac{\log \log x}{\log x}\right).$$

Now we use the inequality  $-\log(1-t) \le \frac{t}{1-t}$  for  $0 \le t < 1$  (see [1]) with  $t = \frac{\log \log x}{\log x}$ , to get

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \le \frac{\log\log x}{\log x} \left(1 - \frac{\log\log x}{\log x}\right)^{-1} \le \frac{1}{m} \frac{\log\log x}{\log x},$$

where

$$m = \min_{x \ge e} \left( 1 - \frac{\log \log x}{\log x} \right) = 1 - \frac{1}{e}$$

Thus, we have

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \le \frac{e}{e-1} \frac{\log\log x}{\log x},$$

which gives the desired bounds. This completes the proof.



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# **3.** Studying the Hyperpower Function $h(x) = x^{x^x}$

Consider the hyperpower function  $h(x) = x^{x^{x^{-1}}}$ . One can define this function as the limit of the sequence  $\{h_n(x)\}_{n \in \mathbb{N}}$  with  $h_1(x) = x$  and  $h_{n+1}(x) = x^{h_n(x)}$ . It is proven that this sequence converges if and only if  $e^{-e} \leq x \leq e^{\frac{1}{e}}$  (see [4] and references therein). This function satisfies the relation  $h(x) = x^{h(x)}$ , which, on taking the logarithm of both sides and a simple calculation yields

$$h(x) = \frac{W(\log(x^{-1}))}{\log(x^{-1})}$$

In this section we obtain some explicit upper and lower bounds for this function. Since the obtained bounds for W(x) hold for large values of x and since for such values of x the value of  $\log(x^{-1})$  is negative, we cannot use these bounds to approximate h(x).

**Theorem 3.1.** Taking  $\lambda = e - 1 - \log(e - 1) = 1.176956974..., for <math>e^{-e} \le x \le e^{\frac{1}{e}}$  we have

(3.1) 
$$\frac{1 + \log(1 - \log x)}{1 - 2\log x} \le h(x) \le \frac{\lambda + \log(1 - \log x)}{1 - 2\log x},$$

where equality holds in the left hand side for x = 1 and in the right hand side for  $x = e^{\frac{1}{e}}$ .

*Proof.* For t > 0, we have  $t \ge \log t + 1$ , which taking  $t = z - \log z$  with z > 0, implies

$$z\left(1-2\log(z^{\frac{1}{z}})\right) \ge \log\left(1-\log(z^{\frac{1}{z}})\right) + 1$$



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and putting  $z^{\frac{1}{z}} = x$ , or equivalently z = h(x), yields that  $h(x)(1 - 2\log x) \ge \log(1 - \log x) + 1$ ; this is the left hand side (3.1), since  $1 - 2\log x$  is positive for  $e^{-e} \le x \le e^{\frac{1}{e}}$ . Note that equality holds for t = z = x = 1.

For the right hand side, we define  $f(z) = z - \log z$  with  $\frac{1}{e} \le z \le e$ . We immediately see that  $1 \le f(z) \le e - 1$ ; in fact it takes its minimum value 1 at z = 1. Also, consider the function  $g(t) = \log t - t + \lambda$  for  $1 \le t \le e - 1$ , with  $\lambda = e - 1 - \log(e - 1)$ . Since  $\frac{d}{dt}g(t) = \frac{1}{t} - 1$  and g(e - 1) = 0, we obtain the inequality  $\log t - t + \lambda \ge 0$  for  $1 \le t \le e - 1$ , and putting  $t = z - \log z$  with  $\frac{1}{e} \le z \le e$  in this inequality, we obtain

$$\log(1 - \log z) + \lambda \ge z \left(1 - 2\log(z^{\frac{1}{z}})\right).$$

Taking  $z^{\frac{1}{z}} = x$ , or equivalently z = h(x) yields the right hand side (3.1). Note that equality holds for  $x = e^{\frac{1}{e}}$  (z = e, t = e - 1). This completes the proof.



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