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AN EXTENSION OF THE ERDÖS-DEBRUNNER INEQUALITY TO GENERAL POWER MEANS

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ABSTRACT. Given the harmonic mean μ of the numbers x_i $(i = 1, 2, 3)$ and a $t \in (0, \min\{x_1, x_2, x_3\}/\mu\})$, we determine the best power mean exponents p and q such that $M_p(x_i - t\mu) \le (1 - t)\mu \le M_q(x_i - t\mu)$, where p and q only depend on t. Also, for $t > 0$ we similarly handle the estimates $M_p(x_i + t\mu) \le (1 + t)\mu \le M_q(x_i + t\mu)$.

Key words and phrases: Erdös-Debrunner inequality, harmonic mean, geometric mean, power means.

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1. INTRODUCTION

Three points D, E, F, one on each of the sides of a triangle ABC, form a triangle DEF that partitions the original one into four sub-triangles. The Erdös-Debrunner inequality says that

$$\min\{A_1, A_2, A_3\} \le A_4,$$

where A_1, A_2, A_3 are the areas of the corner triangles, and A_4 is the area of the central triangle. In [3], Janous conjectured that the optimal improvement would be given by

$$M_{-q}(A_1, A_2, A_3) \le A_4$$

where M_{-q} denotes the $(-q)$-power mean with

$$q = \frac{\ln(3/2)}{\ln 2}$$

(Janous proved the above inequality with $q = 1$. See the classical reference [5] for more on power means). In our paper [4] we confirmed Janous' conjecture. In the course of our proof we revealed some equivalent formulations of this optimal result, one of which is:

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105-07

Theorem 1.1 ([4, Cor. 6]). *Let $p \geq \ln(3/2)/\ln(2)$. Then for all triangles with sides a, b and c and semi-perimeter s, the inequality*

$$\left(\frac{s-a}{a}\right)^p + \left(\frac{s-b}{b}\right)^p + \left(\frac{s-c}{c}\right)^p \geq \frac{3}{2^p}$$

is valid. In terms of power means,

$$M_{-p}\left(\frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c}\right) \leq 2. \tag{1.1}$$

Our aim here is to gain a better understanding of where the number $\ln(3/2)/\ln 2$ in Theorem 1.1 comes from. To do so, we first apply a change of variables to the inequality (1.1). After defining $x_1 := \frac{s}{s-a}$, $x_2 := \frac{s}{s-b}$, $x_3 := \frac{s}{s-c}$, (1.1) takes on a form which for clarity we state as a new theorem (for simplicity of notation, we will denote the p-power mean of the numbers x_1, x_2, x_3 simply by $M_p(x)$).

Theorem 1.2. *For all $x_i > 1$ ($i = 1, 2, 3$) such that*

$$M_{-1}(x) = 3, \tag{1.2}$$

we have

$$M_{-q}(x-1) \leq 2, \tag{1.3}$$

where $q = \ln(3/2)/\ln(2)$.

It is now very easy to check that q is optimal in these results: let $\epsilon > 0$ and consider the special case

$$x_1 = x_2 = 2 + \epsilon, \qquad x_3 = \frac{2+\epsilon}{\epsilon}.$$

(1.2) is obviously satisfied, and (by letting $\epsilon \to 0$)

$$M_{-p}(x-1) \leq 2$$

can only hold if $p \geq q = \ln(3/2)/\ln(2)$.

2. MAIN RESULTS

In the light of the formulation of Theorem 1.2 we see that the new problem is: *Given three numbers with a certain harmonic average, predict the best exponent for a power mean estimate of these numbers after they have been all reduced (or augmented) by a fixed amount.* This point of view leads us to the following generalization (note that Theorem 1.2 is a special case of this after setting $\mu = 3$, $t = 1/3$, where the value of t matches the requirement that $x_i > 1$ for $i = 1, 2, 3$).

Theorem 2.1. *Let $x_i > 0$ ($i = 1, 2, 3$) be such that*

$$M_{-1}(x) = \mu, \tag{2.1}$$

and fix $t \in (0, \min\{x_1, x_2, x_3\}/\mu\})$. Then we have

$$M_0(x - t\mu) \leq (1-t)\mu \leq M_{q_2}(x - t\mu) \quad \text{if} \quad 2/3 \leq t < 1, \tag{2.2}$$

$$M_{-q_1}(x - t\mu) \leq (1-t)\mu \leq M_{q_2}(x - t\mu) \quad \text{if} \quad 1/3 \leq t < 2/3, \tag{2.3}$$

$$M_{-q_1}(x - t\mu) \leq (1-t)\mu \leq M_0(x - t\mu) \quad \text{if} \quad 0 < t < 1/3, \tag{2.4}$$

where

$$q_1 = \frac{\ln(3/2)}{\ln\left(\frac{1-t}{\frac{2}{3}-t}\right)}, \qquad q_2 = \frac{\ln(3/2)}{\ln\left(\frac{t}{t-\frac{1}{3}}\right)}.$$

It is understood that $q_1 = 0$ *when* $t = 2/3$*, and* $q_2 = 0$ *when* $t = 1/3$*.*

The proof of Theorem 2.1 will be rather technical, and accordingly we thought it wise not to pursue further generalizations in this paper, although we are certainly working on it. Similar statements are possible when estimating the means of more than three numbers, and it should also be possible to prove extensions to the case when the hypothesis is not just knowledge of the harmonic mean, but any given mean. Again, we decided not to pursue these more general directions right now as the technicalities would have easily overshadowed the main purpose of this note, even in the simplest next case, that is, $n = 4$.

If one adds $t\mu$ to the x_i, instead of subtracting, we have a result whose proof shows non-linear intricacies even harder than the ones offered by Theorem 2.1:

Theorem 2.2. *Let* $x_i > 0$ *(*$i = 1, 2, 3$*) be such that*

$$M_{-1}(x) = \mu,$$

and fix $t > 0$*. Then we have*

$$M_{-q}\left(x + t\mu\right) \le (1 + t)\mu \le M_0\left(x + t\mu\right),$$

where

$$q = \sqrt{1 + \frac{9t(1+t)}{2}}.$$

Whether q *is best possible is open. However, numerical evidence shows that at least for some* p *with* $p \in \left(1 + \frac{3}{\sqrt{2}}t, q\right)$ *and for some* x_i*,* $M_{-p}(x + t\mu) \le (1 + t)\mu$ *may be false.*

The proofs of Theorems 2.1 and 2.2 will be found in Section 4.

3. APPLICATIONS

As an application of Theorem 2.1 we have the following refinement of the case $n = 3$ of the famous Shapiro cyclic inequality. See [1] for a survey of the topic, and [2] for a recent related result.

Theorem 3.1. *Let* $a_1, a_2, a_3 \ge 0$*, with at most one of the* a_i *being zero. Then, with the index* i *cycling through* $1, 2, 3$*,*

$$M_0\left(\frac{a_i}{a_{i+1} + a_{i+2}}\right) \le \frac{1}{2} \le M_q\left(\frac{a_i}{a_{i+1} + a_{i+2}}\right), \tag{3.1}$$

where $q = \ln(3/2)/\ln(2) \sim 0.58496$*.*

Proof. Defining $x_i := (a_1 + a_2 + a_3)/(a_{i+1} + a_{i+2})$ we see that the harmonic mean $M_{-1}(x)$ equals $3/2$. We apply then Theorem 2.1 (specifically, (2.2)) in the case $\mu = 3/2$, $t = 2/3$ to immediately obtain (3.1). □

For comparison, note that the case $n = 3$ of the original problem posed by Shapiro [6] was stating the simpler inequality

$$\frac{1}{2} \le M_1\left(\frac{a_i}{a_{i+1} + a_{i+2}}\right).$$

Before we embark on the proofs of Theorems 2.1 and 2.2, we want to show a possible use of Theorem 2.2 in a special situation. It is a trivial fact that, given any positive a_1, a_2, a_3, the arithmetic mean of the sums $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_1$ is simply twice the arithmetic mean of the a_i. But what about other power means of the sums $a_i + a_{i+1}$? The next result shows that the power means of $a_i + a_{i+1}$ seem to be related to the classical problem of estimating the difference between the arithmetic and the harmonic mean of the a_i (see [5, 2.14.3] for more on the topic).

Theorem 3.2. *Let a_1, a_2, $a_3 > 0$ and, for simplicity, denote their harmonic and arithmetic means by $\mu_{-1} := M_{-1}(a)$ and $\mu_1 := M_1(a)$, respectively. We then have*

$$M_0(a_i + a_{i+1}) \leq 3\mu_1 - \mu_{-1} \leq M_q(a_i + a_{i+1}),$$

where

$$q = \frac{1}{\mu_{-1}}\sqrt{\frac{(9\mu_1 - 2\mu_{-1})(9\mu_1 - \mu_{-1})}{2}}.$$

Proof. This follows from Theorem 2.2 after first observing that, with $\sigma := a_1 + a_2 + a_3$,

$$M_{-1}\left(\frac{a_i}{\sigma - a_i}\right) = \frac{\mu_{-1}}{3\mu_1 - \mu_{-1}} =: \mu.$$

If we now choose t to satisfy $t\mu = 1$ (i.e., $t = 3\frac{\mu_1}{\mu_{-1}} - 1$), Theorem 2.2 yields (since $\frac{a_i}{\sigma - a_1} + 1 = \frac{\sigma}{\sigma - a_i}$)

$$M_{-q}\left(\frac{\sigma}{\sigma - a_i}\right) \leq \frac{3\mu_1}{3\mu_1 - \mu_{-1}} \leq M_0\left(\frac{\sigma}{\sigma - a_i}\right),$$

and the result follows from simple algebra, the fact that $\sigma = 3\mu_1$, and after finding what the formula for q in Theorem 2.2 translates into in the current case. □

Corollary 3.3. *Let a_1, a_2, $a_3 > 0$, and define*

$$C := (\max_i a_i)/(\min_i a_i).$$

Then

$$M_1(a) - M_{-1}(a) \leq M_{\overline{q}}(a_i + a_{i+1}) - 2M_1(a),$$

where

$$\overline{q} = \frac{1}{4C}\sqrt{\frac{(9C^2 + 10C + 9)(9C^2 + 14C + 9)}{2}}.$$

Proof. This follows from Theorem 3.2 and the following classical result of Specht giving the upper bound of the ratio M_1/M_{-1} in terms of C (see [5, 2.14.3, Theorem 1])

$$\frac{\mu_1}{\mu_{-1}} \leq \frac{(C+1)^2}{4C}.$$

□

Finally, before we get started with the proofs of the main theorems, we present a couple of simpler observations, given here purely for illustrative purposes. First, let us state the trivial (though natural) version of Theorem 1.2 in the case of two variables.

Theorem 3.4. *For all $x_i > 1$ ($i = 1, 2$) such that*

$$M_{-1}(x_1, x_2) = 2 \tag{3.2}$$

we have

$$M_p(x_1 - 1, x_2 - 1) \leq 1 = M_0(x_1 - 1, x_2 - 1) \tag{3.3}$$

for $p < 0$.

Proof. This follows from the the obvious fact that (3.2) implies $x_2 - 1 = 1/(x_1 - 1)$. □

Also as a curiosity and as an example of the multi-variable statements that are possible (in the vein of Theorem 2.1), we have the following

Theorem 3.5. *Let $x_i > 1$ ($i = 1, \dots, n$) be such that*

$$M_{-1}(x) = n. \tag{3.4}$$

Then

$$M_{-1}(x-1) \le n-1 \le M_1(x-1).$$

Note that the first inequality can be rewritten as

$$\frac{1}{n}\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{i=1}^{n}\frac{1}{x_i-1}\right) \le \sum_{i=1}^{n}\frac{1}{(x_i-1)x_i}.$$

Proof. Let $f(h)$ be the function

$$M_{-1}(x_1+h, \dots, x_n+h).$$

A calculation gives that

$$f'(h) = \left(\frac{M_{-1}(x_1+h,\dots,x_n+h)}{M_{-2}(x_1+h,\dots,x_n+h)}\right)^2$$

and this shows that $f'(h) \ge 1$ for all $h \ge -1$. In particular,

$$f(0) - f(-1) \ge 1$$

by the mean value theorem, and this is the first inequality. □

4. PROOFS OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. Without loss of generality we will prove Theorem 2.1 in the case $\mu = 1$. In the first part of the proof we will verify the first inequalities in (2.2), (2.3) and (2.4). To do so, we will apply the method of Lagrange multipliers on the domain $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i > t,\ i = 1, 2, 3\}$ to find the minima of

$$f(x_1, x_2, x_3) := \frac{1}{(x_1-t)^p} + \frac{1}{(x_2-t)^p} + \frac{1}{(x_3-t)^p}$$

under the condition

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 3.$$

Clearly, this investigation is only of interest for $0 < p < 1$. The Lagrange equations simplify to

$$\frac{x_i - t}{x_i^{2/(1+p)}} = c \tag{4.1}$$

for some constant c and $i = 1, 2, 3$. The derivative of the function $h(x) := (x-t)/x^{2/(1+p)}$ (for $x > t$) has the same sign as $2t - (1-p)x$ and so $h(x)$ has precisely one critical point (a maximum) at $x = 2t/(1-p)$. This means that the only possibility we need to study is when, say, $x_1 = x_2$, which can only happen if

$$x_1 = x_2 = \frac{2+\epsilon}{3}, \qquad x_3 = \frac{2+\epsilon}{3\epsilon}, \tag{4.2}$$

for some $\epsilon > 0$ such that $x_i - t > 0$ for $i = 1, 2, 3$ (recall that we are handling the case $\mu = 1$ here, meaning that $\sum_i 1/x_i = 3$). ϵ must therefore satisfy the inequalities

$$3t - 2 < \epsilon \quad \text{and} \quad (3t-1)\epsilon < 2. \tag{4.3}$$

These conditions will force us to distinguish between three cases because of the different possible ranges for ϵ:

Case I: $2/3 \le t < 1$. Here $3t - 2 < \epsilon < 2/(3t-1)$.

Case II: $1/3 < t < 2/3$. In this case $0 < \epsilon < 2/(3t-1)$.

Case III: $0 < t \leq 1/3$. Now ϵ can be any positive number.

With values as in (4.2),

$$f(x_1, x_2, x_3) = f(\epsilon) = 2\frac{3^p}{(2+\epsilon-3t)^p} + \frac{3^p\epsilon^p}{(2+(1-3t)\epsilon)^p}, \tag{4.4}$$

and the cases we just described specify the domain of $f(\epsilon)$ for any given t. The derivative of f with respect to ϵ is

$$f'(\epsilon) = 2\cdot 3^p p\left(\epsilon^{-1+p}(2+\epsilon-3\epsilon t)^{-1-p} - (2+\epsilon-3t)^{-1-p}\right)$$

and so its critical points must satisfy the equation

$$g_1(\epsilon) := 2+\epsilon-3t = (2+\epsilon-3\epsilon t)\epsilon^{\frac{1-p}{1+p}} =: g_2(\epsilon). \tag{4.5}$$

$\epsilon = 1$ is always a critical point of $f(\epsilon)$. After inspecting $f''(1)$ we also see that $\epsilon = 1$ can only be a minimum if $p > 1-2t$, which we will assume from now on. In Cases I and II ($\epsilon > 1/3$), $g_2(\epsilon)$ is always concave on its domain, and so (4.5) can have at most two solutions since $g_1(\epsilon)$ is linear. And since one of these critical points is the local minimum at $\epsilon = 1$, the other one (if any) cannot be a local minimum, too. In Case III (if $0 < t < 1/3$), $g_2(\epsilon)$ is increasing for all $\epsilon > 0$ and, since

$$g_2''(\epsilon) = 2\epsilon^{-3+2/(1+p)}(1+p)^{-2}(1-p)\big((1-3t)\epsilon - 2p\big),$$

we see that $g_2(\epsilon)$ is concave if $\epsilon < 2p/(1-3t)$ and convex if $\epsilon > 2p/(1-3t)$. Since $g_1(0) = 2-3t > 0 = g_2(0)$, we conclude that (4.5) has at most three solutions and thus that $f(\epsilon)$ has at most three critical points. It actually happens that there are exactly three critical points in Case III. In fact, the inequality

$$g_2'(1) = \frac{2}{1+p}\big(2-3t-p\big) < 1$$

is equivalent to $p > 1-2t$, and so it holds by our assumption. Because of this, $g_2(\epsilon)$ must cross $g_1(\epsilon)$ at $\epsilon = 1$ with a slope smaller than 1, and thus $\epsilon = 1$ is the middle of the three critical points of $f(\epsilon)$. We therefore know that $\epsilon = 1$ is the only local minimum of f in all cases.

Summarizing, we have shown that in all possible cases the minima of $f(\epsilon)$ will result from comparing $f(1)$ with the values (or limits) of $f(\epsilon)$ at the endpoints of the allowable intervals for ϵ. We proceed now to do so, while still distinguishing between the same three cases for separate ranges for t.

Case I: $2/3 \leq t < 1$. Here $3t-2 < \epsilon < 2/(3t-1)$, and the values of f close to the endpoints are seen to tend to infinity. Consequently, $f(1)$ yields the absolute minimum of f. We conclude that for these values of t we will have $M_{-p}(x-t) \leq 1-t$ for all $p \in (0,1)$ and, passing to the limit $p \to 0$, the same applies to the geometric average

$$M_0(x-t) = ((x_1-t)(x_2-t)(x_3-t))^{1/3} \leq 1-t.$$

That no higher power mean (that is, of the type $M_r(x)$ with $r > 0$) would work follows from the fact that for our choice of x_1, x_2, x_3 the expression $x_1^r + x_2^r + x_3^3$ grows out of bounds for small enough ϵ.

Case II: $1/3 < t < 2/3$. In this case $0 < \epsilon < 2/(3t-1)$. Values of ϵ tending to the right endpoint will cause f to grow arbitrarily, while $\lim_{\epsilon\to 0} f(\epsilon) = 2\cdot 3^p/(2-3t)^p$. The latter is never smaller than $3/(1-t)^p$ if and only if

$$p \geq \frac{\ln(3/2)}{\ln\left(\frac{3-3t}{2-3t}\right)}. \tag{4.6}$$

We must now measure this condition for p against the one we had obtained at the beginning, $p > 1 - 2t$. We claim that (4.6) is stronger, that is,

$$\frac{\ln(3/2)}{\ln\left(\frac{3-3t}{2-3t}\right)} > 1 - 2t \tag{4.7}$$

when $0 < t < 2/3$. With

$$s(t) := \ln\left(\frac{3-3t}{2-3t}\right)(1-2t),$$

we have

$$s'(t) = \frac{1}{1-t} - \frac{1}{2-3t} - 2\ln\left(1 + \frac{1}{2-3t}\right).$$

Since for all $x > 0$ the classical inequality $\ln(1 + 1/x) > 2/(2x + 1)$ holds (see [5, 3.6.18]), a little algebra shows that

$$s'(t) < -\frac{3-4t}{(1-t)(2-3t)(5-6t)} < 0.$$

Therefore, $s(t)$ is decreasing on $t \in (0, 2/3)$ and is thus always less than $s(0) = \ln(3/2)$ there, proving (4.7). (4.7) being true, to complete the discussion of Case II we may now state that $M_{-q}(x - t) \leq 1 - t$, where

$$q := \frac{\ln(3/2)}{\ln\left(\frac{2-3t}{3-3t}\right)},$$

and this choice of q is optimal.

Case III: $0 < t \leq 1/3$. In this case ϵ can be any positive number, and the limits of $f(\epsilon)$ for $\epsilon \to 0$ and $\epsilon \to \infty$ are given by $2 \cdot 3^p/(2-3t)^p$ and $3^p/(1-3t)^p$. By our discussion of the critical points of $f(\epsilon)$ the absolute minimum of $f(\epsilon)$ is either one of these two values, or $f(1)$. For $3^p/(1-3t)^p$ to always be greater or equal to $3/(1-t)^p$ we need to have

$$p \geq \frac{\ln 3}{\ln\left(\frac{3-3t}{1-3t}\right)}.$$

This condition is actually weaker than (4.6), that is, we always have

$$\frac{\ln(3/2)}{\ln\left(\frac{3-3t}{2-3t}\right)} > \frac{\ln 3}{\ln\left(\frac{3-3t}{1-3t}\right)} \tag{4.8}$$

when $0 < t < 1/3$. A way to convince ourselves of this is to consider the function

$$h(t) := \frac{\ln\left(\frac{3-3t}{1-3t}\right)}{\ln\left(\frac{3-3t}{2-3t}\right)}. \tag{4.9}$$

Notice that its derivative for $t \in (0, 1/3)$ has the same sign as

$$2(a+1)\ln\left(1 + \frac{1}{a+1}\right) - a\ln\left(1 + \frac{2}{a}\right), \tag{4.10}$$

where for convenience we wrote $a := 1 - 3t$ (and thus $a \in (0, 1)$). The latter function of a has the derivative

$$\ln\left(\frac{a^2 + 2a}{(a+1)^2}\right),$$

which is always negative for $a \in (0, 1)$. This implies that the expression in (4.10) is decreasing on $(0, 1)$ and hence it is always greater than its value at $a = 1$, which is $4\ln(3/2) - \ln 3 = \ln(27/16) > 0$. This means that the function of a in (4.10) is always positive for $a \in (0, 1)$, and in turn this implies that $h(t)$ as defined in (4.9) is increasing for $t \in (0, 1/3)$. Finally, since $h(0) = \ln(3)/\ln(3/2) > 0$, $h(t)$ is always greater than $h(0)$, and the inequality (4.8) is established. To wrap up the first part of the proof, we can now state that the first inequalities in (2.2), (2.3) and (2.4) are proved.

Let us now check the second inequalities in (2.2), (2.3) and (2.4), still assuming, for simplicity, that $M_{-1}(x) = 1$. To see for which $p > 0$ we have $M_p(x - t) \geq 1 - t$, we need to minimize

$$g(x_1, x_2.x_3) = (x_1 - t)^p + (x_2 - t)^p + (x_3 - t)^p,$$

and thus the Lagrange equations are now

$$x_i^2(x_i - t)^{p-1} = c$$

for some constant c and $i = 1, 2, 3$. Certainly, since $M_1(x-1) \geq 1-t$ is trivial, we can restrict our attention to $p \in (0, 1)$. Since the function $x^2(x - 1)^{p-1}$ decreases for $x < 2t/(1 + p)$ and increases for $x > 2t/(1 + p)$, we are in a situation similar to the first part of the proof, with only the need to consider the same special situation as in (4.2). In this case, g as a function of ϵ becomes

$$g(\epsilon) = 3^{-p}\left(2(2 + \epsilon - 3t)^p + \frac{(2 + (1 - 3t)\epsilon)^p}{\epsilon^p}\right).$$

Similarly to the way we handled f in the first part of the proof, we see now that the critical points of $g(\epsilon)$ must satisfy the equation

$$\frac{2 + \epsilon - 3t}{2 + \epsilon - 3\epsilon t} = \epsilon^{\frac{1+p}{1-p}}. \tag{4.11}$$

If $0 < t < 1/3$, the left hand side is concave, the right hand side is convex, and so (because of their initial values at $\epsilon = 0$) $\epsilon = 1$ must be the only critical point of $g(\epsilon)$. Since $g(\epsilon)$ is unbounded for ϵ close to 0 or when tending to ∞, we conclude that $\epsilon = 1$ yields the absolute minimum of $g(\epsilon)$ in this case, and thus $M_p(x - t) \geq x - t$. Letting $p \to 0$ shows that if $0 < t \leq 1/3$ we have $x - t \leq M_0(x - t)$, as claimed in (2.4) (the statement for $t = 1/3$ follows by continuity).

When $t \in (1/3, 2/3)$ we rewrite (4.11) in the form

$$2 + \epsilon - 3t = \epsilon^{\frac{1+p}{1-p}}(2 + \epsilon - 3\epsilon t) =: g_3(\epsilon). \tag{4.12}$$

$g_3(\epsilon)$ is increasing for $\epsilon < 2/(3t - 1)$ and decreasing for $\epsilon > 2/(3t - 1)$. From its second derivative we also see that it is convex for $\epsilon < (1 + p)/(3t - 1)$, and concave for $\epsilon > (1 + p)/(3t - 1)$. For $t \in (1/3, 2/3)$ we have $1 < (1 + p)/(3t - 1)$. Hence, $g_3(\epsilon)$ meets the left hand side of (4.12) at $\epsilon = 1$ for the first time, and thus there is exactly one other critical point of $g(\epsilon)$ (at the right of $\epsilon = 1$), and there we must have a local minimum. For small ϵ, $g(\epsilon)$ is arbitrarily large and thus, as we are looking for a minimum, we only need to consider the possibility offered by the right endpoint of the admissible interval (see Case II above), i.e.,

$$g\left(\frac{2}{3t - 1}\right) = 2 \cdot 3^{-p}\left(2 - 3t + \frac{2}{3t - 1}\right)^p.$$

In order to have $M_p(x - t) \geq 1 - t$ we must have that this value be greater or equal to $3(1 - t)^p$, which leads to the condition

$$p \geq \frac{\ln(3/2)}{\ln\left(\frac{3t}{3t-1}\right)}, \tag{4.13}$$

as stated in (2.3). Finally, we consider the case $2/3 < t < 1$, where (as in Case I in the first half of the proof) $3t - 2 < \epsilon < 2/(3t - 1)$. First we observe that since the value $g(3t - 2)$ at the left endpoint must be at least $3(1-t)^p$ in order to have $M_p(x-t) \geq 1-t$, we must have

$$g(3t-2) = \left(\frac{3(1-t)t}{3t-2}\right)^p \geq 3(1-t)^p,$$

that is,

$$p \geq \frac{\ln 3}{\ln\left(\frac{3t}{3t-2}\right)}. \tag{4.14}$$

In complete analogy with (4.8), for $t \in (2/3, 1)$ we have the inequality

$$\frac{\ln(3/2)}{\ln\left(\frac{3t}{3t-1}\right)} > \frac{\ln 3}{\ln\left(\frac{3t}{3t-2}\right)},$$

meaning that the condition in (4.13) trumps the one in (4.14) (we leave the details to the reader). Since convexity and concavity of $g_3(\epsilon)$ (as in (4.12)) are the same as in the previous case, we still have that $g(\epsilon)$ admits at most two critical points inside the admissible interval for ϵ. By inspecting the second derivative of $g(\epsilon)$ at $\epsilon = 1$, we see that its sign is the same as the sign of $1+p-2t$. We will therefore have a minimum at $\epsilon = 1$ if and only if $p > 2t-1$, and this latter condition will certainly hold if

$$p \geq \frac{\ln(3/2)}{\ln\left(\frac{3t}{3t-1}\right)} > 2t-1. \tag{4.15}$$

Once again, in complete analogy with (4.7) (and again using the inequality [5, 3.6.18] to simplify the estimate) we can prove that the function

$$(2t-1)\ln\left(\frac{3t}{3t-1}\right)$$

is strictly increasing in the interval $(2/3, 1)$, and thus (4.15) readily follows. To conclude, since $\epsilon = 1$ is the only minimum of $g(\epsilon)$ in the interval $(3t-2, 2/(3t-1)$, and since we already discussed the conditions (4.13) and (4.14) resulting from the values of $g(\epsilon)$ at the endpoints, our work is done and Theorem 2.1 is now proved. □

Proof of Theorem 2.2. Assume that x_1, x_2, $x_3 > 0$ are given such that $M_{-1}(x) = 1$, and fix $t > 0$. The search for p that satisfy $M_{-p}(x+t) \leq 1+t$ starts out as in the proof of Theorem 2.1. Assume that $p > 1$, since this is the only range that could yield possible non-trivial values of p. We need to find the minima of

$$f(x_1, x_2, x_3) := \frac{1}{(x_1+t)^p} + \frac{1}{(x_2+t)^p} + \frac{1}{(x_3+t)^p}$$

under the condition

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 3$$

and over the domain $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i > 0,\ i = 1, 2, 3\}$. The Lagrange equations simplify to

$$\frac{x_i + t}{x_i^{2/(1+p)}} = c \tag{4.16}$$

for some constant c and $i = 1, 2, 3$, so here, too, we only need to focus on the case when, say, $x_1 = x_2$, which can only happen if

$$x_1 = x_2 = \frac{2+\epsilon}{3}, \qquad x_3 = \frac{2+\epsilon}{3\epsilon}, \tag{4.17}$$

where $\epsilon > 0$ is arbitrary. With these values, $f = f(\epsilon)$ takes on the form

$$f(x_1, x_2, x_3) = f(\epsilon) = 2\frac{3^p}{(2+\epsilon+3t)^p} + \frac{3^p\epsilon^p}{(2+(1+3t)\epsilon)^p}. \tag{4.18}$$

ϵ is a critical point of $f(\epsilon)$ if and only if it satisfies the equation

$$h_1(\epsilon) := (2+\epsilon+3t)\epsilon^{\frac{p-1}{p+1}} = 2 + (1+3t)\epsilon =: h_2(\epsilon). \tag{4.19}$$

Also, $f''(\epsilon) > 0$ if and only if

$$(p+1)(2+\epsilon+3t)^{-2-p} > 2\epsilon^{-2+p}(2+\epsilon+3\epsilon t)^{-2-p}(1+\epsilon-p+3\epsilon t). \tag{4.20}$$

$\epsilon = 1$ is always a critical point, and is a minimum (as it must be, if $M_{-p}(x+t) \le x+t$) is to hold) exactly if $f''(1) > 0$, that is, if $p > 1+2t$. From now on, then, we will assume that $p > 1+2t$.

Define

$$q := \sqrt{1 + \frac{9t(1+t)}{2}},$$

and note that for all $t > 0$ we have $1+2t < q < 1+3t$. If we substitute the identity (4.19) in (4.20) we obtain the condition

$$p(\epsilon) := (p-1)(3t+1)\epsilon^2 - 4(q^2-p)\epsilon + 2(p-1)(2+3t) > 0. \tag{4.21}$$

$p(\epsilon)$ is quadratic in ϵ, and has discriminant Δ equal to

$$\Delta := 72t(1+t)(q^2-p^2).$$

If $p \ge q$ then $\Delta \le 0$, and this means (since then $p(\epsilon) \ge 0$ always) that every critical point of $f(\epsilon)$ is a local minimum: therefore, $\epsilon = 1$ must be the only local minimum in $(0,\infty)$. What is now left to do (in the case $p \ge q$) is to examine the values of $f(\epsilon)$ for $\epsilon \to 0$ and $\epsilon \to \infty$. At both ends of the domain $f(\epsilon)$ must still be greater or equal to $3/(1+t)^p$ for the desired inequality to hold. These two conditions yield, respectively, the inequalities

$$p > q_1 := \frac{\ln(3/2)}{\ln\left(\frac{3+3t}{2+3t}\right)}, \quad p > q_2 := \frac{\ln(3)}{\ln\left(\frac{3+3t}{1+3t}\right)}.$$

Not to overburden the reader, let us just state that an analysis similar to the one we carried out when proving (4.8) will be just as effective in showing that, for all $t > 0$,

$$q_1 < q_2 < 1+2t.$$

Now, since we already saw that $1+2t < q$, we conclude that when $p \ge q$ the inequality $M_{-p}(x+t) \le x+t$ will be true. The first inequality in Theorem 2.2 is thus proved.

As an aside, the case $1+2t < p < q$ seems much harder to handle. Numerical evidence points in the direction that for any such choice of p there are counterexamples where $M_{-p}(x+t) \le x+t$ fails, but we could not prove it. We could understand why this might happen by noticing that if $p < q$ then $f(\epsilon)$ definitely has a chance to have a second local minimum at some location $\epsilon_0 > 1$. To see this, we first observe that $f(\epsilon)$ cannot have more than three critical points: this is because (cf. (4.19)) $h_1(\epsilon)$ is increasing for all $\epsilon > 0$, concave for $\epsilon < (2+3t)/p$ and convex for $\epsilon > (2+3t)/p$. Since $h_2(\epsilon)$ is linear, no more than three solutions of (4.19) are possible. So, if $1+2t < p < q$ then the discriminant of $p(\epsilon)$ is positive, and thus $p(\epsilon)$ has two distinct real roots. Since

$$p(1) = 9(1+t)\,(p-(1+2t)) > 0, \quad p'(1) = 6(1+t)\,(p-(1+3t)) < 0,$$

both roots are greater than 1. Thus, in this case, if $f(\epsilon)$ should have one more local minimum, then it would have to be greater than 1, and, in fact, greater than the larger of the two roots of

$p(\epsilon)$, since no two local minima can be consecutive critical points. The situation is technically murky here, however, and we will not pursue the question further.

Proving the second inequality in Theorem 2.2 is, in comparison, a breeze. First notice that $M_p(x+t) \geq x+t$ will not possibly hold in general for any negative p. On the other hand, if we focus on the geometric mean of the numbers $x_i + t$, the Lagrange method (under the condition $M_{-1}(x) = 1$) will very easily yield the only solution $x_1 = x_2 = x_3$, and hence a quick path to the second inequality in Theorem 2.2. We leave the details to the reader. □

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