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SOME NEW INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS WITH SPECIAL COEFFICIENTS

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ABSTRACT. Some new inequalities for certain trigonometric polynomials with complex semi-convex and complex convex coefficients are given.

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1. INTRODUCTION AND PRELIMINARIES

Petrović [4] proved the following complementary triangle inequality for sequences of complex numbers $\{z_1, z_2, \dots, z_n\}$.

Theorem A. *Let α be a real number and $0 < \theta < \frac{\pi}{2}$. If $\{z_1, z_2, \dots, z_n\}$ are complex numbers such that $\alpha - \theta \le \arg z_\nu \le \alpha + \theta$, $\nu = 1, 2, \dots, n$, then*

$$\left| \sum_{\nu=1}^{n} z_\nu \right| \ge (\cos\theta) \sum_{\nu=1}^{n} |z_\nu|.$$

For $0 < \theta < \frac{\pi}{2}$ denote by $K(\theta)$ the cone $K(\theta) = \{z : |\arg z| \le \theta\}$.

Let $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, for $n = 1, 2, 3, \dots$, where $\{\lambda_n\}$ is a sequence of complex numbers. Then,

$$\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1} = \lambda_n - 2\lambda_{n+1} + \lambda_{n+2}, \quad n = 1, 2, 3, \dots$$

The author Tomovski (see [5]) proved the following inequality for cosine and sine polynomials with complex-valued coefficients.

Theorem B. *Let $x \ne 2k\pi$ for $k = 0, \pm1, \pm2, \dots$*

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(1) *Let* $\{b_k\}$ *be a positive nondecreasing sequence and* $\{u_k\}$ *a sequence of complex numbers such that* $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$. *Then*

$$\left|\sum_{k=n}^{m} u_k f(kx)\right| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}\left[\left(1+\frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_n|\right], \quad (\forall n, m \in \mathbb{N},\ m > n).$$

(2) *Let* $\{b_k\}$ *be a positive nondecreasing sequence and* $\{u_k\}$ *a sequence of complex numbers such that* $\Delta(u_k b_k) \in K(\theta)$. *Then*

$$\left|\sum_{k=n}^{m} u_k f(kx)\right| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}\left[\left(1+\frac{1}{\cos\theta}\right)|u_n| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_m|\right], \quad (\forall n, m \in \mathbb{N},\ m > n).$$

Here $f(x) = \sin x$ *or* $f(x) = \cos x$.

Similarly, the results of Theorem B were given by the author in [5] for sums of type $\sum_{k=n}^{m}(-1)^k u_k f(kx)$, where again $f(x) = \sin x$ or $f(x) = \cos x$.

Mitrinović and Pečarić (see [2, 3]) proved the following inequalities for cosine and sine polynomials with nonnegative coefficients.

Theorem C. *Let* $x \neq 2k\pi$ *for* $k = 0, \pm 1, \pm 2, ..$

(1) *Let* $\{b_k\}$ *be a positive nondecreasing sequence and* $\{a_k\}$ *a nonnegative sequence such that* $\{a_k b_k^{-1}\}$ *is a decreasing sequence. Then*

$$\left|\sum_{k=n}^{m} a_k f(kx)\right| \leq \frac{a_n}{\left|\sin\frac{x}{2}\right|}\left(\frac{b_m}{b_n}\right), \qquad (\forall n, m \in \mathbb{N},\ m > n).$$

(2) *Let* $\{b_k\}$ *be a positive nondecreasing sequence and* $\{a_k\}$ *a nonnegative sequence such that* $\{a_k b_k\}$ *is an increasing sequence. Then*

$$\left|\sum_{k=n}^{m} a_k f(kx)\right| \leq \frac{a_m}{\left|\sin\frac{x}{2}\right|}\left(\frac{b_m}{b_n}\right), \qquad (\forall n, m \in \mathbb{N},\ m > n).$$

Here $f(x) = \sin x$ *or* $f(x) = \cos x$.

The special cases of these inequalities were proved by G.K. Lebed for $b_k = k^s,\ s \geq 0$ (see [1]). Similarly, the results of Theorem C, were given by Mitrinović and Pečarić in [2, 3] for sums of type $\sum_{k=n}^{m}(-1)^k a_k f(kx)$, where again $f(x) = \sin x$ or $f(x) = \cos x$.

The sequence $\{u_k\}$ is said to be ***complex semiconvex*** if there exists a cone $K(\theta)$, such that $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$ or $\Delta^2(u_k b_k) \in K(\theta)$, where $\{b_k\}$ is a positive nondecreasing sequence. For $b_k = 1$, the sequence $\{u_k\}$ shall be called a ***complex convex sequence***.

In this paper we shall give some estimates for cosine and sine polynomials with complex semi-convex and complex convex coefficients.

2. MAIN RESULTS

Theorem 2.1. *Let $\{z_k\}$ be a sequence of complex numbers such that $A = \max_{n \le p \le q \le m} \left| \sum_{j=p}^{q} \sum_{k=i}^{j} z_k \right|$. Further, let $\{b_k\}$ be a positive nondecreasing sequence. If $\{u_k\}$ is a sequence of complex numbers such that $\Delta^2 \left(\frac{u_k}{b_k}\right) \in K(\theta)$, then*

$$\left| \sum_{k=n}^{m} u_k z_k \right| \le A \left[|u_m| + b_m \left(1 + \frac{1}{\cos\theta}\right) \left| \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \right| + \frac{b_m}{\cos\theta} \left| \Delta \left(\frac{u_n}{b_n}\right) \right| \right], \quad (\forall n, m \in \mathbb{N},\ m > n).$$

Proof. Let us estimate the sum $\sum_{k=n}^{m} b_k z_k$.

Since

$$\left| \sum_{k=n}^{m} z_k \right| \le \sum_{j=n+1}^{m} \left| \sum_{k=n}^{j} z_k \right| \le A,$$

we obtain

$$\begin{aligned} \left| \sum_{k=n}^{m} b_k z_k \right| &= \left| b_n \sum_{k=n}^{m} z_k + \sum_{j=n+1}^{m} \left(\sum_{k=j}^{m} z_k \right) (b_j - b_{j-1}) \right| \\ &\le b_n \left| \sum_{k=n}^{m} z_k \right| + \sum_{j=n+1}^{m} \left| \sum_{k=j}^{m} z_k \right| (b_j - b_{j-1}) \\ &\le A (b_n + b_m - b_n) = A b_m. \end{aligned} \tag{*}$$

Then,

$$\begin{aligned} \left| \sum_{k=n}^{m} u_k z_k \right| &= \left| \sum_{k=n}^{m} \frac{u_k}{b_k} (b_k z_k) \right| \\ &= \left| \frac{u_m}{b_m} \sum_{k=n}^{m} b_k z_k + \sum_{j=n}^{m-1} \left(\sum_{k=n}^{j} b_k z_k \right) \Delta \left(\frac{u_j}{b_j}\right) \right| \\ &= \left| \frac{u_m}{b_m} \sum_{k=n}^{m} b_k z_k + \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \sum_{j=n}^{m-1} \sum_{k=n}^{j} b_k z_k + \sum_{r=n}^{m-2} \Delta^2 \left(\frac{u_r}{b_r}\right) \sum_{j=n}^{r} \sum_{k=n}^{j} b_k z_k \right| \\ &\le \frac{|u_m|}{b_m} \left| \sum_{k=n}^{m} b_k z_k \right| + \left| \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \right| \left| \sum_{j=n}^{m-1} \sum_{k=n}^{j} b_k z_k \right| + \sum_{r=n}^{m-2} \left| \Delta^2 \left(\frac{u_r}{b_r}\right) \right| \left| \sum_{j=n}^{r} \sum_{k=n}^{j} b_k z_k \right| \\ &\le A b_m \frac{|u_m|}{b_m} + A b_m \left| \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \right| + \frac{A b_m}{\cos\theta} \left| \sum_{r=n}^{m-2} \Delta^2 \left(\frac{u_r}{b_r}\right) \right| \\ &= A \left[|u_m| + b_m \left| \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \right| + \frac{b_m}{\cos\theta} \left| \Delta \left(\frac{u_n}{b_n}\right) - \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \right| \right] \\ &\le A \left[|u_m| + b_m \left(1 + \frac{1}{\cos\theta}\right) \left| \Delta \left(\frac{u_{m-1}}{b_{m-1}}\right) \right| + \frac{b_m}{\cos\theta} \left| \Delta \left(\frac{u_n}{b_n}\right) \right| \right]. \end{aligned}$$

□

Theorem 2.2. *Let* $\{z_k\}$ *and* $\{b_k\}$ *be defined as in Theorem 2.1. If* $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta^2(u_k b_k) \in K(\theta)$ *, then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \le A\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\left(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|\right)\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

Proof. The sequence $\left\{b_k^{-1}\right\}_{k=n}^{m}$ is nonincreasing, so from (*) we get

$$\left|\sum_{k=n}^{m} b_k^{-1} z_k\right| \le A b_n^{-1}.$$

Now, we have:

$$\begin{aligned}
\left|\sum_{k=n}^{m} u_k z_k\right| &= \left|\sum_{k=n}^{m} (u_k b_k)\, b_k^{-1} z_k\right| \\
&= \left|u_n b_n \sum_{k=n}^{m} b_k^{-1} z_k + \sum_{j=n+1}^{m}\left(\sum_{k=j}^{m} b_k^{-1} z_k\right)(u_j b_j - u_{j-1}b_{j-1})\right| \\
&= \left|u_n b_n \sum_{k=n}^{m} b_k^{-1} z_k - \sum_{j=n+1}^{m-1} \Delta^2(u_{j-1}b_{j-1}) \sum_{r=n}^{j}\sum_{k=r}^{m} b_k^{-1} z_k \right.\\
&\quad \left. + \Delta(u_n b_n)\sum_{k=n}^{m} b_k^{-1} z_k - \Delta(u_{m-1}b_{m-1})\sum_{r=n}^{m}\sum_{k=r}^{m} b_k^{-1} z_k\right| \\
&\le |u_n|\, b_n \left|\sum_{k=n}^{m} b_k^{-1} z_k\right| + \sum_{j=n+1}^{m-1}\left|\Delta^2(u_{j-1}b_{j-1})\right|\left|\sum_{r=n}^{j}\sum_{k=r}^{m} b_k^{-1} z_k\right| \\
&\quad + |\Delta(u_n b_n)|\left|\sum_{k=n}^{m} b_k^{-1} z_k\right| + |\Delta(u_{m-1}b_{m-1})|\left|\sum_{r=n}^{m}\sum_{k=r}^{m} b_k^{-1} z_k\right| \\
&\le |u_n|\, b_n A b_n^{-1} + A b_n^{-1}\sum_{j=n+1}^{m-1}\left|\Delta^2(u_{j-1}b_{j-1})\right| + A b_n^{-1}|\Delta(u_n b_n)| \\
&\quad + A b_n^{-1}|\Delta(u_{m-1}b_{m-1})| \\
&\le A\left[|u_n| + \frac{b_n^{-1}}{\cos\theta}\left|\sum_{j=n+1}^{m-1}\Delta^2(u_{j-1}b_{j-1})\right| \right.\\
&\quad \left. + b_n^{-1}|\Delta(u_n b_n)| + b_n^{-1}|\Delta(u_{m-1}b_{m-1})|\right] \\
&= A\left[|u_n| + \frac{b_n^{-1}}{\cos\theta}|\Delta(u_n b_n) - \Delta(u_{m-1}b_{m-1})| \right.\\
&\quad \left. + b_n^{-1}|\Delta(u_n b_n)| + b_n^{-1}|\Delta(u_{m-1}b_{m-1})|\right] \\
&\le A\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\left(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|\right)\right].
\end{aligned}$$

□

Lemma 2.3. *For all $p,q \in \mathbb{N}$, $p<q$, the following inequalities hold*

$$\left|\sum_{j=p}^{q}\sum_{k=l}^{j} e^{ikx}\right| \leq \frac{q-p+2}{2\sin^2\frac{x}{2}}, \quad x \neq 2k\pi, \quad k=0,\pm1,\pm2,\dots, \tag{2.1}$$

$$\left|\sum_{j=p}^{q}\sum_{k=l}^{j} (-1)^k e^{ikx}\right| \leq \frac{q-p+2}{2\cos^2\frac{x}{2}}, \quad x \neq (2k+1)\pi, \quad k=0,\pm1,\pm2,\dots\ . \tag{2.2}$$

Proof. It is sufficient to prove the first inequality, since the second inequality can be proved analogously.

$$\begin{aligned}\left|\sum_{j=p}^{q}\sum_{k=l}^{j} e^{ikx}\right| &= \left|\sum_{j=p}^{q} e^{ilx}\frac{e^{i(j-l+1)x}-1}{e^{ix}-1}\right| \\ &= \frac{1}{|e^{ix}-1|}\left|\frac{1}{e^{i(l-1)x}}\sum_{j=p}^{q} e^{ijx}-(q-p+1)\right| \\ &\leq \frac{1}{\left|2\sin\frac{x}{2}\right|}\frac{\left|e^{i(q-p+1)}-1\right|}{|e^{ix}-1|}+\frac{q-p+1}{\left|2\sin\frac{x}{2}\right|} \\ &\leq \frac{2}{4\sin^2\frac{x}{2}}+\frac{q-p+1}{2\sin^2\frac{x}{2}}=\frac{q-p+2}{2\sin^2\frac{x}{2}}.\end{aligned}$$

□

By putting $z_k=\exp(ikx)$ in Theorem 2.1 and Theorem 2.2 and using the inequality (2.1) of the above lemma, we have:

Theorem 2.4. (i) *Let $\{b_k\}$ and $\{u_k\}$ be defined as in Theorem 2.1. Then*

$$\left|\sum_{k=n}^{m} u_k \exp(ikx)\right| \leq \frac{m-n+2}{2\sin^2\frac{x}{2}}\left[|u_m|+b_m\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|+\frac{b_m}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right], \quad (\forall n,m\in\mathbb{N},\ m>n).$$

(ii) *Let $\{b_k\}$ and $\{u_k\}$ be defined as in Theorem 2.2. Then*

$$\left|\sum_{k=n}^{m} u_k \exp(ikx)\right| \leq \frac{m-n+2}{2\sin^2\frac{x}{2}}\left[|u_n|+b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\left(|\Delta(u_nb_n)|+|\Delta(u_{m-1}b_{m-1})|\right)\right], \quad (\forall n,m\in\mathbb{N},\ m>n).$$

In both cases $x\neq 2k\pi,\ k=0,\pm1,\pm2,\dots$

Applying the known inequalities $\operatorname{Re} z \leq |z|$ and $\operatorname{Im} z \leq |z|$ for $z\in\mathbb{C}$, we obtain the following result:

Theorem 2.5. *Let $x\neq 2k\pi$ for $k=0,\pm1,\pm2,\dots$.*

(i) *Let* $\{b_k\}$ *and* $\{u_k\}$ *be defined as in Theorem 2.1. Then*

$$\left|\sum_{k=n}^{m} u_k f\left(kx\right)\right| \leq \frac{m-n+2}{2\sin^2\frac{x}{2}}\left[\left|u_m\right| + b_m\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| + \frac{b_m}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

(ii) *Let* $\{b_k\}$ *and* $\{u_k\}$ *be defined as in Theorem 2.2. Then*

$$\left|\sum_{k=n}^{m} u_k f\left(kx\right)\right| \leq \frac{m-n+2}{2\sin^2\frac{x}{2}}\left[\left|u_n\right| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\left(\left|\Delta\left(u_n b_n\right)\right| + \left|\Delta\left(u_{m-1}b_{m-1}\right)\right|\right)\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

Applying inequality (2.2) of Lemma 2.3, we obtain the following results:

Theorem 2.6. *Let* $x \neq (2k+1)\pi$ *for* $k = 0, \pm 1, \pm 2, \ldots$ *and let* $x \mapsto f(x)$ *be defined as in Theorem 2.5.*

(i) *If* $\{b_k\}$ *and* $\{u_k\}$ *are defined as in Theorem 2.1, then*

$$\left|\sum_{k=n}^{m} (-1)^k u_k f\left(kx\right)\right|$$
$$\leq \frac{m-n+2}{2\cos^2\frac{x}{2}}\left[\left|u_m\right| + b_m\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| + \frac{b_m}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

(ii) *If* $\{b_k\}$ *and* $\{u_k\}$ *are defined as in Theorem 2.2, then*

$$\left|\sum_{k=n}^{m} (-1)^k u_k f\left(kx\right)\right|$$
$$\leq \frac{m-n+2}{2\cos^2\frac{x}{2}}\left[\left|u_n\right| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\left(\left|\Delta\left(u_n b_n\right)\right| + \left|\Delta\left(u_{m-1}b_{m-1}\right)\right|\right)\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

For $b_k = 1$, we obtain the following theorem.

Theorem 2.7. *Let* $\{u_k\}$ *be a complex convex sequence.*

(i) *If* $x \neq 2k\pi$ *for* $k = 0, \pm 1, \pm 2, \ldots$*, then we have:*

$$\left|\sum_{k=n}^{m} u_k f\left(kx\right)\right| \leq \frac{m-n+2}{2\sin^2\frac{x}{2}}\left[\left|u_m\right| + \left(1+\frac{1}{\cos\theta}\right)\left|\Delta u_{m-1}\right| + \frac{1}{\cos\theta}\left|\Delta u_n\right|\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

(ii) *If* $x \neq (2k+1)\pi$ *for* $k = 0, \pm 1, \pm 2, \ldots$*, then we have:*

$$\left|\sum_{k=n}^{m} (-1)^k u_k f\left(kx\right)\right| \leq \frac{m-n+2}{2\cos^2\frac{x}{2}}\left[\left|u_n\right| + \left(1+\frac{1}{\cos\theta}\right)\left(\left|\Delta u_n\right| + \left|\Delta u_{m-1}\right|\right)\right],$$
$$(\forall n, m \in \mathbb{N},\ m > n).$$

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