A REFINEMENT OF HÖLDER'S INEQUALITY AND APPLICATIONS

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Abstract: In this paper, it is shown that a refinement of Hölder's inequality can be established using the positive definiteness of the Gram matrix. As applications, some improvements on Minkowski's inequality, Fan Ky's inequality and Hardy's inequality are given.

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1. Introduction

For convenience, we need to introduce the following notations which will be frequently used throughout the paper:

$$(a^r, b^s) = \sum_{n=1}^{\infty} a_n^r b_n^s, \quad \|a\|_r = \left(\sum_{n=1}^{\infty} a_n^r\right)^{\frac{1}{r}}, \quad \|a\|_2 = \|a\|,$$

$$(f^r, g^s) = \int_0^{\infty} f^r(x)\, g^s(x)\, dx, \quad \|f\|_r = \left(\int_0^{\infty} f^r(x)\, dx\right)^{\frac{1}{r}}, \quad \|f\|_2 = \|f\|,$$

and

$$S_r(\alpha, y) = (\alpha^{r/2}, y)\, \|\alpha\|_r^{-r/2},$$

where $a = (a_1, a_2, \dots)$ are sequences of real numbers, $f : [0,\infty) \to [0,\infty)$ are measurable functions and α and y are elements of an inner product space E of real sequences.

Let $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ be sequences of real numbers in $\mathbb{R}^n$. Then Hölder's inequality can be written in the form:

$$(a, b) \le \|a\|_p \|b\|_q. \tag{1.1}$$

The equality in (1.1) holds if and only if $a_i^p = k b_i^q$, $i = 1, 2, \dots$, where k is a constant.

This inequality is important in function theory, functional analysis, Fourier analysis and analytic number theory, etc. However, there are drawbacks in this inequality. For example, let

$$a = (a_1, a_2, \dots, a_n, 0, \dots, 0), \quad b = (0, 0, \dots, b_{n+1}, b_{n+2}, \dots, b_{2n}), \quad a, b \in \mathbb{R}^{2n}.$$

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If we let $a_i = b_j = 1, i = 1, 2, \dots, n; j = n+1, n+2, \dots, 2n$, and substitute them into (1.1), then we have $0 \le n$. In this case, Hölder's inequality is meaningless.

In the present paper we establish a new inequality that improves Hölder's inequality and remedies the defect pointed out above. At the same time, some significant refinements for a number of the classical inequalities can be established. As space is limited, only several applications of the new inequality are given.

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2. Main Results

Let α and β be elements of an inner product space E. Then the inner product of α and β is denoted by (α, β) and the norm of α is given by $\|\alpha\| = \sqrt{(\alpha, \alpha)}$. In our previous papers ([1], [2]), the following result has been obtained by means of the positive definiteness of the Gram matrix.

Lemma 2.1. *Let α, β and γ be three arbitrary vectors of E. If $\|\gamma\| = 1$, then*

$$|(\alpha, \beta)|^2 \leq \|\alpha\|^2 \|\beta\|^2 - (\|\alpha\| |x| - \|\beta\| |y|)^2 , \tag{2.1}$$

where $x = (\beta, \gamma), y = (\alpha, \gamma)$. The equality in (2.1) holds if and only if α and β are linearly dependent, or γ is a linear combination of α and β, and $xy = 0$ but x and y are not simultaneously equal to zero.

For the sake of completeness, we give here a short proof of (2.1), which can also be found in [2].

Proof of Lemma 2.1. Consider the Gram determinant constructed by the vectors α, β and γ:

$$G(\alpha, \beta, \gamma) = \begin{vmatrix} (\alpha, \alpha) & (\alpha, \beta) & (\alpha, \gamma) \\ (\beta, \alpha) & (\beta, \beta) & (\beta, \gamma) \\ (\gamma, \alpha) & (\gamma, \beta) & (\gamma, \gamma) \end{vmatrix} .$$

According to the positive definiteness of Gram matrix we have $G(\alpha, \beta, \gamma) \geq 0$, and $G(\alpha, \beta, \gamma) = 0$ if and only if the vectors α, β and γ are linearly dependent.

Expanding this determinant and using the condition $\|\gamma\| = 1$ we obtain

$$\begin{aligned} G(\alpha, \beta, \gamma) &= \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{\|\alpha\|^2 x^2 - 2(\alpha, \beta)xy + \|\beta\|^2 y^2\} \\ &\leq \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{\|\alpha\|^2 x^2 - 2|(\alpha, \beta)xy| + \|\beta\|^2 y^2\} \\ &\leq \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{\|\alpha\| |x| - \|\beta\| |y|\}^2 \end{aligned}$$

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where $x = (\beta, \gamma)$ and $y = (\alpha, \gamma)$. It follows that the equality holds if and only if the vectors α and β are linearly dependent; or the vector γ is a linear combination of the vector α and β, and $xy = 0$ but x and y are not simultaneously equal to zero. □

Applying Lemma 2.1, we can now establish the following refinement of Hölder's inequality.

Theorem 2.2. *Let* $a_n, b_n \geq 0$, $(n = 1, 2, \dots)$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \|a\|_p < +\infty$ *and* $0 < \|b\|_q < +\infty$, *then*

$$(a, b) \leq \|a\|_p \|b\|_q (1 - r)^m, \tag{2.2}$$

where

$$r = (S_p(a, c) - S_q(b, c))^2, \qquad m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}, \qquad \|c\| = 1$$

and

$$\left(a^{p/2}, c\right)\left(b^{q/2}, c\right) \geq 0.$$

The equality in (2.2) holds if and only if $a^{p/2}$*and* $b^{q/2}$ *are linearly dependent; or if the vector* c *is a linear combination of* $a^{p/2}$*and* $b^{q/2}$, *and* $\left(a^{p/2}, c\right)\left(b^{q/2}, c\right) = 0$, *but the vector* c *is not simultaneously orthogonal to* $a^{p/2}$ *and* $b^{q/2}$.

Proof. Firstly, we consider the case $p \neq q$. Without loss of generality, we suppose that $p > q > 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $R = \frac{p}{2}$, $Q = \frac{p}{p-2}$, then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain

$$(a, b) = \sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \left(a_k b_k^{q/p}\right) b_k^{1-q/p} \tag{2.3}$$

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$$\leq \left(\sum_{k=1}^{\infty} \left(a_k b_k^{q/p} \right)^R \right)^{\frac{1}{R}} \left(\sum_{k=1}^{\infty} \left(b_k^{1-q/p} \right)^Q \right)^{\frac{1}{Q}}$$
$$= \left(a^{p/2}, b^{q/2} \right)^{2/p} \|b\|_q^{q(1-2/p)} .$$

The equality in (2.3) holds if and only if $a^{p/2}$and $b^{q/2}$ are linearly dependent. In fact, the equality in (2.3) holds if and only if for any k, there exists c_0 $(c_0 \neq 0)$ such that

$$\left(a_k b_k^{q/p} \right)^R = c_0 \left(b_k^{1-q/p} \right)^Q .$$

It is easy to deduce that $a_k^{p/2} = c_0 b_k^{q/2}$.

If α, β and γ in (2.1) are replaced by $a^{p/2}, b^{q/2}$ and c respectively, then we have

(2.4) $$\left(a^{p/2}, b^{q/2} \right)^2 \leq \|a\|_p^p \|b\|_q^q (1 - r) ,$$

where $r = (S_p(a,c) - S_q(b,c))^2$. Substituting (2.4) into (2.3), we obtain after simplifications

(2.5) $$(a, b) \leq \|a\|_p \|b\|_q (1 - r)^{\frac{1}{p}} .$$

It is known from Lemma 2.1 that the equality in (2.5) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent; or if the vector c is a linear combination of $a^{p/2}$and $b^{q/2}$, and $\left(a^{p/2}, c \right) \left(b^{q/2}, c \right) = 0$, but the vector c is not simultaneously orthogonal to $a^{p/2}$ and $b^{q/2}$.

Note the symmetry of p and q. The inequality (2.2) follows from (2.5).

Secondly, consider the case $p = 2$. By Lemma 2.1, we obtain:

(2.6) $$(a, b) \leq \|a\| \|b\| (1 - r)^{\frac{1}{2}} ,$$

where $r = \left(\frac{(a,c)}{\|a\|} - \frac{(b,c)}{\|b\|} \right)^2$, $\|c\| = 1$ and $(a,c)(b,c) \geq 0$. The equality in (2.6) holds if and only if a and b are linearly dependent, or the vector c is a linear combination

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of a and b, and $(a,c)(b,c) = 0$, but (a,c) and (b,c) are not simultaneously equal to zero.

The proof of the theorem is thus completed. □

Consider the example given in the Introduction. Let $c = (c_1, c_2, \dots, c_{2n})$, $c \in \mathbb{R}^{2n}$, where $c_i = \frac{1}{\sqrt{n}}$, $i = 1, 2, \dots, n$ and $c_j = 0$, $j = n+1, n+2, \dots, 2n$. It is easy to deduce that $\|c\| = 1$ and $r = 1$. Substituting them into (2.2), it follows that the equality is valid.

The following theorem provides a similar result to Theorem 2.2.

Theorem 2.3. *Let $f(x), g(x) \geq 0$ $(x \in (0, +\infty))$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_q < +\infty$, then*

$$(f,g) \leq \|f\|_p \|g\|_q (1-r)^m, \tag{2.7}$$

where

$$r = (S_p(f,h) - S_q(g,h))^2, \qquad m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\},$$

$$\|h\| = 1, \quad \textit{i.e.} \quad \|h\| = \left(\int_0^\infty h^2(x)dx\right)^{\frac{1}{2}} = 1$$

and

$$\left(f^{p/2}, h\right)\left(g^{q/2}, h\right) \geq 0.$$

The equality in (2.3) holds if and only if $f^{p/2}$and $g^{q/2}$ are linearly dependent; or the vector h is a linear combination of $f^{p/2}$and $g^{q/2}$, and $\left(f^{p/2}, h\right)\left(g^{q/2}, h\right) = 0$, but the vector h is not simultaneously orthogonal to $f^{p/2}$ and $g^{q/2}$.

Its proof is similar to that of Theorem 2.2. Hence it is omitted.

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3. Applications

3.1. A Refinement of Minkowski's Inequality

We firstly give a refinement of Minkowski's inequality for the discrete form.

Theorem 3.1. *Let* $a_k, b_k \geq 0$, $p > 1$. *If* $0 < \|a\|_p < +\infty$ *and* $0 < \|b\|_p < +\infty$, *then*

$$\|a+b\|_p < \left(\|a\|_p + \|b\|_p\right)(1-r)^m, \tag{3.1}$$

where

$$\|a+b\|_p = \left(\sum_{k=1}^{\infty} (a_k+b_k)^p\right)^{\frac{1}{p}},$$

$$r = \min\left\{r(a), r(b)\right\}, \qquad m = \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\},$$

$$r(x) = \left\{\frac{\left(x^{p/2}, c\right)}{\|x\|_p^{p/2}} - \frac{\left((a+b)^{p/2}, c\right)}{\|a+b\|_p^{p/2}}\right\}^2, \quad x = a, b;$$

$$\left((a+b)^{p/2}, c\right) = \sum_{k=1}^{\infty} (a_k+b_k)^{p/2} c_k,$$

and c *is a variable unit-vector.*

Proof. Let $m = \min\left\{\frac{1}{p}, 1-\frac{1}{p}\right\}$,

$$\|a+b\|_p = \left(\sum_{k=1}^{\infty} (a_k+b_k)^p\right)^{\frac{1}{p}}.$$

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By Theorem 2.2, we have

$$\sum_{k=1}^{\infty} a_k \left(a_k + b_k\right)^{p-1} \le \|a\|_p \left(\sum_{k=1}^{\infty} \left(a_k + b_k\right)^p\right)^{1-\frac{1}{p}} \left(1 - r(a)\right)^m \tag{3.2}$$

and

$$\sum_{k=1}^{\infty} b_k \left(a_k + b_k\right)^{p-1} \le \|b\|_p \left(\sum_{k=1}^{\infty} \left(a_k + b_k\right)^p\right)^{1-\frac{1}{p}} \left(1 - r(b)\right)^m, \tag{3.3}$$

where

$$r\left(x\right) = \left\{\frac{\left(x^{p/2}, c\right)}{\|x\|_p^{p/2}} - \frac{\left((a+b)^{p/2}, c\right)}{\|a+b\|_p^{p/2}}\right\}^2, \qquad x = a, b,$$

$$\|a+b\|_p^{p/2} = \left(\sum_{k=1}^{\infty} \left(a_k + b_k\right)^p\right)^{\frac{1}{2}},$$

$$\left((a+b)^{p/2}, c\right) = \sum_{k=1}^{\infty} \left(a_k + b_k\right)^{p/2} c_k,$$

and c is a variable unit-vector.

Adding (3.5) and (3.3) we obtain, after simplifying:

$$\|a+b\|_p \le \|a\|_p \left(1 - r(a)\right)^m + \|b\|_p \left(1 - r(b)\right)^m. \tag{3.4}$$

Let $r = \min\left\{r\left(a\right), r\left(b\right)\right\}$, then the inequality (3.1) follows. This completes the proof of Theorem 3.1. □

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If we choose a unit-vector c such that its ith component is 1 and the rest is zero, i.e. $c = (0, 0, \dots, 0, \underset{(i)}{1}, 0, \dots)$, then

$$r(x) = \left\{ \frac{x_i^{p/2}}{\|x\|_p^{p/2}} - \frac{(a_i + b_i)^{p/2}}{\|a+b\|_p^{p/2}} \right\}^2 \qquad x = a, b.$$

Similarly, we can establish a refinement of Minkowski's integral inequality.

Theorem 3.2. *Let* $f(x), g(x) \geq 0$, $p > 1$. *If* $0 < \|f\|_p < +\infty$ *and* $0 < \|g\|_p < +\infty$, *then*

$$\|f + g\|_p < \left(\|f\|_p + \|g\|_p\right)(1 - r)^m, \tag{3.5}$$

where

$$\|f + g\|_p = \left(\int_0^\infty (f(x) + g(x))^p \, dx\right)^{\frac{1}{p}},$$

$$r = \min\{r(f), r(g)\}, \quad m = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\},$$

$$r(t) = \left\{ \frac{(t^{p/2}, h)}{\|t\|_p^{p/2}} - \frac{((f+g)^{p/2}, h)}{\|f+g\|_p^{p/2}} \right\}^2, \quad t = f, g,$$

$$((f+g)^{p/2}, h) = \int_0^\infty (f(x) + g(x))^{p/2} h(x) \, dx,$$

and h *is a variable unit-vector, i.e.*

$$\|h\| = \left\{\int_0^\infty h^2(x) \, dx\right\}^{\frac{1}{2}} = 1.$$

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Its proof is similar to that of Theorem 3.1. Hence it is omitted.

Remark 1. The variable unit-vector h can be chosen in accordance with our requirements. For example, we may choose h such that

$$h(x) = \sqrt{\frac{2}{\pi(1+x^2)}}.$$

3.2. A Strengthening of Fan Ky's Inequality

Theorem 3.3. *Let A, B and C be three positive definite matrices of order n, $0 \le \lambda \le 1$. Then*

$$|A|^{\lambda}|B|^{1-\lambda} \le |\lambda A + (1-\lambda)B| \left(1 - \left(\frac{|AC|^{\frac{1}{4}}}{\left|\frac{1}{2}(A+C)\right|^{\frac{1}{2}}} - \frac{|BC|^{\frac{1}{4}}}{\left|\frac{1}{2}(B+C)\right|^{\frac{1}{2}}}\right)^2\right)^m, \tag{3.6}$$

where $|C| = \pi^n, \quad m = \min\{\lambda, 1-\lambda\}.$

Proof. When $\lambda = 0, 1$, the inequality (3.3) is obviously valid. Hence we need only consider the case $0 < \lambda < 1$.

If D is a positive definite matrix of order n, then it is known from [4] that

$$J_n = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-(x,Dx)} dx = \frac{\pi^{n/2}}{|D|^{\frac{1}{2}}}, \tag{3.7}$$

where $x = (x_1, x_2, \dots, x_n)$, and $dx = dx_1 dx_2 \cdots dx_n$.

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Let $F(x) = e^{-\lambda(x,Ax)}$ and $G(x) = e^{-(1-\lambda)(x,Bx)}$. If $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$, according to (3.4) and (2.7) we have

(3.8)
$$\begin{aligned}
&\frac{\pi^{n/2}}{|\lambda A + (1-\lambda) B|^{\frac{1}{2}}} \\
&= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(x)\, G(x)\, dx \\
&\leq \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} G^q(x)\, dx \right\}^{\frac{1}{q}} (1-r)^m \\
&= \frac{\pi^{n/2} (1-r)^m}{\left(|A|^{\lambda} |B|^{1-\lambda} \right)^{\frac{1}{2}}},
\end{aligned}$$

where

$$\begin{aligned}
r &= \left(S_{\frac{1}{\lambda}}(F, H) - S_{\frac{1}{1-\lambda}}(G, H) \right)^2 \\
&= \left\{ \left(F^{\frac{1}{2\lambda}}, H \right) \|F\|_{\frac{1}{\lambda}}^{-\frac{1}{2\lambda}} - \left(G^{\frac{1}{2(1-\lambda)}}, H \right) \|G\|_{\frac{1}{1-\lambda}}^{-\frac{1}{2(1-\lambda)}} \right\},
\end{aligned}$$

where $H = e^{-\frac{1}{2}(x,Cx)}$, C is a positive definite matrix of order n, and

$$\|H\| = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H^2(x)\, dx \right\}^{\frac{1}{2}} = 1.$$

By the definition of the variable unit-vector H, it is easy to deduce that $|C| = \pi^n$.

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Hence we have

$$\left(F^{\frac{1}{2\lambda}},H\right)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}F^{\frac{1}{2\lambda}}\left(x\right)H\left(x\right)dx$$
$$=\frac{\pi^{n/2}}{\left|\frac{1}{2}\left(A+C\right)\right|^{\frac{1}{2}}}=\left\{\frac{|C|}{\left|\frac{1}{2}\left(A+C\right)\right|}\right\}^{\frac{1}{2}}$$

and

$$\|F\|_{1/\lambda}^{1/2\lambda}=\left\{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}F^{1/\lambda}\left(x\right)dx\right\}^{\frac{1}{2}}=\left\{\frac{\pi^{n/2}}{|A|^{1/2}}\right\}^{\frac{1}{2}}=\left\{\frac{|C|}{|A|}\right\}^{\frac{1}{4}},$$

whence

$$S_{1/\lambda}\left(F,H\right)=\frac{|AC|^{\frac{1}{4}}}{\left|\frac{1}{2}(A+C)\right|^{\frac{1}{2}}}.$$

Similarly,

$$S_{1/(1-\lambda)}\left(G,H\right)=\frac{|BC|^{\frac{1}{4}}}{\left|\frac{1}{2}\left(B+C\right)\right|^{\frac{1}{2}}},$$

therefore we obtain

$$r=\left(\frac{|AC|^{\frac{1}{4}}}{\left|\frac{1}{2}\left(A+C\right)\right|^{\frac{1}{2}}}-\frac{|BC|^{\frac{1}{4}}}{\left|\frac{1}{2}\left(B+C\right)\right|^{\frac{1}{2}}}\right)^{2}. \tag{3.9}$$

It follows from (3.8) and (3.9) that the inequality (3.3) is valid. □

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3.3. An Improvement of Hardy's Inequality

We give firstly a refinement of Hardy's inequality for the discrete form.

Theorem 3.4. *Let* $a_n \geq 0, \quad \beta_n = \frac{1}{n}\sum_{k=1}^{n} a_k$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$. *If* $0<\|a\|_p<+\infty$, *then*

$$\|\beta\|_p \leq \left(\frac{p}{p-1}\right)\|a\|_p (1-r)^m, \tag{3.10}$$

where

$$r = \left(\frac{(a^{p/2},c)}{\|a\|_p^{p/2}} - \frac{(\beta^{p/2},c)}{\|\beta\|_p^{p/2}}\right)^2,$$

c *is a variable unit-vector and* $m=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. Firstly, we estimate the difference of the following two terms:

$$\begin{aligned}\beta_n^p - \frac{p}{p-1}\beta_n^{p-1}a_n &= \beta_n^p - \frac{p}{p-1}\left(n\beta_n-(n-1)\beta_{n-1}\right)\beta_n^{p-1} \\ &= \beta_n^p\left(1-\frac{np}{p-1}\right)+\frac{(n-1)p}{p-1}\left((\beta_n^p)^{p-1}\beta_{n-1}^p\right)^{\frac{1}{p}}.\end{aligned} \tag{3.11}$$

Applying the arithmetic-geometric mean inequality to the second term on the right-hand side of (3.11) we get

$$\left((\beta_n^p)^{p-1}\beta_{n-1}^p\right)^{\frac{1}{p}} \leq \frac{1}{p}\left((p-1)\beta_n^p+\beta_{n-1}^p\right). \tag{3.12}$$

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It follows from (3.11) and (3.12) that

$$\beta_n^p - \frac{p}{p-1}\beta_n^{p-1}a_n \le \beta_n^p\left(1-\frac{np}{p-1}\right) + \frac{(n-1)}{p-1}\left((p-1)\beta_n^p + \beta_{n-1}^p\right) = \frac{1}{p-1}\left((n-1)\beta_{n-1}^p - n\beta_n^p\right).$$

Summing the above inequality with respect to n, we have

$$\sum_{n=1}^{N}\beta_n^p - \frac{p}{p-1}\sum_{n=1}^{N}\beta_n^{p-1}a_n \le -\frac{1}{p-1}\left(N\beta_N^p\right) \le 0.$$

Hence

$$\sum_{n=1}^{N}\beta_n^p \le \frac{p}{p-1}\sum_{n=1}^{N}\beta_n^{p-1}a_n.$$

Letting $N \to \infty$, we get

$$\sum_{n=1}^{\infty}\beta_n^p \le \frac{p}{p-1}\sum_{n=1}^{\infty}\beta_n^{p-1}a_n. \tag{3.13}$$

Applying the inequality (2.2) to the right-hand side of (3.13) we obtain

$$\frac{p}{p-1}\sum_{n=1}^{\infty}a_n\beta_n^{p-1} \le \frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\beta_n^{(p-1)q}\right)^{\frac{1}{q}}(1-r)^m = \frac{p}{p-1}\|a\|_p\left(\|\beta\|_p^p\right)^{\frac{1}{q}}(1-r)^m, \tag{3.14}$$

where $r = (S_p(a,c) - S_q(\beta^{p-1},c))^2$, c is a variable unit-vector and $m = \min\left\{\frac{1}{p},\frac{1}{q}\right\}$.

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We obtain from (3.13) and (3.14) after simplification

$$\|\beta\|_p \le \left(\frac{p}{p-1}\right) \|a\|_p (1-r)^m. \tag{3.15}$$

It is easy to deduce that

$$S_p(a,c) = \frac{(a^{p/2},c)}{\|a\|_p^{p/2}} \qquad \text{and} \qquad S_q(\beta^{p-1},c) = \frac{(\beta^{(p-1)q/2},c)}{\|\beta^{p-1}\|_q^{q/2}} = \frac{(\beta^{p/2},c)}{\|\beta\|_p^{p/2}}.$$

Hence

$$r = \left((a^{p/2},c)\|a\|_p^{-p/2} - (\beta^{p/2},c)\|\beta\|_p^{-p/2} \right)^2,$$

where c is a variable unit-vector. The proof of the theorem is completed. □

A variable unit-vector c can be chosen in accordance with our requirements. For example, we may choose $c \in \mathbb{R}^\infty$ such that $c = (1,0,0,\dots)$. Obviously, $\|c\| = 1$ and

$$r = a_1^p \left(\|a\|_p^{-p/2} - \|\beta\|_p^{-p/2} \right)^2.$$

Similarly, we can establish a refinement of Hardy's integral inequality.

Theorem 3.5. *Let* $f(x) \ge 0$, $g(x) = \frac{1}{x}\int_0^x f(t)dt$, $\frac{1}{p}+\frac{1}{q} = 1$ *and* $p > 1$. *If* $0 < \int_0^\infty f(t)dt < +\infty$, *then*

$$\|g\|_p < \frac{p}{p-1} \|f\|_p (1-r)^m, \tag{3.16}$$

where

$$r = \left(\frac{(f^{p/2},h)}{\|f\|_p^{p/2}} - \frac{(g^{p/2},h)}{\|g\|_p^{p/2}} \right)^2,$$

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h is a variable unit-vector, i.e.

$$\|h\| = \left(\int_0^\infty h^2(t)dt\right)^{\frac{1}{2}} = 1 \quad \textit{and} \quad m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}.$$

Proof. Using integration by parts and then applying (2.2) we obtain that

$$\|g\|_p^p = \int_0^\infty g^p(t)dt = \frac{p}{p-1}\left(f, g^{p-1}\right) \tag{3.17}$$
$$\leq \frac{p}{p-1}\|f\|_p \left\|g^{p-1}\right\|_q (1-r)^m$$
$$= \frac{p}{p-1}\|f\|_p \|g\|_p^{p-1} (1-r)^m,$$

where $r = \left(S_p(f,h) - S_q(g^{p-1},h)\right)^2$, $m = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and h is a variable unit-vector. It is easy to deduce that

$$S_p(f,h) = \frac{\left(f^{p/2}, h\right)}{\|f\|_p^{p/2}} \qquad \text{and} \qquad S_q\left(g^{p-1},h\right) = \frac{\left(g^{p/2}, h\right)}{\|g\|_p^{p/2}}.$$

It follows that the inequality (3.16) is valid. The theorem is thus proved. □

A variable unit-vector h can be chosen in accordance with our requirements. For example, we may choose h such that $h(x) = e^{-x/2}$. Obviously, we then have

$$\|h\| = \left(\int_0^\infty h^2(t)dt\right)^{\frac{1}{2}} = 1.$$

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