

A REFINEMENT OF HÖLDER'S INEQUALITY AND APPLICATIONS

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Abstract: In this paper, it is shown that a refinement of Hölder's inequality can be established using the positive definiteness of the Gram matrix. As applications, some improvements on Minkowski's inequality, Fan Ky's inequality and Hardy's inequality are given.

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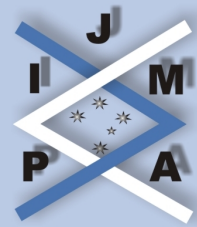
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1. Introduction

For convenience, we need to introduce the following notations which will be frequently used throughout the paper:

$$(a^r, b^s) = \sum_{n=1}^{\infty} a_n^r b_n^s, \quad \|a\|_r = \left(\sum_{n=1}^{\infty} a_n^r \right)^{\frac{1}{r}}, \quad \|a\|_2 = \|a\|,$$

$$(f^r, g^s) = \int_0^{\infty} f^r(x) g^s(x) dx, \quad \|f\|_r = \left(\int_0^{\infty} f^r(x) dx \right)^{\frac{1}{r}}, \quad \|f\|_2 = \|f\|,$$

and

$$S_r(\alpha, y) = (\alpha^{r/2}, y) \|\alpha\|_r^{-r/2},$$

where $a = (a_1, a_2, \dots)$ are sequences of real numbers, $f : [0, \infty) \rightarrow [0, \infty)$ are measurable functions and α and y are elements of an inner product space E of real sequences.

Let $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ be sequences of real numbers in \mathbb{R}^n . Then Hölder's inequality can be written in the form:

$$(1.1) \quad (a, b) \leq \|a\|_p \|b\|_q.$$

The equality in (1.1) holds if and only if $a_i^p = k b_i^q$, $i = 1, 2, \dots$, where k is a constant.

This inequality is important in function theory, functional analysis, Fourier analysis and analytic number theory, etc. However, there are drawbacks in this inequality. For example, let

$$a = (a_1, a_2, \dots, a_n, 0, \dots, 0), \quad b = (0, 0, \dots, b_{n+1}, b_{n+2}, \dots, b_{2n}), \quad a, b \in \mathbb{R}^{2n}.$$

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If we let $a_i = b_j = 1, i = 1, 2, \dots, n; j = n + 1, n + 2, \dots, 2n$, and substitute them into (1.1), then we have $0 \leq n$. In this case, Hölder's inequality is meaningless.

In the present paper we establish a new inequality that improves Hölder's inequality and remedies the defect pointed out above. At the same time, some significant refinements for a number of the classical inequalities can be established. As space is limited, only several applications of the new inequality are given.



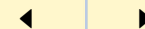
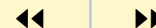
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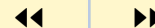
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2. Main Results

Let α and β be elements of an inner product space E . Then the inner product of α and β is denoted by (α, β) and the norm of α is given by $\|\alpha\| = \sqrt{(\alpha, \alpha)}$. In our previous papers ([1], [2]), the following result has been obtained by means of the positive definiteness of the Gram matrix.

Lemma 2.1. *Let α, β and γ be three arbitrary vectors of E . If $\|\gamma\| = 1$, then*

$$(2.1) \quad |(\alpha, \beta)|^2 \leq \|\alpha\|^2 \|\beta\|^2 - (\|\alpha\| |x| - \|\beta\| |y|)^2,$$

where $x = (\beta, \gamma)$, $y = (\alpha, \gamma)$. The equality in (2.1) holds if and only if α and β are linearly dependent, or γ is a linear combination of α and β , and $xy = 0$ but x and y are not simultaneously equal to zero.

For the sake of completeness, we give here a short proof of (2.1), which can also be found in [2].

Proof of Lemma 2.1. Consider the Gram determinant constructed by the vectors α, β and γ :

$$G(\alpha, \beta, \gamma) = \begin{vmatrix} (\alpha, \alpha) & (\alpha, \beta) & (\alpha, \gamma) \\ (\beta, \alpha) & (\beta, \beta) & (\beta, \gamma) \\ (\gamma, \alpha) & (\gamma, \beta) & (\gamma, \gamma) \end{vmatrix}.$$

According to the positive definiteness of Gram matrix we have $G(\alpha, \beta, \gamma) \geq 0$, and $G(\alpha, \beta, \gamma) = 0$ if and only if the vectors α, β and γ are linearly dependent.

Expanding this determinant and using the condition $\|\gamma\| = 1$ we obtain

$$\begin{aligned} G(\alpha, \beta, \gamma) &= \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{\|\alpha\|^2 x^2 - 2(\alpha, \beta)xy + \|\beta\|^2 y^2\} \\ &\leq \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{\|\alpha\|^2 x^2 - 2|(\alpha, \beta)xy| + \|\beta\|^2 y^2\} \\ &\leq \|\alpha\|^2 \|\beta\|^2 - (\alpha, \beta)^2 - \{\|\alpha\| |x| - \|\beta\| |y|\}^2 \end{aligned}$$



where $x = (\beta, \gamma)$ and $y = (\alpha, \gamma)$. It follows that the equality holds if and only if the vectors α and β are linearly dependent; or the vector γ is a linear combination of the vector α and β , and $xy = 0$ but x and y are not simultaneously equal to zero. \square

Applying Lemma 2.1, we can now establish the following refinement of Hölder's inequality.

Theorem 2.2. Let $a_n, b_n \geq 0$, ($n = 1, 2, \dots$), $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|a\|_p < +\infty$ and $0 < \|b\|_q < +\infty$, then

$$(2.2) \quad (a, b) \leq \|a\|_p \|b\|_q (1 - r)^m,$$

where

$$r = (S_p(a, c) - S_q(b, c))^2, \quad m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \quad \|c\| = 1$$

and

$$(a^{p/2}, c) (b^{q/2}, c) \geq 0.$$

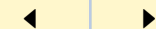
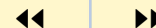
The equality in (2.2) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent; or if the vector c is a linear combination of $a^{p/2}$ and $b^{q/2}$, and $(a^{p/2}, c) (b^{q/2}, c) = 0$, but the vector c is not simultaneously orthogonal to $a^{p/2}$ and $b^{q/2}$.

Proof. Firstly, we consider the case $p \neq q$. Without loss of generality, we suppose that $p > q > 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $R = \frac{p}{2}$, $Q = \frac{p}{p-2}$, then $\frac{1}{R} + \frac{1}{Q} = 1$. By Hölder's inequality we obtain

$$(2.3) \quad (a, b) = \sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \left(a_k b_k^{q/p} \right) b_k^{1-q/p}$$

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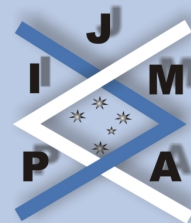
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$$\begin{aligned} &\leq \left(\sum_{k=1}^{\infty} \left(a_k b_k^{q/p} \right)^R \right)^{\frac{1}{R}} \left(\sum_{k=1}^{\infty} \left(b_k^{1-q/p} \right)^Q \right)^{\frac{1}{Q}} \\ &= \left(a^{p/2}, b^{q/2} \right)^{2/p} \|b\|_q^{q(1-2/p)}. \end{aligned}$$

The equality in (2.3) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent. In fact, the equality in (2.3) holds if and only if for any k , there exists c_0 ($c_0 \neq 0$) such that

$$\left(a_k b_k^{q/p} \right)^R = c_0 \left(b_k^{1-q/p} \right)^Q.$$

It is easy to deduce that $a_k^{p/2} = c_0 b_k^{q/2}$.

If α, β and γ in (2.1) are replaced by $a^{p/2}, b^{q/2}$ and c respectively, then we have

$$(2.4) \quad \left(a^{p/2}, b^{q/2} \right)^2 \leq \|a\|_p^p \|b\|_q^q (1-r),$$

where $r = (S_p(a, c) - S_q(b, c))^2$. Substituting (2.4) into (2.3), we obtain after simplifications

$$(2.5) \quad (a, b) \leq \|a\|_p \|b\|_q (1-r)^{\frac{1}{p}}.$$

It is known from Lemma 2.1 that the equality in (2.5) holds if and only if $a^{p/2}$ and $b^{q/2}$ are linearly dependent; or if the vector c is a linear combination of $a^{p/2}$ and $b^{q/2}$, and $(a^{p/2}, c)(b^{q/2}, c) = 0$, but the vector c is not simultaneously orthogonal to $a^{p/2}$ and $b^{q/2}$.

Note the symmetry of p and q . The inequality (2.2) follows from (2.5).

Secondly, consider the case $p = 2$. By Lemma 2.1, we obtain:

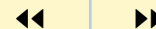
$$(2.6) \quad (a, b) \leq \|a\| \|b\| (1-r)^{\frac{1}{2}},$$

where $r = \left(\frac{(a,c)}{\|a\|} - \frac{(b,c)}{\|b\|} \right)^2$, $\|c\| = 1$ and $(a, c)(b, c) \geq 0$. The equality in (2.6) holds if and only if a and b are linearly dependent, or the vector c is a linear combination



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of a and b , and $(a, c)(b, c) = 0$, but (a, c) and (b, c) are not simultaneously equal to zero.

The proof of the theorem is thus completed. \square

Consider the example given in the Introduction. Let $c = (c_1, c_2, \dots, c_{2n})$, $c \in \mathbb{R}^{2n}$, where $c_i = \frac{1}{\sqrt{n}}$, $i = 1, 2, \dots, n$ and $c_j = 0$, $j = n + 1, n + 2, \dots, 2n$. It is easy to deduce that $\|c\| = 1$ and $r = 1$. Substituting them into (2.2), it follows that the equality is valid.

The following theorem provides a similar result to Theorem 2.2.

Theorem 2.3. Let $f(x), g(x) \geq 0$ ($x \in (0, +\infty)$), $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_q < +\infty$, then

$$(2.7) \quad (f, g) \leq \|f\|_p \|g\|_q (1 - r)^m,$$

where

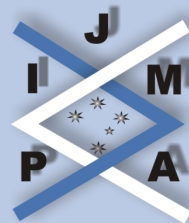
$$r = (S_p(f, h) - S_q(g, h))^2, \quad m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\},$$
$$\|h\| = 1, \quad \text{i.e.} \quad \|h\| = \left(\int_0^\infty h^2(x) dx \right)^{\frac{1}{2}} = 1$$

and

$$(f^{p/2}, h) (g^{q/2}, h) \geq 0.$$

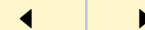
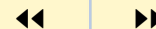
The equality in (2.3) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent; or the vector h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{p/2}, h) (g^{q/2}, h) = 0$, but the vector h is not simultaneously orthogonal to $f^{p/2}$ and $g^{q/2}$.

Its proof is similar to that of Theorem 2.2. Hence it is omitted.



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3. Applications

3.1. A Refinement of Minkowski's Inequality

We firstly give a refinement of Minkowski's inequality for the discrete form.

Theorem 3.1. Let $a_k, b_k \geq 0$, $p > 1$. If $0 < \|a\|_p < +\infty$ and $0 < \|b\|_p < +\infty$, then

$$(3.1) \quad \|a + b\|_p < \left(\|a\|_p + \|b\|_p \right) (1 - r)^m,$$

where

$$\|a + b\|_p = \left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{\frac{1}{p}},$$
$$r = \min \{r(a), r(b)\}, \quad m = \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\},$$
$$r(x) = \left\{ \frac{(x^{p/2}, c)}{\|x\|_p^{p/2}} - \frac{((a + b)^{p/2}, c)}{\|a + b\|_p^{p/2}} \right\}^2, \quad x = a, b;$$
$$((a + b)^{p/2}, c) = \sum_{k=1}^{\infty} (a_k + b_k)^{p/2} c_k,$$

and c is a variable unit-vector.

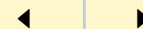
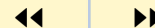
Proof. Let $m = \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\}$,

$$\|a + b\|_p = \left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{\frac{1}{p}}.$$



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By Theorem 2.2, we have

$$(3.2) \quad \sum_{k=1}^{\infty} a_k (a_k + b_k)^{p-1} \leq \|a\|_p \left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{1-\frac{1}{p}} (1 - r(a))^m$$

and

$$(3.3) \quad \sum_{k=1}^{\infty} b_k (a_k + b_k)^{p-1} \leq \|b\|_p \left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{1-\frac{1}{p}} (1 - r(b))^m,$$

where

$$r(x) = \left\{ \frac{(x^{p/2}, c)}{\|x\|_p^{p/2}} - \frac{((a+b)^{p/2}, c)}{\|a+b\|_p^{p/2}} \right\}^2, \quad x = a, b,$$

$$\|a+b\|_p^{p/2} = \left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{\frac{1}{2}},$$

$$((a+b)^{p/2}, c) = \sum_{k=1}^{\infty} (a_k + b_k)^{p/2} c_k,$$

and c is a variable unit-vector.

Adding (3.5) and (3.3) we obtain, after simplifying:

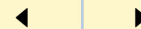
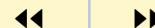
$$(3.4) \quad \|a+b\|_p \leq \|a\|_p (1 - r(a))^m + \|b\|_p (1 - r(b))^m.$$

Let $r = \min\{r(a), r(b)\}$, then the inequality (3.1) follows. This completes the proof of Theorem 3.1. \square



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If we choose a unit-vector c such that its i th component is 1 and the rest is zero, i.e. $c = (0, 0, \dots, 0, \underset{(i)}{1}, 0, \dots)$, then

$$r(x) = \left\{ \frac{x_i^{p/2}}{\|x\|_p^{p/2}} - \frac{(a_i + b_i)^{p/2}}{\|a + b\|_p^{p/2}} \right\}^2 \quad x = a, b.$$

Similarly, we can establish a refinement of Minkowski's integral inequality.

Theorem 3.2. Let $f(x), g(x) \geq 0$, $p > 1$. If $0 < \|f\|_p < +\infty$ and $0 < \|g\|_p < +\infty$, then

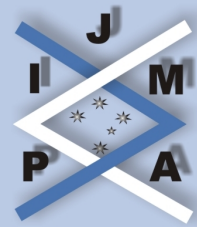
$$(3.5) \quad \|f + g\|_p < \left(\|f\|_p + \|g\|_p \right) (1 - r)^m,$$

where

$$\|f + g\|_p = \left(\int_0^\infty (f(x) + g(x))^p dx \right)^{\frac{1}{p}},$$
$$r = \min \{r(f), r(g)\}, \quad m = \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\},$$
$$r(t) = \left\{ \frac{(t^{p/2}, h)}{\|t\|_p^{p/2}} - \frac{((f + g)^{p/2}, h)}{\|f + g\|_p^{p/2}} \right\}^2, \quad t = f, g,$$
$$((f + g)^{p/2}, h) = \int_0^\infty (f(x) + g(x))^{p/2} h(x) dx,$$

and h is a variable unit-vector, i.e.

$$\|h\| = \left\{ \int_0^\infty h^2(x) dx \right\}^{\frac{1}{2}} = 1.$$



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Its proof is similar to that of Theorem 3.1. Hence it is omitted.

Remark 1. The variable unit-vector h can be chosen in accordance with our requirements. For example, we may choose h such that

$$h(x) = \sqrt{\frac{2}{\pi(1+x^2)}}.$$

3.2. A Strengthening of Fan Ky's Inequality

Theorem 3.3. Let A, B and C be three positive definite matrices of order n , $0 \leq \lambda \leq 1$. Then

$$(3.6) \quad |A|^\lambda |B|^{1-\lambda} \leq |\lambda A + (1-\lambda)B| \left(1 - \left(\frac{|AC|^{\frac{1}{4}}}{|\frac{1}{2}(A+C)|^{\frac{1}{2}}} - \frac{|BC|^{\frac{1}{4}}}{|\frac{1}{2}(B+C)|^{\frac{1}{2}}} \right)^2 \right)^m,$$

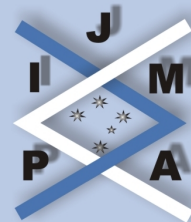
where $|C| = \pi^n$, $m = \min\{\lambda, 1-\lambda\}$.

Proof. When $\lambda = 0, 1$, the inequality (3.3) is obviously valid. Hence we need only consider the case $0 < \lambda < 1$.

If D is a positive definite matrix of order n , then it is known from [4] that

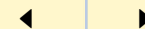
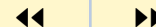
$$(3.7) \quad J_n = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-(x, Dx)} dx = \frac{\pi^{n/2}}{|D|^{\frac{1}{2}}},$$

where $x = (x_1, x_2, \dots, x_n)$, and $dx = dx_1 dx_2 \cdots dx_n$.



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Let $F(x) = e^{-\lambda(x, Ax)}$ and $G(x) = e^{-(1-\lambda)(x, Bx)}$. If $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$, according to (3.4) and (2.7) we have

$$\begin{aligned}
 (3.8) \quad & \frac{\pi^{n/2}}{|\lambda A + (1-\lambda)B|^{\frac{1}{2}}} \\
 &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(x) G(x) dx \\
 &\leq \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} G^q(x) dx \right\}^{\frac{1}{q}} (1-r)^m \\
 &= \frac{\pi^{n/2} (1-r)^m}{\left(|A|^\lambda |B|^{1-\lambda} \right)^{\frac{1}{2}}},
 \end{aligned}$$

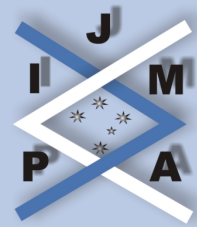
where

$$\begin{aligned}
 r &= \left(S_{\frac{1}{\lambda}}(F, H) - S_{\frac{1}{1-\lambda}}(G, H) \right)^2 \\
 &= \left\{ \left(F^{\frac{1}{2\lambda}}, H \right) \|F\|_{\frac{1}{\lambda}}^{-\frac{1}{2\lambda}} - \left(G^{\frac{1}{2(1-\lambda)}}, H \right) \|G\|_{\frac{1}{1-\lambda}}^{-\frac{1}{2(1-\lambda)}} \right\},
 \end{aligned}$$

where $H = e^{-\frac{1}{2}(x, Cx)}$, C is a positive definite matrix of order n , and

$$\|H\| = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H^2(x) dx \right\}^{\frac{1}{2}} = 1.$$

By the definition of the variable unit-vector H , it is easy to deduce that $|C| = \pi^n$.



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Hence we have

$$\begin{aligned} \left(F^{\frac{1}{2\lambda}}, H \right) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^{\frac{1}{2\lambda}}(x) H(x) dx \\ &= \frac{\pi^{n/2}}{\left| \frac{1}{2}(A+C) \right|^{\frac{1}{2}}} = \left\{ \frac{|C|}{\left| \frac{1}{2}(A+C) \right|} \right\}^{\frac{1}{2}} \end{aligned}$$

and

$$\|F\|_{1/\lambda}^{1/2\lambda} = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F^{1/\lambda}(x) dx \right\}^{\frac{1}{2}} = \left\{ \frac{\pi^{n/2}}{|A|^{1/2}} \right\}^{\frac{1}{2}} = \left\{ \frac{|C|}{|A|} \right\}^{\frac{1}{4}},$$

whence

$$S_{1/\lambda}(F, H) = \frac{|AC|^{\frac{1}{4}}}{\left| \frac{1}{2}(A+C) \right|^{\frac{1}{2}}}.$$

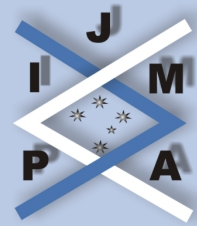
Similarly,

$$S_{1/(1-\lambda)}(G, H) = \frac{|BC|^{\frac{1}{4}}}{\left| \frac{1}{2}(B+C) \right|^{\frac{1}{2}}},$$

therefore we obtain

$$(3.9) \quad r = \left(\frac{|AC|^{\frac{1}{4}}}{\left| \frac{1}{2}(A+C) \right|^{\frac{1}{2}}} - \frac{|BC|^{\frac{1}{4}}}{\left| \frac{1}{2}(B+C) \right|^{\frac{1}{2}}} \right)^2.$$

It follows from (3.8) and (3.9) that the inequality (3.3) is valid. \square



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3.3. An Improvement of Hardy's Inequality

We give firstly a refinement of Hardy's inequality for the discrete form.

Theorem 3.4. Let $a_n \geq 0$, $\beta_n = \frac{1}{n} \sum_{k=1}^n a_k$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \|a\|_p < +\infty$, then

$$(3.10) \quad \|\beta\|_p \leq \left(\frac{p}{p-1} \right) \|a\|_p (1-r)^m,$$

where

$$r = \left(\frac{(a^{p/2}, c)}{\|a\|_p^{p/2}} - \frac{(\beta^{p/2}, c)}{\|\beta\|_p^{p/2}} \right)^2,$$

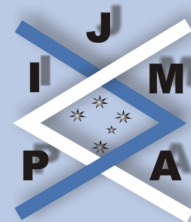
c is a variable unit-vector and $m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

Proof. Firstly, we estimate the difference of the following two terms:

$$(3.11) \quad \beta_n^p - \frac{p}{p-1} \beta_n^{p-1} a_n = \beta_n^p - \frac{p}{p-1} (n\beta_n - (n-1)\beta_{n-1}) \beta_n^{p-1} \\ = \beta_n^p \left(1 - \frac{np}{p-1} \right) + \frac{(n-1)p}{p-1} ((\beta_n^p)^{p-1} \beta_{n-1}^p)^{\frac{1}{p}}.$$

Applying the arithmetic-geometric mean inequality to the second term on the right-hand side of (3.11) we get

$$(3.12) \quad ((\beta_n^p)^{p-1} \beta_{n-1}^p)^{\frac{1}{p}} \leq \frac{1}{p} ((p-1)\beta_n^p + \beta_{n-1}^p).$$



It follows from (3.11) and (3.12) that

$$\begin{aligned} \beta_n^p - \frac{p}{p-1} \beta_n^{p-1} a_n &\leq \beta_n^p \left(1 - \frac{np}{p-1}\right) + \frac{(n-1)}{p-1} ((p-1)\beta_n^p + \beta_{n-1}^p) \\ &= \frac{1}{p-1} ((n-1)\beta_{n-1}^p - n\beta_n^p). \end{aligned}$$

Summing the above inequality with respect to n , we have

$$\sum_{n=1}^N \beta_n^p - \frac{p}{p-1} \sum_{n=1}^N \beta_n^{p-1} a_n \leq -\frac{1}{p-1} (N\beta_N^p) \leq 0.$$

Hence

$$\sum_{n=1}^N \beta_n^p \leq \frac{p}{p-1} \sum_{n=1}^N \beta_n^{p-1} a_n.$$

Letting $N \rightarrow \infty$, we get

$$(3.13) \quad \sum_{n=1}^{\infty} \beta_n^p \leq \frac{p}{p-1} \sum_{n=1}^{\infty} \beta_n^{p-1} a_n.$$

Applying the inequality (2.2) to the right-hand side of (3.13) we obtain

$$\begin{aligned} (3.14) \quad \frac{p}{p-1} \sum_{n=1}^{\infty} a_n \beta_n^{p-1} &\leq \frac{p}{p-1} \left(\sum_{n=1}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \beta_n^{(p-1)q} \right)^{\frac{1}{q}} (1-r)^m \\ &= \frac{p}{p-1} \|a\|_p \left(\|\beta\|_p^p \right)^{\frac{1}{q}} (1-r)^m, \end{aligned}$$

where $r = (S_p(a, c) - S_q(\beta^{p-1}, c))^2$, c is a variable unit-vector and $m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$.

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We obtain from (3.13) and (3.14) after simplification

$$(3.15) \quad \|\beta\|_p \leq \left(\frac{p}{p-1}\right) \|a\|_p (1-r)^m.$$

It is easy to deduce that

$$S_p(a, c) = \frac{(a^{p/2}, c)}{\|a\|_p^{p/2}} \quad \text{and} \quad S_q(\beta^{p-1}, c) = \frac{(\beta^{(p-1)q/2}, c)}{\|\beta^{p-1}\|_q^{q/2}} = \frac{(\beta^{p/2}, c)}{\|\beta\|_p^{p/2}}.$$

Hence

$$r = \left((a^{p/2}, c) \|a\|_p^{-p/2} - (\beta^{p/2}, c) \|\beta\|_p^{-p/2} \right)^2,$$

where c is a variable unit-vector. The proof of the theorem is completed. \square

A variable unit-vector c can be chosen in accordance with our requirements. For example, we may choose $c \in \mathbb{R}^\infty$ such that $c = (1, 0, 0, \dots)$. Obviously, $\|c\| = 1$ and

$$r = a_1^p \left(\|a\|_p^{-p/2} - \|\beta\|_p^{-p/2} \right)^2.$$

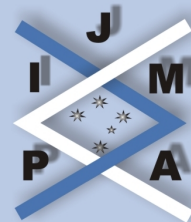
Similarly, we can establish a refinement of Hardy's integral inequality.

Theorem 3.5. Let $f(x) \geq 0$, $g(x) = \frac{1}{x} \int_0^x f(t)dt$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. If $0 < \int_0^\infty f(t)dt < +\infty$, then

$$(3.16) \quad \|g\|_p < \frac{p}{p-1} \|f\|_p (1-r)^m,$$

where

$$r = \left(\frac{(f^{p/2}, h)}{\|f\|_p^{p/2}} - \frac{(g^{p/2}, h)}{\|g\|_p^{p/2}} \right)^2,$$



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h is a variable unit-vector, i.e.

$$\|h\| = \left(\int_0^\infty h^2(t) dt \right)^{\frac{1}{2}} = 1 \quad \text{and} \quad m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}.$$

Proof. Using integration by parts and then applying (2.2) we obtain that

$$\begin{aligned} (3.17) \quad \|g\|_p^p &= \int_0^\infty g^p(t) dt = \frac{p}{p-1} (f, g^{p-1}) \\ &\leq \frac{p}{p-1} \|f\|_p \|g^{p-1}\|_q (1-r)^m \\ &= \frac{p}{p-1} \|f\|_p \|g\|_p^{p-1} (1-r)^m, \end{aligned}$$

where $r = (S_p(f, h) - S_q(g^{p-1}, h))^2$, $m = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ and h is a variable unit-vector. It is easy to deduce that

$$S_p(f, h) = \frac{(f^{p/2}, h)}{\|f\|_p^{p/2}} \quad \text{and} \quad S_q(g^{p-1}, h) = \frac{(g^{p/2}, h)}{\|g\|_p^{p/2}}.$$

It follows that the inequality (3.16) is valid. The theorem is thus proved. \square

A variable unit-vector h can be chosen in accordance with our requirements. For example, we may choose h such that $h(x) = e^{-x/2}$. Obviously, we then have

$$\|h\| = \left(\int_0^\infty h^2(t) dt \right)^{\frac{1}{2}} = 1.$$

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