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SOME BOAS-BELLMAN TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES

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Abstract

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Abstract

Some inequalities in 2-inner product spaces generalizing Bessel's result that are similar to the Boas-Bellman inequality from inner product spaces, are given. Applications for determinantal integral inequalities are also provided.

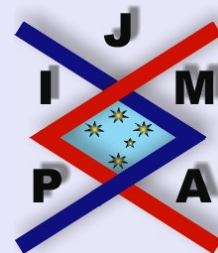
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1. Introduction

Let $(H; (\cdot, \cdot))$ be an inner product space over the real or complex number field \mathbb{K} . If $(e_i)_{1 \leq i \leq n}$ are orthonormal vectors in the inner product space H , i.e., $(e_i, e_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [9, p. 391]):

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2$$

for any $x \in H$.

For other results related to Bessel's inequality, see [5] – [7] and Chapter XV in the book [9].

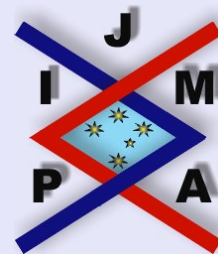
In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality (see also [9, p. 392]).

Theorem 1.1. *If x, y_1, \dots, y_n are elements of an inner product space $(H; (\cdot, \cdot))$, then the following inequality:*

$$\sum_{i=1}^n |(x, y_i)|^2 \leq \|x\|^2 \left[\max_{1 \leq i \leq n} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right]$$

holds.

It is the main aim of the present paper to point out the corresponding version of Boas-Bellman inequality in 2-inner product spaces. Some natural generalizations and related results are also pointed out. Applications for determinantal integral inequalities are provided.



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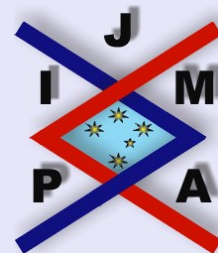
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For a comprehensive list of fundamental results on 2-inner product spaces and linear 2-normed spaces, see the recent books [3] and [8] where further references are given.



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2. Bessel's Inequality in 2-Inner Product Spaces

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [3]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot | \cdot)$ is a \mathbb{K} -valued function defined on $X \times X \times X$ satisfying the following conditions:

(2I₁) $(x, x | z) \geq 0$ and $(x, x | z) = 0$ if and only if x and z are linearly dependent;

(2I₂) $(x, x | z) = (z, z | x)$,

(2I₃) $(y, x | z) = \overline{(x, y | z)}$,

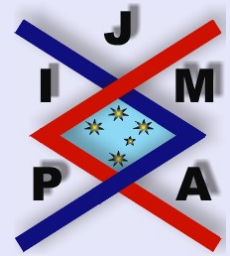
(2I₄) $(\alpha x, y | z) = \alpha(x, y | z)$ for any scalar $\alpha \in \mathbb{K}$,

(2I₅) $(x + x', y | z) = (x, y | z) + (x', y | z)$.

$(\cdot, \cdot | \cdot)$ is called a *2-inner product* on X and $(X, (\cdot, \cdot | \cdot))$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner products $(\cdot, \cdot | \cdot)$ can be immediately obtained as follows [4]:

(1) If $\mathbb{K} = \mathbb{R}$, then (2I₃) reduces to

$$(y, x | z) = (x, y | z).$$



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(2) From $(2I_3)$ and $(2I_4)$, we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and

$$(2.1) \quad (x, \alpha y|z) = \bar{\alpha}(x, y|z).$$

(3) Using $(2I_2) - (2I_5)$, we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2 \operatorname{Re}(x, y|z)$$

and

$$(2.2) \quad \operatorname{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)].$$

In the real case, (2.2) reduces to

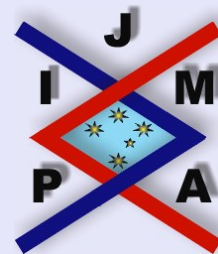
$$(2.3) \quad (x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$,

$$(2.4) \quad (x, y|\alpha z) = \alpha^2(x, y|z).$$

In the complex case, using (2.1) and (2.2), we have

$$\operatorname{Im}(x, y|z) = \operatorname{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$



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which, in combination with (2.2), yields

$$(2.5) \quad (x, y|z) = \frac{1}{4}[(z, z|x+y) - (z, z|x-y)] + \frac{i}{4}[(z, z|x+iy) - (z, z|x-iy)].$$

Using the above formula and (2.1), we have, for any $\alpha \in \mathbb{C}$,

$$(2.6) \quad (x, y|\alpha z) = |\alpha|^2(x, y|z).$$

However, for $\alpha \in \mathbb{R}$, (2.6) reduces to (2.4). Also, from (2.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(2.7) \quad (y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0.$$

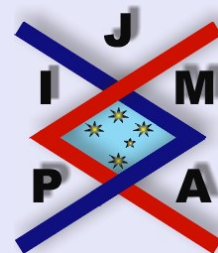
For $x = z$, (2.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(2.8) \quad (z, y|z) = (y, z|z) = 0,$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y, z \in X$.



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Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (2.7), it follows that

$$(2.9) \quad |(x, y|z)|^2 \leq (x, x|z)(y, y|z).$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot | \cdot))$, we can define a function $\|\cdot\|\cdot\|$ on $X \times X$ by

$$(2.10) \quad \|x|z\| = \sqrt{(x, x|z)}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

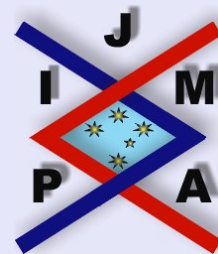
(2N₁) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

(2N₂) $\|z|x\| = \|x|z\|$,

(2N₃) $\|\alpha x|z\| = |\alpha| \|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

(2N₄) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot\|\cdot\|$ defined on $X \times X$ and satisfying the conditions (2N₁) – (2N₄) is called a 2-norm on X and $(X, \|\cdot\|\cdot\|)$ is called a linear 2-normed



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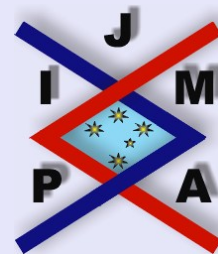
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space [8]. Whenever a 2-inner product space $(X, (\cdot, \cdot | \cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot\| \cdot \|\cdot\|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot, \cdot | \cdot))$ be a 2-inner product space over the real or complex number field \mathbb{K} . If $(e_i)_{1 \leq i \leq n}$ are linearly independent vectors in the 2-inner product space X , and, for a given $z \in X$, $(e_i, e_j | z) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(e_i)_{1 \leq i \leq n}$ is z -orthonormal), then the following inequality is the corresponding Bessel's inequality (see for example [4]) for the z -orthonormal family $(e_i)_{1 \leq i \leq n}$ in the 2-inner product space $(X; (\cdot, \cdot | \cdot))$:

$$(2.11) \quad \sum_{i=1}^n |(x, e_i | z)|^2 \leq \|x|z\|^2$$

for any $x \in X$. For more details about this inequality, see the recent paper [4] and the references therein.



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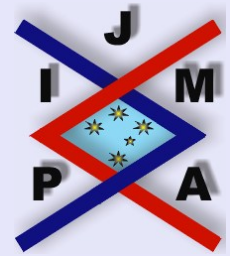
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3. Some Inequalities for 2-Norms

We start with the following lemma which is also interesting in itself.

Lemma 3.1. *Let $z_1, \dots, z_n, z \in X$ and $\mu_1, \dots, \mu_n \in \mathbb{K}$. Then one has the inequality:*

$$(3.1) \quad \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\mu_i|^2 \sum_{i=1}^n \|z_i |z|\|^2; \\ \left(\sum_{i=1}^n |\mu_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n \|z_i |z|\|^{2\beta} \right)^{\frac{1}{\beta}}, \\ \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\mu_i|^2 \max_{1 \leq i \leq n} \|z_i |z|\|^2, \\ \max_{1 \leq i \neq j \leq n} \{|\mu_i \mu_j|\} \sum_{1 \leq i \neq j \leq n} |(z_i, z_j |z)|; \\ + \begin{cases} \left[\left(\sum_{i=1}^n |\mu_i|^\gamma \right)^2 - \left(\sum_{i=1}^n |\mu_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j |z)|^\delta \right)^{\frac{1}{\delta}}, \\ \quad \text{where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[\left(\sum_{i=1}^n |\mu_i| \right)^2 - \sum_{i=1}^n |\mu_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(z_i, z_j |z)|. \end{cases} \end{cases}$$



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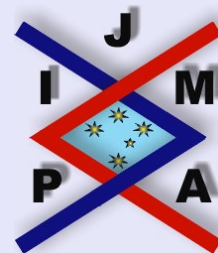
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Proof. We observe that

$$\begin{aligned}
 (3.2) \quad \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 &= \left(\sum_{i=1}^n \mu_i z_i, \sum_{j=1}^n \mu_j z_j |z| \right) \\
 &= \sum_{i=1}^n \sum_{j=1}^n \mu_i \overline{\mu_j} (z_i, z_j |z|) \\
 &= \left| \sum_{i=1}^n \sum_{j=1}^n \mu_i \overline{\mu_j} (z_i, z_j |z|) \right| \\
 &\leq \sum_{i=1}^n \sum_{j=1}^n |\mu_i| |\overline{\mu_j}| |(z_i, z_j |z|)| \\
 &= \sum_{i=1}^n |\mu_i|^2 \|z_i |z|\|^2 + \sum_{1 \leq i \neq j \leq n} |\mu_i| |\mu_j| |(z_i, z_j |z|)|.
 \end{aligned}$$

Using Hölder's inequality, we may write that

$$(3.3) \quad \sum_{i=1}^n |\mu_i|^2 \|z_i |z|\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\mu_i|^2 \sum_{i=1}^n \|z_i |z|\|^2; \\ \left(\sum_{i=1}^n |\mu_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n \|z_i |z|\|^{2\beta} \right)^{\frac{1}{\beta}}, \\ \quad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\mu_i|^2 \max_{1 \leq i \leq n} \|z_i |z|\|^2. \end{cases}$$



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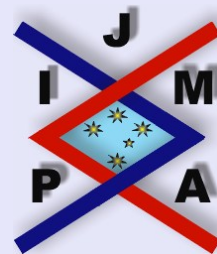
By Hölder's inequality for double sums, we also have

$$\begin{aligned}
 (3.4) \quad & \sum_{1 \leq i \neq j \leq n} |\mu_i| |\mu_j| |(z_i, z_j | z)| \\
 & \leq \begin{cases} \max_{1 \leq i \neq j \leq n} |\mu_i \mu_j| \sum_{1 \leq i \neq j \leq n} |(z_i, z_j | z)|; \\ \left(\sum_{1 \leq i \neq j \leq n} |\mu_i|^\gamma |\mu_j|^\gamma \right)^{\frac{1}{\gamma}} \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j | z)|^\delta \right)^{\frac{1}{\delta}}, \\ \text{where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \sum_{1 \leq i \neq j \leq n} |\mu_i| |\mu_j| \max_{1 \leq i \neq j \leq n} |(z_i, z_j | z)|, \end{cases} \\
 & = \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\mu_i \mu_j|\} \sum_{1 \leq i \neq j \leq n} |(z_i, z_j | z)|; \\ \left[\left(\sum_{i=1}^n |\mu_i|^\gamma \right)^2 - \left(\sum_{i=1}^n |\mu_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j | z)|^\delta \right)^{\frac{1}{\delta}}, \\ \text{where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[\left(\sum_{i=1}^n |\mu_i| \right)^2 - \sum_{i=1}^n |\mu_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(z_i, z_j | z)|. \end{cases}
 \end{aligned}$$

Utilizing (3.3) and (3.4) in (3.2), we may deduce the desired result (3.1). \square

Remark 1. Inequality (3.1) contains in fact 9 different inequalities which may be obtained combining the first 3 ones with the last 3 ones.

A particular result of interest is embodied in the following inequality.



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Corollary 3.2. *With the assumptions in Lemma 3.1, we have*

$$\begin{aligned}
 (3.5) \quad & \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 \\
 & \leq \sum_{i=1}^n |\mu_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i |z|\|^2 \right. \\
 & \quad \left. + \frac{\left[\left(\sum_{i=1}^n |\mu_i|^2 \right)^2 - \sum_{i=1}^n |\mu_i|^4 \right]^{\frac{1}{2}}}{\sum_{i=1}^n |\mu_i|^2} \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j |z)|^2 \right)^{\frac{1}{2}} \right\} \\
 & \leq \sum_{i=1}^n |\mu_i|^2 \left\{ \max_{1 \leq i \leq n} \|z_i |z|\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j |z)|^2 \right)^{\frac{1}{2}} \right\}.
 \end{aligned}$$

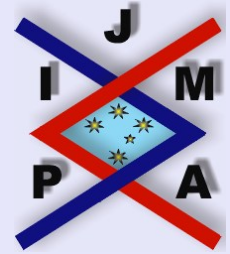
The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $\gamma = \delta = 2$.

The second inequality in (3.5) follows by the fact that

$$\left[\left(\sum_{i=1}^n |\mu_i|^2 \right)^2 - \sum_{i=1}^n |\mu_i|^4 \right]^{\frac{1}{2}} \leq \sum_{i=1}^n |\mu_i|^2.$$

Applying the following Cauchy-Bunyakovsky-Schwarz inequality

$$(3.6) \quad \left(\sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq n,$$



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we may write that

$$(3.7) \quad \left(\sum_{i=1}^n |\mu_i|^\gamma \right)^2 - \sum_{i=1}^n |\mu_i|^{2\gamma} \leq (n-1) \sum_{i=1}^n |\mu_i|^{2\gamma} \quad (n \geq 1)$$

and

$$(3.8) \quad \left(\sum_{i=1}^n |\mu_i| \right)^2 - \sum_{i=1}^n |\mu_i|^2 \leq (n-1) \sum_{i=1}^n |\mu_i|^2 \quad (n \geq 1).$$

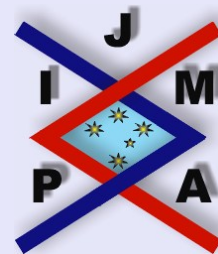
Also, it is obvious that:

$$(3.9) \quad \max_{1 \leq i \neq j \leq n} \{|\mu_i \mu_j|\} \leq \max_{1 \leq i \leq n} |\mu_i|^2.$$

Consequently, we may state the following coarser upper bounds for $\|\sum_{i=1}^n \mu_i z_i |z|\|^2$ that may be useful in applications.

Corollary 3.3. *With the assumptions in Lemma 3.1, we have the inequalities:*

$$(3.10) \quad \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\mu_i|^2 \sum_{i=1}^n \|z_i |z|\|^2; \\ \left(\sum_{i=1}^n |\mu_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n \|z_i |z|\|^{2\beta} \right)^{\frac{1}{\beta}}, \\ \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \sum_{i=1}^n |\mu_i|^2 \max_{1 \leq i \leq n} \|z_i |z|\|^2, \end{cases}$$



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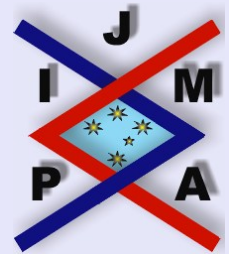
$$+ \left\{ \begin{array}{l} \max_{1 \leq i \leq n} |\mu_i|^2 \sum_{1 \leq i \neq j \leq n} |(z_i, z_j|z)|; \\ (n-1)^{\frac{1}{\gamma}} \left(\sum_{i=1}^n |\mu_i|^{2\gamma} \right)^{\frac{1}{\gamma}} \left(\sum_{1 \leq i \neq j \leq n} |z_i(z_i, z_j|z)|^\delta \right)^{\frac{1}{\delta}}, \\ \text{where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ (n-1) \sum_{i=1}^n |\mu_i|^2 \max_{1 \leq i \neq j \leq n} |(z_i, z_j|z)|. \end{array} \right.$$

The proof is obvious by Lemma 3.1 on applying the inequalities (3.7) – (3.9).

Remark 2. The following inequalities which are incorporated in (3.10) are of special interest:

$$(3.11) \quad \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 \leq \max_{1 \leq i \leq n} |\mu_i|^2 \left[\sum_{i=1}^n \|z_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(z_i, z_j|z)| \right];$$

$$(3.12) \quad \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 \leq \left(\sum_{i=1}^n |\mu_i|^{2p} \right)^{\frac{1}{p}} \left[\left(\sum_{i=1}^n \|z_i\|^2 \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |(z_i, z_j|z)|^q \right)^{\frac{1}{q}} \right],$$



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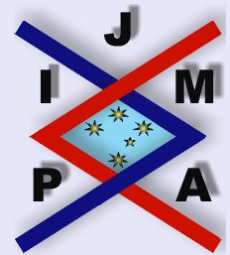
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where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; and

$$(3.13) \quad \left\| \sum_{i=1}^n \mu_i z_i |z| \right\|^2 \leq \sum_{i=1}^n |\mu_i|^2 \left[\max_{1 \leq i \leq n} \|z_i |z|\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(z_i, z_j |z)| \right].$$



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4. Some Inequalities for Fourier Coefficients

The following results holds

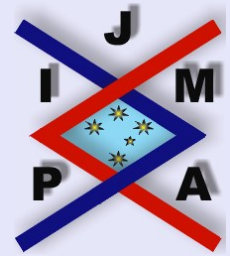
Theorem 4.1. Let x, y_1, \dots, y_n, z be vectors of a 2-inner product space $(X; (\cdot, \cdot))$ and $c_1, \dots, c_n \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$). Then one has the inequalities:

$$(4.1) \quad \left| \sum_{i=1}^n c_i(x, y_i|z) \right|^2 \leq \|x|z\|^2 \times \begin{cases} \max_{1 \leq i \leq n} |c_i|^2 \sum_{i=1}^n \|y_i|z\|^2; \\ \left(\sum_{i=1}^n |c_i|^{2\alpha} \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n \|y_i|z\|^{2\beta} \right)^{\frac{1}{\beta}}, \\ \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \|y_i|z\|^2; \end{cases}$$

$$+ \|x|z\|^2 \times \begin{cases} \max_{1 \leq i \neq j \leq n} \{|c_i c_j|\} \sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|; \\ \left[\left(\sum_{i=1}^n |c_i|^\gamma \right)^2 - \left(\sum_{i=1}^n |c_i|^{2\gamma} \right) \right]^{\frac{1}{\gamma}} \\ \times \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^\delta \right)^{\frac{1}{\delta}}, \text{ where } \gamma > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[\left(\sum_{i=1}^n |c_i| \right)^2 - \sum_{i=1}^n |c_i|^2 \right] \max_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|. \end{cases}$$

Proof. We note that

$$\sum_{i=1}^n c_i(x, y_i|z) = \left(x, \sum_{i=1}^n \bar{c}_i y_i|z \right).$$



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Using Schwarz's inequality in 2-inner product spaces, we have

$$\left| \sum_{i=1}^n c_i(x, y_i|z) \right|^2 \leq \|x|z\|^2 \left\| \sum_{i=1}^n \overline{c_i} y_i|z \right\|^2.$$

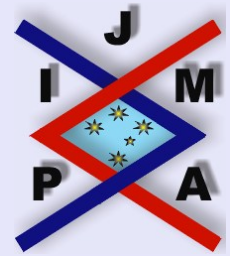
Now using Lemma 3.1 with $\mu_i = \overline{c_i}$, $z_i = y_i$ ($i = 1, \dots, n$), we deduce the desired inequality (4.1). \square

The following particular inequalities that may be obtained by the Corollaries 3.2, 3.3, and Remark 2, hold.

Corollary 4.2. *With the assumptions in Theorem 4.1, one has the inequalities:*

$$(4.2) \quad \left| \sum_{i=1}^n c_i(x, y_i|z) \right|^2 \leq \|x|z\|^2 \times \left\{ \begin{array}{l} \sum_{i=1}^n |c_i|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\}; \\ \max_{1 \leq i \leq n} |c_i|^2 \left\{ \sum_{i=1}^n \|y_i|z\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}; \\ \left(\sum_{i=1}^n |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left(\sum_{i=1}^n \|y_i|z\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^q \right)^{\frac{1}{q}} \right\}, \\ \sum_{i=1}^n |c_i|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}. \end{array} \right.$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$;



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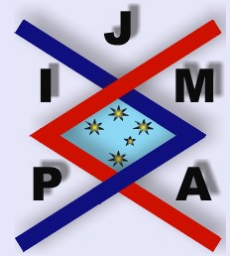
If one chooses $c_i = \overline{(x, y_i|z)}$ ($i = 1, \dots, n$) in (4.1), then it is possible to obtain 9 different inequalities between the Fourier coefficients $(x, y_i|z)$ and the 2-norms and 2-inner products of the vectors y_i ($i = 1, \dots, n$). We restrict ourselves only to those inequalities that may be obtained from (4.2).

From the first inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we get

$$\left(\sum_{i=1}^n |(x, y_i|z)|^2 \right)^2 \leq \|x|z\|^2 \sum_{i=1}^n |(x, y_i|z)|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\},$$

which is clearly equivalent to the following *Boas-Bellman type inequality* for 2-inner products:

$$(5.1) \quad \sum_{i=1}^n |(x, y_i|z)|^2 \leq \|x|z\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\}.$$



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From the second inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we get

$$\left(\sum_{i=1}^n |(x, y_i|z)|^2 \right)^2 \leq \|x|z\|^2 \max_{1 \leq i \leq n} |(x, y_i|z)|^2 \left\{ \sum_{i=1}^n \|y_i|z\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}.$$

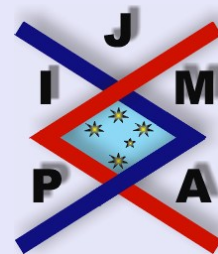
Taking the square root in this inequality, we obtain

$$(5.2) \quad \sum_{i=1}^n |(x, y_i|z)|^2 \leq \|x|z\| \max_{1 \leq i \leq n} |(x, y_i|z)| \left\{ \sum_{i=1}^n \|y_i|z\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}^{\frac{1}{2}}$$

for any x, y_1, \dots, y_n, z vectors in the 2-inner product space $(X; (\cdot, \cdot|z))$.

If we assume that $(e_i)_{1 \leq i \leq n}$ is an orthonormal family in X with respect with the vector z , i.e., $(e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$, then by (5.1) we deduce Bessel's inequality (2.11), while from (5.2) we have

$$(5.3) \quad \sum_{i=1}^n |(x, e_i|z)|^2 \leq \sqrt{n} \|x|z\| \max_{1 \leq i \leq n} |(x, e_i|z)|, \quad x \in X.$$



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From the third inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we deduce

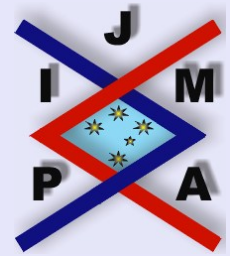
$$\begin{aligned} & \left(\sum_{i=1}^n |(x, y_i|z)|^2 \right)^2 \\ & \leq \|x|z\|^2 \left(\sum_{i=1}^n |(x, y_i|z)|^{2p} \right)^{\frac{1}{p}} \\ & \times \left\{ \left(\sum_{i=1}^n \|y_i|z\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Taking the square root in this inequality, we get

$$\begin{aligned} (5.4) \quad & \sum_{i=1}^n |(x, y_i|z)|^2 \\ & \leq \|x|z\| \left(\sum_{i=1}^n |(x, y_i|z)|^{2p} \right)^{\frac{1}{2p}} \\ & \times \left\{ \left(\sum_{i=1}^n \|y_i|z\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^q \right)^{\frac{1}{q}} \right\}^{\frac{1}{2}} \end{aligned}$$

for any $x, y_1, \dots, y_n, z \in X$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The above inequality (5.4) becomes, for an orthonormal family $(e_i)_{1 \leq i \leq n}$



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with respect of the vector z ,

$$(5.5) \quad \sum_{i=1}^n |(x, e_i|z)|^2 \leq n^{\frac{1}{q}} \|x|z\| \left(\sum_{i=1}^n |(x, e_i|z)|^{2p} \right)^{\frac{1}{2p}}, \quad x \in X.$$

Finally, the choice $c_i = \overline{(x, y_i|z)}$ ($i = 1, \dots, n$) will produce in the last inequality in (4.2)

$$\begin{aligned} & \left(\sum_{i=1}^n |(x, y_i|z)|^2 \right)^2 \\ & \leq \|x|z\|^2 \sum_{i=1}^n |(x, y_i|z)|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}, \end{aligned}$$

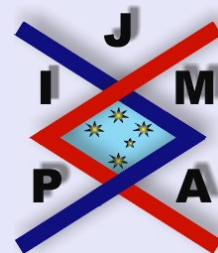
which gives the following inequality

$$(5.6) \quad \sum_{i=1}^n |(x, y_i|z)|^2 \leq \|x|z\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}$$

for any $x, y_1, \dots, y_n, z \in X$.

It is obvious that (5.6) will give for z -orthonormal families, the Bessel inequality mentioned in (2.11) from the Introduction.

Remark 3. *Observe that, both the Boas-Bellman type inequality for 2-inner products incorporated in (5.1) and the inequality (5.6) become in the particular case of z -orthonormal families, the regular Bessel's inequality. Consequently, a comparison of the upper bounds is necessary.*



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It suffices to consider the quantities

$$A_n := \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}}$$

and

$$B_n := (n - 1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|,$$

where $n \geq 1$, and $y_1, \dots, y_n, z \in X$.

If we choose $n = 3$, we have

$$A_3 = \sqrt{2} \left((y_1, y_2|z)^2 + (y_2, y_3|z)^2 + (y_3, y_1|z)^2 \right)^{\frac{1}{2}}$$

and

$$B_3 = 2 \max \{ |(y_1, y_2|z)|, |(y_2, y_3|z)|, |(y_3, y_1|z)| \},$$

where $y_1, y_2, y_3, z \in X$.

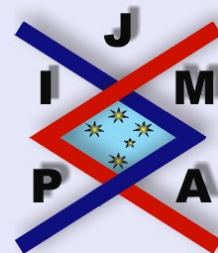
If we consider $a := |(y_1, y_2|z)| \geq 0, b := |(y_2, y_3|z)| \geq 0$ and $c := |(y_3, y_1|z)| \geq 0$, then we have to compare

$$A_3 := \sqrt{2} (a^2 + b^2 + c^2)^{\frac{1}{2}}$$

with

$$B_3 = 2 \max \{ a, b, c \}.$$

If we assume that $b = c = 1$, then $A_3 := \sqrt{2} (a^2 + 2)^{\frac{1}{2}}, B_3 = 2 \max \{ a, 1 \}$. Finally, for $a = 1$, we get $A_3 = \sqrt{6}, B_3 = 2$ showing that $A_3 > B_3$, while for $a = 2$ we have $A_3 = \sqrt{12}, B_3 = 4$ showing that $B_3 > A_3$.



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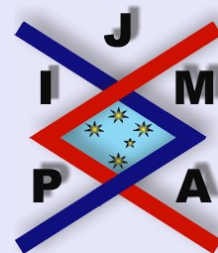
In conclusion, we may state that the bounds

$$M_1 := \|x|z\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\}$$

and

$$M_2 := \|x|z\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i|z\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j|z)| \right\}$$

for the Bessel's sum $\sum_{i=1}^n |(x, y_i|z)|^2$ cannot be compared in general, meaning that sometimes one is better than the other.



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6. Applications for Determinantal Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a σ -algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2 - \rho$ -integrable on Ω , i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \rightarrow [0, \infty)$ is a measurable function on Ω .

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by the formula

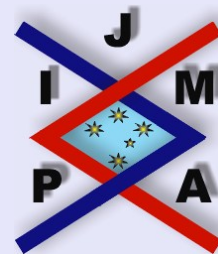
$$(6.1) \quad (f, g|h)_\rho := \frac{1}{2} \int_\Omega \int_\Omega \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t),$$

where, by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix},$$

we denote the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$



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generating the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$(6.2) \quad \|f|h\|_\rho := \left(\frac{1}{2} \int_\Omega \int_\Omega \rho(s)\rho(t) \left| \begin{array}{cc} f(s) & f(t) \\ h(s) & h(t) \end{array} \right|^2 d\mu(s) d\mu(t) \right)^{\frac{1}{2}}.$$

A simple calculation with integrals reveals that

$$(6.3) \quad (f, g|h) = \left| \begin{array}{cc} \int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right|$$

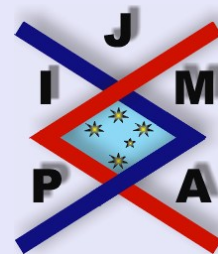
and

$$(6.4) \quad \|f|h\|_\rho = \left| \begin{array}{cc} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu \end{array} \right|^{\frac{1}{2}},$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho fg d\mu$.

Using the representations (6.2), (6.3) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities, as follows.

Proposition 6.1. *Let $f, g_1, \dots, g_n, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \rightarrow [0, \infty)$ is a*



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measurable function on Ω . Then we have the inequality

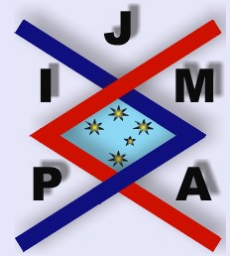
$$\begin{aligned} & \sum_{i=1}^n \left| \begin{array}{cc} \int_{\Omega} \rho f g_i d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho g_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right|^2 \\ & \leq \left| \begin{array}{cc} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho f h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right| \times \left\{ \max_{1 \leq i \leq n} \left| \begin{array}{cc} \int_{\Omega} \rho g_i^2 d\mu & \int_{\Omega} \rho g_i h d\mu \\ \int_{\Omega} \rho g_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right| \right. \\ & \left. + \left(\sum_{1 \leq i \neq j \leq n}^n \left| \begin{array}{cc} \int_{\Omega} \rho g_j g_i d\mu & \int_{\Omega} \rho g_j h d\mu \\ \int_{\Omega} \rho g_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right|^2 \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

The proof follows by the inequality (5.1) applied for the 2-inner product and 2-norm defined in (??) and (6.1), and utilizing the identities (6.2) and (6.3).

If one uses the inequality (5.6), then the following result may also be stated.

Proposition 6.2. Let $f, g_1, \dots, g_n, h \in L^2_{\rho}(\Omega)$, where $\rho : \Omega \rightarrow [0, \infty)$ is a measurable function on Ω . Then we have the inequality

$$\begin{aligned} & \sum_{i=1}^n \left| \begin{array}{cc} \int_{\Omega} \rho f g_i d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho g_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right|^2 \\ & \leq \left| \begin{array}{cc} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho f h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right| \times \left\{ \max_{1 \leq i \leq n} \left| \begin{array}{cc} \int_{\Omega} \rho g_i^2 d\mu & \int_{\Omega} \rho g_i h d\mu \\ \int_{\Omega} \rho g_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right| \right. \\ & \left. + (n-1) \max_{1 \leq i \neq j \leq n} \left| \begin{array}{cc} \int_{\Omega} \rho g_j g_i d\mu & \int_{\Omega} \rho g_j h d\mu \\ \int_{\Omega} \rho g_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right| \right\}. \end{aligned}$$



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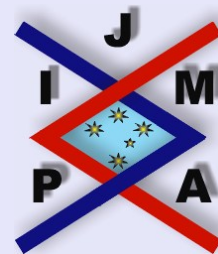
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References

- [1] R. BELLMAN, Almost orthogonal series, *Bull. Amer. Math. Soc.*, **50** (1944), 517–519.
- [2] R.P. BOAS, A general moment problem, *Amer. J. Math.*, **63** (1941), 361–370.
- [3] Y.J. CHO, P.C.S. LIN, S.S. KIM AND A. MISIAK, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Inc., New York, 2001.
- [4] Y.J. CHO, M. MATIĆ AND J.E. PEČARIĆ, On Gram's determinant in 2-inner product spaces, *J. Korean Math. Soc.*, **38**(6) (2001), 1125–1156.
- [5] S.S. DRAGOMIR AND J. SÁNDOR, On Bessel's and Gram's inequality in prehilbertian spaces, *Periodica Math. Hung.*, **29**(3) (1994), 197–205.
- [6] S.S. DRAGOMIR AND B. MOND, On the Boas-Bellman generalisation of Bessel's inequality in inner product spaces, *Italian J. of Pure & Appl. Math.*, **3** (1998), 29–35.
- [7] S.S. DRAGOMIR, B. MOND AND J.E. PEČARIĆ, Some remarks on Bessel's inequality in inner product spaces, *Studia Univ. Babeş-Bolyai, Mathematica*, **37**(4) (1992), 77–86.
- [8] R.W. FREESE AND Y.J. CHO, *Geometry of Linear 2-Normed Spaces*, Nova Science Publishers, Inc., New York, 2001.
- [9] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.



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