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SOME BOAS-BELLMAN TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES

S.S. DRAGOMIR, Y.J. CHO, S.S. KIM, AND A. SOFO

School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428, MCMC
Victoria 8001, Australia.
EMail: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir

Department of Mathematics
College of Education
Gyeongsang National University
Chinju 660-701, Korea.
EMail: yjcho@gsnu.ac.kr

Department of Mathematics
Dongeui University
Pusan 614-714, Korea.
EMail: sskim@deu.ac.kr

School of Computer Science and Mathematics
Victoria University of Technology
PO Box 14428, MCMC
Victoria 8001, Australia.
EMail: sofo@matilda.vu.edu.au
URL: http://rgmia.vu.edu.au/sofo

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Abstract

Some inequalities in 2-inner product spaces generalizing Bessel's result that are similar to the Boas-Bellman inequality from inner product spaces, are given. Applications for determinantal integral inequalities are also provided.

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Contents

1. Introduction

Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\le i\le n}$ are orthonormal vectors in the inner product space H, i.e., $(e_i,e_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [9, p. 391]):

$$\sum_{i=1}^{n}|(x,e_i)|^2\le\|x\|^2$$

for any $x\in H$.

For other results related to Bessel's inequality, see [5] – [7] and Chapter XV in the book [9].

In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality (see also [9, p. 392]).

Theorem 1.1. *If $x,y_1,\dots,y_n$ are elements of an inner product space $(H;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\le\|x\|^2\left[\max_{1\le i\le n}\|y_i\|^2+\left(\sum_{1\le i\ne j\le n}|(y_i,y_j)|^2\right)^{\frac{1}{2}}\right]$$

holds.

It is the main aim of the present paper to point out the corresponding version of Boas-Bellman inequality in 2-inner product spaces. Some natural generalizations and related results are also pointed out. Applications for determinantal integral inequalities are provided.

For a comprehensive list of fundamental results on 2-inner product spaces and linear 2-normed spaces, see the recent books [3] and [8] where further references are given.

2. Bessel's Inequality in 2-Inner Product Spaces

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [3]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

$(2I_1)$ $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent;

$(2I_2)$ $(x, x|z) = (z, z|x)$,

$(2I_3)$ $(y, x|z) = \overline{(x, y|z)}$,

$(2I_4)$ $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

$(2I_5)$ $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner products $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows [4]:

(1) If $\mathbb{K} = \mathbb{R}$, then $(2I_3)$ reduces to

$$(y, x|z) = (x, y|z).$$

(2) From $(2I_3)$ and $(2I_4)$, we have

$$(0,y|z)=0, \quad (x,0|z)=0$$

and

$$(x,\alpha y|z)=\bar{\alpha}(x,y|z). \tag{2.1}$$

(3) Using $(2I_2)$ – $(2I_5)$, we have

$$(z,z|x\pm y)=(x\pm y,x\pm y|z)=(x,x|z)+(y,y|z)\pm 2\operatorname{Re}(x,y|z)$$

and

$$\operatorname{Re}(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)]. \tag{2.2}$$

In the real case, (2.2) reduces to

$$(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)] \tag{2.3}$$

and, using this formula, it is easy to see that, for any $\alpha\in\mathbb{R}$,

$$(x,y|\alpha z)=\alpha^2(x,y|z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\operatorname{Im}(x,y|z)=\operatorname{Re}[-i(x,y|z)]=\frac{1}{4}[(z,z|x+iy)-(z,z|x-iy)],$$

which, in combination with (2.2), yields

$$(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)]+\frac{i}{4}[(z,z|x+iy)-(z,z|x-iy)]. \tag{2.5}$$

Using the above formula and (2.1), we have, for any $\alpha\in\mathbb{C}$,

$$(x,y|\alpha z)=|\alpha|^2(x,y|z). \tag{2.6}$$

However, for $\alpha\in\mathbb{R}$, (2.6) reduces to (2.4). Also, from (2.6) it follows that

$$(x,y|0)=0.$$

(4) For any three given vectors $x,y,z\in X$, consider the vector $u=(y,y|z)x-(x,y|z)y$. By $(2I_1)$, we know that $(u,u|z)\geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u,u|z)\geq 0$ can be rewritten as

$$(y,y|z)[(x,x|z)(y,y|z)-|(x,y|z)|^2]\geq 0. \tag{2.7}$$

For $x=z$, (2.7) becomes

$$-(y,y|z)|(z,y|z)|^2\geq 0,$$

which implies that

$$(z,y|z)=(y,z|z)=0, \tag{2.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y,z\in X$.

Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (2.7), it follows that

$$|(x, y|z)|^2 \le (x, x|z)(y, y|z). \tag{2.9}$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (2.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot \,|\, \cdot))$, we can define a function $\| \cdot \,|\, \cdot \|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{2.10}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:

$(2N_1)$ $\|x|z\| \ge 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

$(2N_2)$ $\|z|x\| = \|x|z\|$,

$(2N_3)$ $\|\alpha x|z\| = |\alpha| \|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

$(2N_4)$ $\|x + x'|z\| \le \|x|z\| + \|x'|z\|$.

Any function $\| \cdot \,|\, \cdot \|$ defined on $X \times X$ and satisfying the conditions $(2N_1)$ – $(2N_4)$ is called a 2-*norm* on X and $(X, \| \cdot \,|\, \cdot \|)$ is called a *linear* 2-*normed* *space* [8]. Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot|\cdot\|)$ with the 2-norm defined by (2.10).

Let $(X; (\cdot, \cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X$, $(e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(e_i)_{1\leq i\leq n}$ is $z-$orthonormal), then the following inequality is the corresponding Bessel's inequality (see for example [4]) for the $z-$orthonormal family $(e_i)_{1\leq i\leq n}$ in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$:

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \leq \|x|z\|^2 \tag{2.11}$$

for any $x \in X$. For more details about this inequality, see the recent paper [4] and the references therein.

3. Some Inequalities for 2-Norms

We start with the following lemma which is also interesting in itself.

Lemma 3.1. *Let* $z_1, \dots, z_n, z \in X$ *and* $\mu_1, \dots, \mu_n \in \mathbb{K}$. *Then one has the inequality:*

$$\left\|\sum_{i=1}^{n} \mu_i z_i | z\right\|^2 \tag{3.1}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\mu_i|^2 \sum_{i=1}^n \|z_i|z\|^2 ; \\ \left(\sum_{i=1}^n |\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n \|z_i|z\|^{2\beta}\right)^{\frac{1}{\beta}}, \\ \qquad\qquad \textit{where } \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^n |\mu_i|^2 \max\limits_{1\leq i\leq n} \|z_i|z\|^2 , \end{cases}$$

$$+ \begin{cases} \max\limits_{1\leq i\neq j\leq n} \{|\mu_i \mu_j|\} \sum_{1\leq i\neq j\leq n} |(z_i, z_j|z)| ; \\ \left[\left(\sum_{i=1}^n |\mu_i|^{\gamma}\right)^2 - \left(\sum_{i=1}^n |\mu_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \left(\sum\limits_{1\leq i\neq j\leq n} |(z_i, z_j|z)|^{\delta}\right)^{\frac{1}{\delta}}, \\ \qquad\qquad \textit{where } \ \gamma > 1, \ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left[\left(\sum_{i=1}^n |\mu_i|\right)^2 - \sum_{i=1}^n |\mu_i|^2\right] \max\limits_{1\leq i\neq j\leq n} |(z_i, z_j|z)| . \end{cases}$$

Proof. We observe that

$$\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2 = \left(\sum_{i=1}^{n}\mu_i z_i, \sum_{j=1}^{n}\mu_j z_j|z\right) \tag{3.2}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i\overline{\mu_j}\left(z_i, z_j|z\right)$$
$$= \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i\overline{\mu_j}\left(z_i, z_j|z\right)\right|$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n}|\mu_i|\left|\overline{\mu_j}\right|\left|(z_i, z_j|z)\right|$$
$$= \sum_{i=1}^{n}|\mu_i|^2\left\|z_i|z\right\|^2 + \sum_{1\leq i\neq j\leq n}|\mu_i|\left|\mu_j\right|\left|(z_i, z_j|z)\right|.$$

Using Hölder's inequality, we may write that

$$\sum_{i=1}^{n}|\mu_i|^2\left\|z_i|z\right\|^2 \leq \begin{cases} \max\limits_{1\leq i\leq n}|\mu_i|^2\sum\limits_{i=1}^{n}\left\|z_i|z\right\|^2; \\ \left(\sum\limits_{i=1}^{n}|\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\left\|z_i|z\right\|^{2\beta}\right)^{\frac{1}{\beta}}, \\ \qquad \text{where } \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum\limits_{i=1}^{n}|\mu_i|^2\max\limits_{1\leq i\leq n}\left\|z_i|z\right\|^2. \end{cases} \tag{3.3}$$

By Hölder's inequality for double sums, we also have

$$\sum_{1\le i\ne j\le n}|\mu_i||\mu_j||(z_i,z_j|z)| \le \begin{cases} \max\limits_{1\le i\ne j\le n}|\mu_i\mu_j|\sum\limits_{1\le i\ne j\le n}|(z_i,z_j|z)|; \\ \left(\sum\limits_{1\le i\ne j\le n}|\mu_i|^\gamma|\mu_j|^\gamma\right)^{\frac{1}{\gamma}}\left(\sum\limits_{1\le i\ne j\le n}|(z_i,z_j|z)|^\delta\right)^{\frac{1}{\delta}}, \\ \qquad \text{where } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \sum\limits_{1\le i\ne j\le n}|\mu_i||\mu_j|\max\limits_{1\le i\ne j\le n}|(z_i,z_j|z)|, \end{cases}$$
$$= \begin{cases} \max\limits_{1\le i\ne j\le n}\{|\mu_i\mu_j|\}\sum\limits_{1\le i\ne j\le n}|(z_i,z_j|z)|; \\ \left[\left(\sum\limits_{i=1}^n|\mu_i|^\gamma\right)^2-\left(\sum\limits_{i=1}^n|\mu_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}}\left(\sum\limits_{1\le i\ne j\le n}|(z_i,z_j|z)|^\delta\right)^{\frac{1}{\delta}}, \\ \qquad \text{where } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left[\left(\sum\limits_{i=1}^n|\mu_i|\right)^2-\sum\limits_{i=1}^n|\mu_i|^2\right]\max\limits_{1\le i\ne j\le n}|(z_i,z_j|z)|. \end{cases} \tag{3.4}$$

Utilizing (3.3) and (3.4) in (3.2), we may deduce the desired result (3.1). □

Remark 1. *Inequality (3.1) contains in fact 9 different inequalities which may be obtained combining the first 3 ones with the last 3 ones.*

A particular result of interest is embodied in the following inequality.

Corollary 3.2. *With the assumptions in Lemma 3.1, we have*

$$\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2 \tag{3.5}$$
$$\leq \sum_{i=1}^{n}|\mu_i|^2\left\{\max_{1\leq i\leq n}\|z_i|z\|^2 + \frac{\left[\left(\sum_{i=1}^{n}|\mu_i|^2\right)^2-\sum_{i=1}^{n}|\mu_i|^4\right]^{\frac{1}{2}}}{\sum_{i=1}^{n}|\mu_i|^2}\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j|z)|^2\right)^{\frac{1}{2}}\right\}$$
$$\leq \sum_{i=1}^{n}|\mu_i|^2\left\{\max_{1\leq i\leq n}\|z_i|z\|^2+\left(\sum_{1\leq i\neq j\leq n}|(z_i,z_j|z)|^2\right)^{\frac{1}{2}}\right\}.$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $\gamma=\delta=2$.

The second inequality in (3.5) follows by the fact that

$$\left[\left(\sum_{i=1}^{n}|\mu_i|^2\right)^2-\sum_{i=1}^{n}|\mu_i|^4\right]^{\frac{1}{2}}\leq\sum_{i=1}^{n}|\mu_i|^2.$$

Applying the following Cauchy-Bunyakovsky-Schwarz inequality

$$\left(\sum_{i=1}^{n}a_i\right)^2\leq n\sum_{i=1}^{n}a_i^2,\quad a_i\in\mathbb{R}_+,\ 1\leq i\leq n, \tag{3.6}$$

we may write that

$$\left(\sum_{i=1}^{n}|\mu_i|^{\gamma}\right)^2-\sum_{i=1}^{n}|\mu_i|^{2\gamma}\leq(n-1)\sum_{i=1}^{n}|\mu_i|^{2\gamma}\qquad(n\geq 1)\tag{3.7}$$

and

$$\left(\sum_{i=1}^{n}|\mu_i|\right)^2-\sum_{i=1}^{n}|\mu_i|^{2}\leq(n-1)\sum_{i=1}^{n}|\mu_i|^{2}\qquad(n\geq 1).\tag{3.8}$$

Also, it is obvious that:

$$\max_{1\leq i\neq j\leq n}\{|\mu_i\mu_j|\}\leq\max_{1\leq i\leq n}|\mu_i|^2.\tag{3.9}$$

Consequently, we may state the following coarser upper bounds for $\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2$ that may be useful in applications.

Corollary 3.3. *With the assumptions in Lemma 3.1, we have the inequalities:*

$$\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2\leq\begin{cases}\max\limits_{1\leq i\leq n}|\mu_i|^2\sum_{i=1}^{n}\|z_i|z\|^2;\\[2ex] \left(\sum_{i=1}^{n}|\mu_i|^{2\alpha}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{n}\|z_i|z\|^{2\beta}\right)^{\frac{1}{\beta}},\\ \qquad\qquad \textit{where}\ \ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \sum_{i=1}^{n}|\mu_i|^2\max\limits_{1\leq i\leq n}\|z_i|z\|^2,\end{cases}\tag{3.10}$$

$$
+\begin{cases}
\max\limits_{1\le i\le n}\left|\mu_i\right|^2\sum_{1\le i\ne j\le n}\left|(z_i,z_j|z)\right|; \\
(n-1)^{\frac{1}{\gamma}}\left(\sum_{i=1}^{n}\left|\mu_i\right|^{2\gamma}\right)^{\frac{1}{\gamma}}\left(\sum_{1\le i\ne j\le n}\left|z\left(_{i},z_j|z\right)\right|^{\delta}\right)^{\frac{1}{\delta}}, \\
\qquad\qquad\qquad \text{where } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\
(n-1)\sum_{i=1}^{n}\left|\mu_i\right|^2\max\limits_{1\le i\ne j\le n}\left|(z_i,z_j|z)\right|.
\end{cases}
$$

The proof is obvious by Lemma 3.1 on applying the inequalities (3.7) – (3.9).

Remark 2. *The following inequalities which are incorporated in (3.10) are of special interest:*

$$
\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2\le\max_{1\le i\le n}\left|\mu_i\right|^2\left[\sum_{i=1}^{n}\left\|z_i|z\right\|^2+\sum_{1\le i\ne j\le n}\left|(z_i,z_j|z)\right|\right]; \tag{3.11}
$$

$$
\begin{aligned}
&\left\|\sum_{i=1}^{n}\mu_i z_i|z\right\|^2 \\
&\le\left(\sum_{i=1}^{n}\left|\mu_i\right|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n}\left\|z_i|z\right\|^{2q}\right)^{\frac{1}{q}}\right. \\
&\left.\quad+(n-1)^{\frac{1}{p}}\left(\sum_{1\le i\ne j\le n}\left|(z_i,z_j|z)\right|^{q}\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{3.12}
$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; *and*

$$\left\| \sum_{i=1}^{n} \mu_i z_i | z \right\|^2 \leq \sum_{i=1}^{n} |\mu_i|^2 \left[\max_{1 \leq i \leq n} \|z_i | z\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(z_i, z_j | z)| \right]. \tag{3.13}$$

4. Some Inequalities for Fourier Coefficients

The following results holds

Theorem 4.1. *Let* $x, y_1, \ldots, y_n, z$ *be vectors of a 2-inner product space* $(X; (\cdot, \cdot|\cdot))$ *and* $c_1, \ldots, c_n \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \tag{4.1}$$

$$\leq \|x|z\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n} |c_i|^2 \sum_{i=1}^n \|y_i|z\|^2; \\ \left(\sum_{i=1}^n |c_i|^{2\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n \|y_i|z\|^{2\beta}\right)^{\frac{1}{\beta}}, \\ \qquad\qquad \text{where } \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum\limits_{i=1}^{n} |c_i|^2 \max\limits_{1\leq i\leq n} \|y_i|z\|^2; \end{cases}$$

$$+ \|x|z\|^2 \times \begin{cases} \max\limits_{1\leq i\neq j\leq n} \{|c_i c_j|\} \sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)|; \\ \left[\left(\sum_{i=1}^n |c_i|^{\gamma}\right)^2 - \left(\sum_{i=1}^n |c_i|^{2\gamma}\right)\right]^{\frac{1}{\gamma}} \\ \times \left(\sum\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^{\delta}\right)^{\frac{1}{\delta}}, \text{ where } \gamma>1, \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left[\left(\sum_{i=1}^n |c_i|\right)^2 - \sum_{i=1}^n |c_i|^2\right] \max\limits_{1\leq i\neq j\leq n} |(y_i, y_j|z)|. \end{cases}$$

Proof. We note that

$$\sum_{i=1}^{n} c_i (x, y_i|z) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i | z\right).$$

Using Schwarz's inequality in 2-inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2 \le \|x|z\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i|z\right\|^2.$$

Now using Lemma 3.1 with $\mu_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \dots, n)$, we deduce the desired inequality (4.1). □

The following particular inequalities that may be obtained by the Corollaries 3.2, 3.3, and Remark 2, hold.

Corollary 4.2. *With the assumptions in Theorem 4.1, one has the inequalities:*

$$(4.2)\quad \left|\sum_{i=1}^{n} c_i (x, y_i|z)\right|^2$$

$$\le \|x|z\|^2 \times \begin{cases} \sum_{i=1}^{n} |c_i|^2 \left\{\max\limits_{1\le i\le n} \|y_i|z\|^2 + \left(\sum\limits_{1\le i\ne j\le n} |(y_i, y_j|z)|^2\right)^{\frac{1}{2}}\right\}; \\ \max\limits_{1\le i\le n} |c_i|^2 \left\{\sum_{i=1}^{n} \|y_i|z\|^2 + \sum\limits_{1\le i\ne j\le n} |(y_i, y_j|z)|\right\}; \\ \left(\sum_{i=1}^{n} |c_i|^{2p}\right)^{\frac{1}{p}} \left\{\left(\sum_{i=1}^{n} \|y_i|z\|^{2q}\right)^{\frac{1}{q}} \right. \\ \qquad \left. + (n-1)^{\frac{1}{p}} \left(\sum\limits_{1\le i\ne j\le n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}}\right\}, \\ \qquad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^{n} |c_i|^2 \left\{\max\limits_{1\le i\le n} \|y_i|z\|^2 + (n-1) \max\limits_{1\le i\ne j\le n} |(y_i, y_j|z)|\right\}. \end{cases}$$

5. Some Boas-Bellman Type Inequalities in 2-Inner Product Spaces

If one chooses $c_i = \overline{(x, y_i|z)}$ $(i = 1, \ldots, n)$ in (4.1), then it is possible to obtain 9 different inequalities between the Fourier coefficients $(x, y_i|z)$ and the 2-norms and 2-inner products of the vectors y_i $(i = 1, \ldots, n)$. We restrict ourselves only to those inequalities that may be obtained from (4.2).

From the first inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we get

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \leq \|x|z\|^2 \sum_{i=1}^{n} |(x, y_i|z)|^2 \left\{ \max_{1\leq i\leq n} \|y_i|z\|^2 + \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2\right)^{\frac{1}{2}} \right\},$$

which is clearly equivalent to the following *Boas-Bellman type inequality* for 2-inner products:

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \leq \|x|z\|^2 \left\{ \max_{1\leq i\leq n} \|y_i|z\|^2 + \left(\sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|^2\right)^{\frac{1}{2}} \right\}. \tag{5.1}$$

From the second inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we get

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \leq \|x|z\|^2 \max_{1\leq i\leq n} |(x, y_i|z)|^2 \left\{\sum_{i=1}^{n} \|y_i|z\|^2 + \sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|\right\}.$$

Taking the square root in this inequality, we obtain

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \leq \|x|z\| \max_{1\leq i\leq n} |(x, y_i|z)| \left\{\sum_{i=1}^{n} \|y_i|z\|^2 + \sum_{1\leq i\neq j\leq n} |(y_i, y_j|z)|\right\}^{\frac{1}{2}} \tag{5.2}$$

for any $x, y_1, \ldots, y_n, z$ vectors in the 2-inner product space $(X; (\cdot, \cdot|\cdot))$.

If we assume that $(e_i)_{1\leq i\leq n}$ is an orthonormal family in X with respect with the vector z, i.e., $(e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$, then by (5.1) we deduce Bessel's inequality (2.11), while from (5.2) we have

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \leq \sqrt{n}\, \|x|z\| \max_{1\leq i\leq n} |(x, e_i|z)|, \quad x \in X. \tag{5.3}$$

From the third inequality in (4.2) for $c_i = \overline{(x, y_i|z)}$, we deduce

$$\left(\sum_{i=1}^{n} |(x, y_i|z)|^2\right)^2 \le \|x|z\|^2 \left(\sum_{i=1}^{n} |(x, y_i|z)|^{2p}\right)^{\frac{1}{p}} \times \left\{ \left(\sum_{i=1}^{n} \|y_i|z\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\le i\ne j\le n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}} \right\}$$

for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Taking the square root in this inequality, we get

$$\sum_{i=1}^{n} |(x, y_i|z)|^2 \le \|x|z\| \left(\sum_{i=1}^{n} |(x, y_i|z)|^{2p}\right)^{\frac{1}{2p}} \times \left\{ \left(\sum_{i=1}^{n} \|y_i|z\|^{2q}\right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1\le i\ne j\le n} |(y_i, y_j|z)|^q\right)^{\frac{1}{q}} \right\}^{\frac{1}{2}} \tag{5.4}$$

for any $x, y_1, \ldots, y_n, z \in X$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The above inequality (5.4) becomes, for an orthornormal family $(e_i)_{1\le i\le n}$

with respect of the vector z,

$$\sum_{i=1}^{n}|(x,e_i|z)|^2 \le n^{\frac{1}{q}}\|x|z\|\left(\sum_{i=1}^{n}|(x,e_i|z)|^{2p}\right)^{\frac{1}{2p}}, \quad x\in X. \tag{5.5}$$

Finally, the choice $c_i = \overline{(x, y_i|z)}$ $(i = 1, \dots, n)$ will produce in the last inequality in (4.2)

$$\left(\sum_{i=1}^{n}|(x,y_i|z)|^2\right)^2 \le \|x|z\|^2\sum_{i=1}^{n}|(x,y_i|z)|^2\left\{\max_{1\le i\le n}\|y_i|z\|^2+(n-1)\max_{1\le i\ne j\le n}|(y_i,y_j|z)|\right\},$$

which gives the following inequality

$$\sum_{i=1}^{n}|(x,y_i|z)|^2 \le \|x|z\|^2\left\{\max_{1\le i\le n}\|y_i|z\|^2+(n-1)\max_{1\le i\ne j\le n}|(y_i,y_j|z)|\right\} \tag{5.6}$$

for any $x, y_1, \dots, y_n, z \in X$.

It is obvious that (5.6) will give for $z-$orthonormal families, the Bessel inequality mentioned in (2.11) from the Introduction.

Remark 3. *Observe that, both the Boas-Bellman type inequality for 2-inner products incorporated in (5.1) and the inequality (5.6) become in the particular case of $z-$orthonormal families, the regular Bessel's inequality. Consequently, a comparison of the upper bounds is necessary.*

It suffices to consider the quantities

$$A_n := \left(\sum_{1 \le i \ne j \le n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}}$$

and

$$B_n := (n-1) \max_{1 \le i \ne j \le n} |(y_i, y_j|z)|,$$

where $n \ge 1$, *and* $y_1, \dots, y_n, z \in X$.

If we choose $n = 3$, *we have*

$$A_3 = \sqrt{2} \left((y_1, y_2|z)^2 + (y_2, y_3|z)^2 + (y_3, y_1|z)^2 \right)^{\frac{1}{2}}$$

and

$$B_3 = 2 \max \{ |(y_1, y_2|z)|, |(y_2, y_3|z)|, |(y_3, y_1|z)| \},$$

where $y_1, y_2, y_3, z \in X$.

If we consider $a := |(y_1, y_2|z)| \ge 0, b := |(y_2, y_3|z)| \ge 0$ *and* $c := |(y_3, y_1|z)| \ge 0$, *then we have to compare*

$$A_3 := \sqrt{2} \left(a^2 + b^2 + c^2 \right)^{\frac{1}{2}}$$

with

$$B_3 = 2 \max \{a, b, c\}.$$

If we assume that $b = c = 1$, *then* $A_3 := \sqrt{2} \left(a^2 + 2 \right)^{\frac{1}{2}}$, $B_3 = 2 \max \{a, 1\}$. *Finally, for* $a = 1$, *we get* $A_3 = \sqrt{6}$, $B_3 = 2$ *showing that* $A_3 > B_3$, *while for* $a = 2$ *we have* $A_3 = \sqrt{12}$, $B_3 = 4$ *showing that* $B_3 > A_3$.

In conclusion, we may state that the bounds

$$M_1 := \|x|z\|^2 \left\{ \max_{1\le i\le n} \|y_i|z\|^2 + \left(\sum_{1\le i\ne j\le n} |(y_i, y_j|z)|^2 \right)^{\frac{1}{2}} \right\}$$

and

$$M_2 := \|x|z\|^2 \left\{ \max_{1\le i\le n} \|y_i|z\|^2 + (n-1) \max_{1\le i\ne j\le n} |(y_i, y_j|z)| \right\}$$

for the Bessel's sum $\sum_{i=1}^{n} |(x, y_i|z)|^2$ *cannot be compared in general, meaning that sometimes one is better than the other.*

6. Applications for Determinantal Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L_{\rho}^{2}(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_{\Omega} \rho(s) |f(s)|^{2} d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L_{\rho}^{2}(\Omega)$ by the formula

$$(f, g|h)_{\rho} := \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t), \tag{6.1}$$

where, by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix},$$

we denote the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

generating the 2-norm on $L_\rho^2(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}^2 d\mu(s)\,d\mu(t)\right)^{\frac{1}{2}}. \tag{6.2}$$

A simple calculation with integrals reveals that

$$(f,g|h)_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{6.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{\frac{1}{2}}, \tag{6.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

Using the representations (6.2), (6.3) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities, as follows.

Proposition 6.1. *Let* $f, g_1, \dots, g_n, h \in L_\rho^2(\Omega)$, *where* $\rho : \Omega \to [0,\infty)$ *is a* *measurable function on Ω. Then we have the inequality*

$$\sum_{i=1}^{n}\left|\begin{array}{cc}\int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right|^2 \leq \left|\begin{array}{cc}\int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right| \times \left\{ \max_{1\leq i\leq n}\left|\begin{array}{cc}\int_\Omega \rho g_i^2 d\mu & \int_\Omega \rho g_i h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right| + \left(\sum_{1\leq i\neq j\leq n}\left|\begin{array}{cc}\int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right|^2\right)^{\frac{1}{2}}\right\}.$$

The proof follows by the inequality (5.1) applied for the 2-inner product and 2-norm defined in (**??**) and (6.1) , and utilizing the identities (6.2) and (6.3) .

If one uses the inequality (5.6), then the following result may also be stated.

Proposition 6.2. *Let $f, g_1, \dots, g_n, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω. Then we have the inequality*

$$\sum_{i=1}^{n}\left|\begin{array}{cc}\int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right|^2 \leq \left|\begin{array}{cc}\int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right| \times \left\{ \max_{1\leq i\leq n}\left|\begin{array}{cc}\int_\Omega \rho g_i^2 d\mu & \int_\Omega \rho g_i h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right| + (n-1)\max_{1\leq i\neq j\leq n}\left|\begin{array}{cc}\int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h d\mu \\ \int_\Omega \rho g_i h d\mu & \int_\Omega \rho h^2 d\mu\end{array}\right|\right\}.$$

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