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NOTE ON QI'S INEQUALITY AND BOUGOFFA'S INEQUALITY

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ABSTRACT. In this paper, an answer to a problem proposed by L. Bougoffa is given. A consolidation of Qi's inequality and Bougoffa's inequality is obtained.

Key words and phrases: Qi's inequality, Bougoffa's inequality, Integral inequality, Cauchy's Mean Value Theorem.

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1. INTRODUCTION

In the paper [7] F. Qi proposed the following open problem, which has attracted much attention from some mathematicians (cf. [1, 5, 6, 8]).

Problem 1.1. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1}. \tag{1.1}$$

hold for $t > 1$?

Similar to Problem 1.1, in the paper [2] L. Bougoffa proposed the following:

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Problem 1.2. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \le \left(\int_a^b f(x)dx\right)^{1-t}. \tag{1.2}$$

hold for $t < 1$?

By using Hölder's inequality, L. Bougoffa obtained an answer to Problem 1.2 as follows

Proposition 1.1. *For a given positive integer* $p \ge 2$*, if* $0 < m \le f(x) \le M$ *on* $[a,b]$ *with* $M \le m^{(p-1)^2}/(b-a)^p$*, then*

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \le \left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}. \tag{1.3}$$

We can see that the condition

$$0 < m \le f(x) \le M \text{ on } [a,b] \text{ with } M \le m^{(p-1)^2}/(b-a)^p \tag{1.4}$$

is not satisfied when $\min\limits_{[a,b]} f(x) = 0$.

In this paper, we firstly give an answer to Problem 1.2, in which we allow $\min\limits_{[a,b]} f(x) = 0$ and p unnecessarily to be an integer. Secondly, we obtain a consolidation of Qi's inequality and Bougoffa's inequality.

2. MAIN RESULTS AND PROOFS

Theorem 2.1. *Let* $p > 2$ *be a positive number and* $f(x)$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $f(a) = 0$*. If* $[f^{p-2}]'(x) \ge p^p(p-2)/(p-1)^{p+1}$ *for* $x \in (a,b)$*, then*

$$\int_a^b [f(x)]^{\frac{1}{p}} dx \le \left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}. \tag{2.1}$$

If $0 \le [f^{p-2}]'(x) \le p^p(p-2)/(p-1)^{p+1}$ *for* $x \in (a,b)$*, then the inequality (2.1) reverses.*

Proof. If $f \equiv 0$ on $[a,b]$, then it is trivial that the equation in (2.1) holds. Suppose now that f is not identically 0 on $[a,b]$ and $[f^{p-2}]'(x) \ge 0$ for $x \in (a,b)$, we may assume $f(x) > 0, x \in (a,b]$. This implies that both sides of (2.1) are not 0.

If $[f^{p-2}]'(x) \ge p^p(p-2)/(p-1)^{p+1}$ for $x \in (a,b)$, then $f(x) > 0$ for $x \in (a,b]$. Thus both sides of (2.1) are not 0. By using Cauchy's Mean Value Theorem twice, we have

$$\begin{aligned} \frac{\int_a^b [f(x)]^{\frac{1}{p}} dx}{\left(\int_a^b f(x)dx\right)^{1-\frac{1}{p}}} &= \frac{[f(b_1)]^{\frac{1}{p}-1}}{(1-\frac{1}{p})\left(\int_a^{b_1} f(x)dx\right)^{-\frac{1}{p}}} \quad (a < b_1 < b) \\ &= \left(\frac{\int_a^{b_1} f(x)dx}{(1-\frac{1}{p})^p [f(b_1)]^{p-1}}\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{(1-\frac{1}{p})^p (p-1)[f(b_2)]^{p-3} f'(b_2)}\right)^{\frac{1}{p}} \quad (a < b_2 < b_1) \\ &= \left(\frac{1}{\frac{(p-1)^{p+1}}{p^p(p-2)}[f^{p-2}]'(b_2)}\right)^{\frac{1}{p}} \\ &\le 1. \end{aligned} \tag{2.2}$$

So the inequality (2.1) holds.

If $0 \le [f^{p-2}]'(x) \le p^p(p-2)/(p-1)^{p+1}$, then $\frac{(p-1)^{p+1}}{p^p(p-2)}[f^{p-2}]'(b_2) \le 1$, which, together with (2.2), implies that the inequality (2.1) reverses. □

In the paper [3], Y. Chen and J. Kimball gave an answer to Problem 1.1 as follows

Proposition 2.2. *Let $p > 2$ be a positive number and $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = 0$. If $[f^{\frac{1}{p-2}}]'(x) \ge (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a,b)$, then*

$$\left(\int_a^b f(x)dx\right)^{p-1} \le \int_a^b [f(x)]^p dx. \tag{2.3}$$

If $0 \le [f^{\frac{1}{p-2}}]'(x) \le (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a,b)$, then the inequality (2.3) reverses.

Thus, combining Theorem 2.1 and Proposition 2.2, we can obtain another result of this paper, which gives a consolidation of Qi's inequality and Bougoffa's inequality. To our best knowledge, this result is not found in the literature.

Theorem 2.3. *Let $p > 2$ be a positive number and $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = 0$.*

(1) *If $[f^{p-2}]'(x) \ge p^p(p-2)/(p-1)^{p+1}$ and $[f^{\frac{1}{p-2}}]'(x) \ge (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a,b)$, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{p}} dx\right)^p \le \left(\int_a^b f(x)dx\right)^{p-1} \le \int_a^b [f(x)]^p dx. \tag{2.4}$$

(2) *If $0 \le [f^{p-2}]'(x) \le p^p(p-2)/(p-1)^{p+1}$ and $0 \le [f^{\frac{1}{p-2}}]'(x) \le (p-1)^{\frac{1}{p-2}-1}$ for $x \in (a,b)$, then the inequality (2.4) reverses.*

Corollary 2.4. *Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = 0$.*

(1) *If $f'(x) \ge \frac{27}{16}$ for $x \in (a,b)$, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{3}} dx\right)^3 \le \left(\int_a^b f(x)dx\right)^2 < \int_a^b [f(x)]^3 dx. \tag{2.5}$$

(2) *If $0 \le f'(x) \le 1$ for $x \in (a,b)$, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{3}} dx\right)^3 > \left(\int_a^b f(x)dx\right)^2 \ge \int_a^b [f(x)]^3 dx. \tag{2.6}$$

Proof. Set $p = 3$ in Theorem 2.3. □

In order to illustrate a possible practical use of Corollary 2.4, we shall give two simple examples in which we can apply inequality (2.5) and (2.6).

Example 2.1. Let $f(x) = e^x - e$ on $[1,2]$, we see that $f'(x) > e > \frac{27}{16}$ for $x \in (1,2)$, other conditions of Corollary 2.4 are fulfilled and straightforward computation yields

$$\left(\int_1^2 (e^x - e)^{\frac{1}{3}} dx\right)^3 \approx 1.56 < \left(\int_1^2 (e^x - e)dx\right)^2 \approx 3.81 < \int_1^2 (e^x - e)^3 dx \approx 18.74.$$

Example 2.2. Let $f(x) = \frac{e^x - e}{10}$ on $[1,2]$, then $\frac{e}{10} \le f'(x) \le \frac{e^2}{10}$, other conditions of Corollary 2.4 are fulfilled and direct calculation produces that

$$\left[\int_1^2 \left(\frac{e^x - e}{10}\right)^{\frac{1}{3}} dx\right]^3 \approx 0.156 > \left(\int_1^2 \frac{e^x - e}{10} dx\right)^2 \approx 0.038 > \int_1^2 \left(\frac{e^x - e}{10}\right)^3 dx \approx 0.019.$$

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