Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 3, Article 61, 2004

ON THE ℓ_p NORM OF GCD AND RELATED MATRICES

PENTTI HAUKKANEN

DEPARTMENT OF MATHEMATICS, STATISTICS AND PHILOSOPHY,
FIN-33014 UNIVERSITY OF TAMPERE, FINLAND
mapehau@uta.fi

Received 12 March, 2004; accepted 06 April, 2004
Communicated by J. Sándor

ABSTRACT. We estimate the ℓ_p norm of the $n \times n$ matrix, whose ij entry is $(i,j)^s/[i,j]^r$, where $r, s \in \mathbb{R}$, (i,j) is the greatest common divisor of i and j and $[i,j]$ is the least common multiple of i and j.

Key words and phrases: GCD matrix, LCM matrix, Smith's determinant, ℓ_p norm, O-estimate.

2000 *Mathematics Subject Classification.* 11C20; 15A36; 11A25.

1. INTRODUCTION

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its ij entry, that is, $(S)_f = (f((x_i, x_j)))$. Analogously, let $[S]_f$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij entry, that is, $[S]_f = (f([x_i, x_j]))$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrix on S associated with f, respectively. H. J. S. Smith [7] calculated $\det(S)_f$ when S is a factor-closed set and $\det[S]_f$ in a more special case. Since Smith, a large number of results on GCD and LCM matrices have been presented in the literature. For general accounts see e.g. [3, 4, 5].

Norms of GCD matrices have not been studied much in the literature. Some results are obtained in [2, 8, 9, 10, 11]. In this paper we provide further results.

Let $p \in \mathbb{Z}^+$. The ℓ_p norm of an $n \times n$ matrix M is defined as

$$\|M\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |m_{ij}|^p \right)^{\frac{1}{p}}.$$

Let $r, s \in \mathbb{R}$. Let A denote the $n \times n$ matrix, whose i, j entry is given as

$$a_{ij} = \frac{(i,j)^s}{[i,j]^r}, \tag{1.1}$$

ISSN (electronic): 1443-5756

094-04

where (i,j) is the greatest common divisor of i and j and $[i,j]$ is the least common multiple of i and j. For $s=1, r=0$ and $s=0, r=-1$, respectively, the matrix A is the GCD and the LCM matrix on $\{1,2,\dots,n\}$. For $s=1, r=1$ the matrix A is the Hadamard product of the GCD matrix and the reciprocal LCM matrix on $\{1,2,\dots,n\}$. In this paper we estimate the ℓ_p norm of the matrix A given in (1.1) for all $r,s\in\mathbb{R}$ and $p\in\mathbb{Z}^+$.

2. PRELIMINARIES

In this section we review the basic results on arithmetical functions needed in this paper. For more comprehensive treatments on arithmetical functions and their estimates we refer to [1] and [6].

The Dirichlet convolution $f*g$ of two arithmetical functions f and g is defined as

$$(f*g)(n)=\sum_{d|n} f(d)g(n/d).$$

Let N^u, $u\in\mathbb{R}$, denote the arithmetical function defined as $N^u(n)=n^u$ for all $n\in\mathbb{Z}^+$, and let E denote the arithmetical function defined as $E(n)=1$ for all $n\in\mathbb{Z}^+$. The divisor function σ_u, $u\in\mathbb{R}$, is defined as

$$\sigma_u(n)=\sum_{d|n} d^u=(N^u*E)(n). \tag{2.1}$$

The Jordan totient function $J_k(n)$, $k\in\mathbb{Z}^+$, is defined as the number of k-tuples $a_1,a_2,\dots,a_k \pmod{n}$ such that the greatest common divisor of $a_1,a_2,\dots,a_k$ and n is 1. By convention, $J_k(1)=1$. The Möbius function μ is the inverse of E under the Dirichlet convolution. It is well known that $J_k=N^k*\mu$. This suggests we define $J_u=N^u*\mu$ for all $u\in\mathbb{R}$. Since μ is the inverse of E under the Dirichlet convolution, we have

$$n^u=\sum_{d|n} J_u(d). \tag{2.2}$$

It is easy to see that

$$J_u(n)=n^u\prod_{p|n}(1-p^{-u}).$$

We thus have

$$0\le J_u(n)\le n^u \text{ for } u\ge 0. \tag{2.3}$$

Lemma 2.1.

(a) *If* $s>-1$, *then* $\sum_{k\le n} k^s=O(n^{s+1})$.
(b) $\sum_{k\le n} k^{-1}=O(\log n)$.
(c) *If* $s<-1$, *then* $\sum_{k\le n} k^s=O(1)$.

Lemma 2.1 follows from the well-known asymptotic formulas for n^s, see [1, Chapter 3].

Lemma 2.2. *Suppose that* $t>1$.

(a) *If* $u-t>-1$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(n^{u-t+1})$.
(b) *If* $u-t=-1$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(\log n)$.
(b) *If* $u-t<-1$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(1)$.

Proof. For all u and t we have

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=\sum_{k\le n}k^{-t}\sum_{d|k}d^u=\sum_{dq\le n}d^{u-t}q^{-t}=\sum_{d\le n}d^{u-t}\sum_{q\le n/d}q^{-t}. \tag{2.4}$$

Now, let $t>1$. Then, by Lemma 2.1(c),

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(1)\sum_{d\le n}d^{u-t}.$$

If $u-t>-1$, then on the basis of Lemma 2.1(a)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(1)O(n^{u-t+1}).$$

If $u-t=-1$, then on the basis of Lemma 2.1(b)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(1)O(\log n).$$

If $u-t<-1$, then on the basis of Lemma 2.1(c)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(1)O(1).$$

□

Lemma 2.3.

(a) *If* $u>0$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k}=O(n^u\log n)$.
(b) *If* $u=0$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k}=O(\log^2 n)$.
(c) *If* $u<0$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k}=O(\log n)$.

Proof. According to (2.4) with $t=1$ we have

$$\sum_{k\le n}\frac{\sigma_u(k)}{k}=\sum_{d\le n}d^{u-1}\sum_{q\le n/d}q^{-1}.$$

Thus on the basis of Lemma 2.1(b)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(\log n)\sum_{d\le n}d^{u-1}.$$

If $u>0$, then on the basis of Lemma 2.1(a)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(\log n)O(n^u).$$

If $u=0$, then on the basis of Lemma 2.1(b)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(\log n)O(\log n).$$

If $u<0$, then on the basis of Lemma 2.1(c)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t}=O(\log n)O(1).$$

□

Lemma 2.4. *Suppose that* $t < 1$.

(a) *If* $u > 0$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1+u-t})$.
(b) *If* $u = 0$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1-t}\log n)$.
(c) *If* $u < 0$, *then* $\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1-t})$.

Proof. According to (2.4) we have

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = \sum_{d\le n} d^{u-t}\sum_{q\le n/d} q^{-t}.$$

Thus on the basis of Lemma 2.1(a)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1-t})\sum_{d\le n} d^{u-1}.$$

If $u > 0$, then on the basis of Lemma 2.1(a)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1-t})O(n^u).$$

If $u = 0$, then on the basis of Lemma 2.1(b)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1-t})O(\log n).$$

If $u < 0$, then on the basis of Lemma 2.1(c)

$$\sum_{k\le n}\frac{\sigma_u(k)}{k^t} = O(n^{1-t})O(1).$$

□

3. RESULTS

In Theorems 3.1, 3.2 and 3.3 we estimate the ℓ_p norm of the matrix A given in (1.1). Their proofs are based on the formulas in Section 2 and the following observations.

Since $(i,j)[i,j] = ij$, we have for all p, r, s

$$\|A\|_p^p = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(i,j)^{sp}}{[i,j]^{rp}} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(i,j)^{(r+s)p}}{i^{rp}j^{rp}}. \tag{3.1}$$

From (2.2) we obtain

$$\begin{aligned}\|A\|_p^p &= \sum_{i=1}^{n}\frac{1}{i^{rp}}\sum_{j=1}^{n}\frac{1}{j^{rp}}\sum_{d|(i,j)} J_{(r+s)p}(d)\\ &= \sum_{i=1}^{n}\frac{1}{i^{rp}}\sum_{d|i} J_{(r+s)p}(d)\sum_{\substack{j=1\\ d|j}}^{n}\frac{1}{j^{rp}}\\ &= \sum_{i=1}^{n}\frac{1}{i^{rp}}\sum_{d|i}\frac{J_{(r+s)p}(d)}{d^{rp}}\sum_{j=1}^{[n/d]}\frac{1}{j^{rp}}. \qquad (3.2)\end{aligned}$$

Theorem 3.1. *Suppose that* $r > 1/p$.

(1) *If* $s > r - 1/p$, *then* $\|A\|_p = O(n^{s-r+1/p})$.
(2) *If* $s = r - 1/p$, *then* $\|A\|_p = O(\log^{1/p} n)$.

(3) *If* $s < r - 1/p$, *then* $\|A\|_p = O(1)$.

Proof. Let $r > 1/p$ or $rp > 1$. Then, by (3.2) and Lemma 2.1(c),

$$\|A\|_p^p = O(1) \sum_{i=1}^{n} \frac{1}{i^{rp}} \sum_{d|i} \frac{|J_{(r+s)p}(d)|}{d^{rp}}.$$

Assume that $r + s \geq 0$. Then, by (2.3) and (2.1),

$$\|A\|_p^p = O(1) \sum_{i=1}^{n} \frac{\sigma_{sp}(i)}{i^{rp}}.$$

Case 1. Let $s > r - 1/p$ or $sp - rp > -1$. Then, by Lemma 2.2(a),

$$\|A\|_p^p = O(1)O(n^{sp-rp+1}) = O(n^{sp-rp+1}).$$

Case 2. Let $s = r - 1/p$ or $sp - rp = -1$. Then, by Lemma 2.2(b),

$$\|A\|_p^p = O(1)O(\log n) = O(\log n).$$

Case 3. Let $s < r - 1/p$ or $sp - rp < -1$. Then, by Lemma 2.2(c),

$$\|A\|_p^p = O(1)O(1) = O(1).$$

Now, assume that $r + s < 0$. Since $r > 1/p$, we have $s < r - 1/p$ and thus we consider Case 3. Since $r + s < 0$, then $(i, j)^{(r+s)p} \leq 1$ and thus on the basis of (3.1) we have

$$\|A\|_p^p \leq \sum_{i=1}^{n} i^{-rp} \sum_{j=1}^{n} j^{-rp}.$$

Since $rp > 1$, we obtain from Lemma 2.1(c)

$$\|A\|_p^p = O(1)O(1) = O(1).$$

□

Theorem 3.2. *Suppose that* $r = 1/p$.

(1) *If* $s > 0$, *then* $\|A\|_p = O(n^s \log^{2/p} n)$.
(2) *If* $s = 0$, *then* $\|A\|_p = O(\log^{3/p} n)$.
(3) *If* $s < 0$, *then* $\|A\|_p = O(\log^{2/p} n)$.

Proof. From (3.2) with $rp = 1$ we obtain

$$\|A\|_p^p = \sum_{i=1}^{n} \frac{1}{i} \sum_{d|i} \frac{J_{sp+1}(d)}{d} \sum_{j=1}^{[n/d]} \frac{1}{j}.$$

By Lemma 2.1(b),

$$\|A\|_p^p = O(\log n) \sum_{i=1}^{n} \frac{1}{i} \sum_{d|i} \frac{|J_{sp+1}(d)|}{d}.$$

Assume that $sp + 1 \geq 0$. Then, by (2.3) and (2.1),

$$\|A\|_p^p = O(\log n) \sum_{i=1}^{n} \frac{\sigma_{sp}(i)}{i}.$$

Case 4. Assume that $s > 0$ or $sp > 0$. Then, by Lemma 2.3(a),

$$\|A\|_p^p = O(\log n)O(n^{sp} \log n) = O(n^{sp} \log^2 n).$$

Case 5. Assume that $s = 0$ or $sp = 0$. Then, by Lemma 2.3(b),

$$\|A\|_p^p = O(\log n)O(\log^2 n) = O(\log^3 n).$$

Case 6. Assume that $s < 0$ or $sp < 0$. Then, by Lemma 2.3(c),

$$\|A\|_p^p = O(\log n)O(\log n) = O(\log^2 n).$$

Now, assume that $sp + 1 < 0$. Then $s < 0$ and thus we consider Case 6. Since $sp + 1 < 0$ and $rp = 1$, then $(i, j)^{(r+s)p} \leq 1$ and thus on the basis of (3.1) we have

$$\|A\|_p^p \leq \sum_{i=1}^{n} i^{-rp} \sum_{j=1}^{n} j^{-rp}.$$

Since $rp = 1$, we obtain from Lemma 2.1(b)

$$\|A\|_p^p = O(\log n)O(\log n) = O(\log^2 n).$$

□

Theorem 3.3. *Suppose that* $r < 1/p$.

(1) *If* $s > -r + 1/p$, *then* $\|A\|_p = O(n^{s-r+1/p})$.
(2) *If* $s = -r + 1/p$, *then* $\|A\|_p = O(n^{-2r+2/p} \log^{1/p} n)$.
(3) *If* $s < -r + 1/p$, *then* $\|A\|_p = O(n^{-2r+2/p})$.

Proof. Let $r < 1/p$ or $rp < 1$. By (3.2) and Lemma 2.1(a),

$$\|A\|_p^p = O(n^{1-rp}) \sum_{i=1}^{n} \frac{1}{i^{rp}} \sum_{d|i} \frac{|J_{(r+s)p}(d)|}{d}.$$

Assume that $r + s \geq 0$. Then, by (2.3) and (2.1),

$$\|A\|_p^p = O(n^{1-rp}) \sum_{i=1}^{n} \frac{\sigma_{(r+s)p-1}(i)}{i^{rp}}.$$

Case 7. Let $s > -r + 1/p$ or $(r + s)p - 1 > 0$. Then, by Lemma 2.4(a),

$$\|A\|_p^p = O(n^{1-rp})O(n^{1+(r+s)p-1-rp}) = O(n^{1+sp-rp}).$$

Case 8. Let $s = -r + 1/p$ or $(r + s)p - 1 = 0$. Then, by Lemma 2.4(b),

$$\|A\|_p^p = O(n^{1-rp})O(n^{1-rp} \log n) = O(n^{2-2rp} \log n).$$

Case 9. Let $s < -r + 1/p$ or $(r + s)p - 1 < 0$. Then, by Lemma 2.4(c),

$$\|A\|_p^p = O(n^{1-rp})O(n^{1-rp}) = O(n^{2-2rp}).$$

Now, assume that $r + s < 0$. Then $s < -r + 1/p$ and thus we consider Case 9. Since $r + s < 0$, then $(i, j)^{(r+s)p} \leq 1$ and thus on the basis of (3.1) we have

$$\|A\|_p^p \leq \sum_{i=1}^{n} i^{-rp} \sum_{j=1}^{n} j^{-rp}.$$

Since $rp < 1$, we obtain from Lemma 2.1(a)

$$\|A\|_p^p = O(n^{1-rp})O(n^{1-rp}) = O(n^{2-2rp}).$$

□

Corollary 3.4.

(a) $\|(i, j)\|_p = O(n^{1+1/p})$ *when* $p \geq 2$.

(b) $\|(i,j)\|_p = O(n^2 \log n)$ *when* $p = 1$.
(c) $\|[i,j]\|_p = O(n^{2+2/p})$ *when* $p \geq 1$.
(d) $\|(i,j)/[i,j]\|_p = O(n^{1/p})$ *when* $p \geq 2$.
(e) $\|(i,j)/[i,j]\|_p = O(n \log^2 n)$ *when* $p = 1$.

Proof.

(a) Take $r = 0, s = 1$, $p \geq 2$ in Case 7 of Theorem 3.3.
(b) Take $r = 0, s = 1$, $p = 1$ in Case 8 of Theorem 3.3.
(c) Take $r = -1, s = 0$, $p \geq 1$ in Case 9 of Theorem 3.3.
(d) Take $r = s = 1$, $p \geq 2$ in Case 1 of Theorem 3.1.
(e) Take $r = s = p = 1$ in Case 4 of Theorem 3.2.

□

REFERENCES

[1] T.M. APOSTOL, *Introduction to Analytic Number Theory*, UTM, Springer–Verlag, New York, 1976.

[2] D. BOZKURT AND S. SOLAK, On the norms of GCD matrices, *Math. Comput. Appl.*, **7**(3) (2002), 205–210.

[3] P. HAUKKANEN AND J. SILLANPÄÄ, Some analogues of Smith's determinant, *Linear and Multilinear Algebra*, **41** (1996), 233–244.

[4] P. HAUKKANEN, J. WANG AND J. SILLANPÄÄ, On Smith's determinant, *Linear Algebra Appl.*, **258** (1997), 251–269.

[5] I. KORKEE AND P. HAUKKANEN, On meet and join matrices associated with incidence functions, *Linear Algebra Appl.*, **372** (2003), 127–153.

[6] D.S. MITRINOVIĆ, J. SÁNDOR AND B. CRSTICI, *Handbook of Number Theory*, Kluwer Academic Publishers, MIA Vol. **351**, 1996.

[7] H.J.S. SMITH, On the value of a certain arithmetical determinant, *Proc. London Math. Soc.*, **7** (1875/76), 208–212.

[8] S. SOLAK, R. TÜRKMEN AND D. BOZKURT, On GCD, LCM and Hilbert matrices and their applications, *Appl. Math. Comput.*, **146**(2-3) (2003), 595–600.

[9] S. SOLAK, R. TÜRKMEN AND D. BOZKURT, On the norms of GCD, Toeplitz and Hankel matrices related to Fibonacci numbers, *Int. Math. J.*, **3**(2) (2003), 195–200.

[10] D. TAŞCI , The bounds for Perron roots of GCD, GMM, and AMM matrices, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **46**(1-2) (1997), 165–171.

[11] R. TÜRKMEN AND D. BOZKURT, A note on the norms of the GCD matrix, *Math. Comput. Appl.*, **9**(2) (2004), 303–308.