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NORM INEQUALITIES FOR SEQUENCES OF OPERATORS RELATED TO THE SCHWARZ INEQUALITY

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Abstract

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Abstract

Some norm inequalities for sequences of linear operators defined on Hilbert spaces that are related to the classical Schwarz inequality are given. Applications for vector inequalities are also provided.

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1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a real or complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators that map H into H .

In many estimates one needs to use upper bounds for the norm of the linear combination of bounded linear operators A_1, \dots, A_n with the scalars $\alpha_1, \dots, \alpha_n$, where separate information for scalars and operators are provided. In this situation, the classical approach is to use a Hölder type inequality as stated below

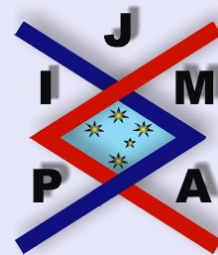
$$\left\| \sum_{i=1}^n \alpha_i A_i \right\| \left(\leq \sum_{i=1}^n |\alpha_i| \|A_i\| \right) \leq \begin{cases} \max_{1 \leq i \leq n} \{|\alpha_i|\} \sum_{i=1}^n \|A_i\|; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n \|A_i\|^q)^{\frac{1}{q}} \text{ if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} \{\|A_i\|\} \sum_{i=1}^n |\alpha_i|. \end{cases}$$

Notice that, the case when $p = q = 2$, which provides the Cauchy-Bunyakovsky-Schwarz inequality

$$(1.1) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\| \leq \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n \|A_i\|^2 \right)^{\frac{1}{2}}$$

is of special interest and of larger utility.

In the previous paper [1], in order to improve (1.1), we have established the following norm inequality for the operators $A_1, \dots, A_n \in B(H)$ and scalars



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$\alpha_1, \dots, \alpha_n \in \mathbb{K}$:

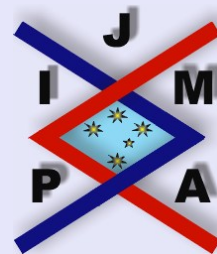
$$(1.2) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^n \|A_i\|^2; \\ (\sum_{i=1}^n |\alpha_i|^{2p})^{\frac{1}{p}} (\sum_{i=1}^n \|A_i\|^{2q})^{\frac{1}{q}} \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \|A_i\|^2 \end{cases}$$

$$+ \begin{cases} \max_{1 \leq i \neq j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|; \\ \left[(\sum_{i=1}^n |\alpha_i|^r)^2 - \sum_{i=1}^n |\alpha_i|^{2r} \right]^{\frac{1}{r}} \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^s \right)^{\frac{1}{s}} \\ \quad \text{if } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \left[(\sum_{i=1}^n |\alpha_i|)^2 - \sum_{i=1}^n |\alpha_i|^2 \right] \max_{1 \leq i \neq j \leq n} \|A_i A_j^*\|, \end{cases}$$

where (1.2) should be seen as all the 9 possible configurations.

Some particular inequalities of interest that can be obtained from (1.2) and provide alternative bounds for the classical Cauchy-Bunyakovsky-Schwarz (CBS) inequality are the following [1]:

$$(1.3) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\| \leq \max_{1 \leq i \leq n} |\alpha_i| \left(\sum_{i,j=1}^n \|A_i A_j^*\| \right)^{\frac{1}{2}},$$



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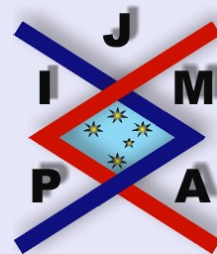


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$$(1.4) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[\max_{1 \leq i \leq n} \|A_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \right],$$

$$(1.5) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left[\max_{1 \leq i \leq n} \|A_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^2 \right)^{\frac{1}{2}} \right]$$

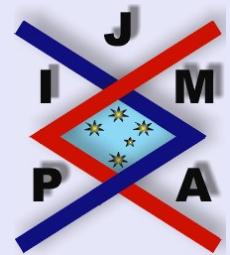
and

$$(1.6) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\left(\sum_{i=1}^n \|A_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \right],$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular, for $p = q = 2$, we have from (1.6)

$$(1.7) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \left(\sum_{i=1}^n |\alpha_i|^4 \right)^{\frac{1}{2}} \left[\left(\sum_{i=1}^n \|A_i\|^4 \right)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}} \left(\sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^2 \right)^{\frac{1}{2}} \right].$$

The aim of the present paper is to establish other upper bounds of interest for the quantity $\|\sum_{i=1}^n \alpha_i A_i\|$, where, as above, $\alpha_1, \dots, \alpha_n$ are real or complex numbers, while A_1, \dots, A_n are bounded linear operators on the Hilbert space $(H; \langle \cdot, \cdot \rangle)$. These are compared with the (CBS) inequality (1.1) and shown that some times they are better. Applications for vector inequalities are also given.



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2. Some General Results

The following result containing 9 different inequalities may be stated:

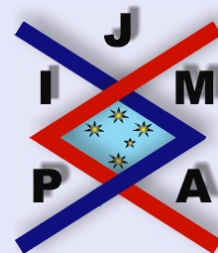
Theorem 2.1. Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$. Then

$$(2.1) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \begin{cases} A \\ B \\ C \end{cases}$$

where

$$(2.2) \quad A := \begin{cases} \max_{1 \leq k \leq n} |\alpha_k|^2 \sum_{i,j=1}^n \|A_i A_j^*\|, \\ \max_{1 \leq k \leq n} |\alpha_k| \left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\| \right)^s \right)^{\frac{1}{s}}, \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |\alpha_k| \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n \|A_i A_j^*\| \right), \end{cases}$$

$$(2.2) \quad B := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \\ \left(\sum_{i=1}^n |\alpha_i|^t \right)^{\frac{1}{t}} \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \\ \quad \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \sum_{i=1}^n |\alpha_i| \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \right\}, \end{cases}$$



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for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

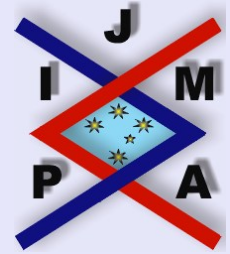
$$C := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{k=1}^n |\alpha_k| \sum_{i=1}^n \max_{1 \leq j \leq n} \{ \|A_i A_j^*\| \}, \\ \left(\sum_{i=1}^n |\alpha_i|^m \right)^{\frac{1}{m}} \sum_{k=1}^n |\alpha_k| \left[\sum_{i=1}^n \left(\max_{1 \leq j \leq n} \{ \|A_i A_j^*\| \} \right)^l \right]^{\frac{1}{l}}, \\ \left(\sum_{k=1}^n |\alpha_k| \right)^2 \max_{1 \leq i, j \leq n} \{ \|A_i A_j^*\| \}. \end{cases} \quad \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1;$$

Proof. We observe, in the operator partial order of $B(H)$, we have that

$$(2.3) \quad 0 \leq \left(\sum_{i=1}^n \alpha_i A_i \right) \left(\sum_{i=1}^n \alpha_i A_i \right)^* \\ = \sum_{i=1}^n \alpha_i A_i \sum_{j=1}^n \bar{\alpha}_j A_j^* = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j A_i A_j^*.$$

Taking the norm in (2.3) and noticing that $\|UU^*\| = \|U\|^2$ for any $U \in B(H)$, we have:

$$\left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 = \left\| \sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j A_i A_j^* \right\| \\ \leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| \|A_i A_j^*\|$$



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$$= \sum_{i=1}^n |\alpha_i| \left(\sum_{j=1}^n |\alpha_j| \|A_i A_j^*\| \right) =: M.$$

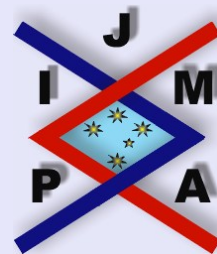
Utilising Hölder's discrete inequality we have that

$$\sum_{j=1}^n |\alpha_j| \|A_i A_j^*\| \leq \begin{cases} \max_{1 \leq k \leq n} |\alpha_k| \sum_{j=1}^n \|A_i A_j^*\|, \\ \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \text{ where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n |\alpha_k| \max_{1 \leq j \leq n} \|A_i A_j^*\|, \end{cases}$$

for any $i \in \{1, \dots, n\}$.

This provides the following inequalities:

$$M \leq \begin{cases} \max_{1 \leq k \leq n} |\alpha_k| \sum_{i=1}^n |\alpha_i| \left(\sum_{j=1}^n \|A_i A_j^*\| \right) =: M_1 \\ \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n |\alpha_i| \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} := M_p \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{k=1}^n |\alpha_k| \sum_{i=1}^n |\alpha_i| \left(\max_{1 \leq j \leq n} \|A_i A_j^*\| \right) := M_\infty. \end{cases}$$



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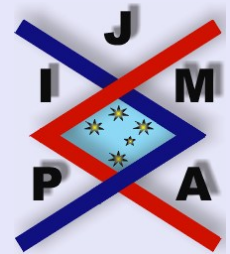
Utilising Hölder's inequality for $r, s > 1, \frac{1}{r} + \frac{1}{s} = 1$, we have:

$$\sum_{i=1}^n |\alpha_i| \left(\sum_{j=1}^n \|A_i A_j^*\| \right) \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{j=1}^n \|A_i A_j^*\| \\ \left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\| \right)^s \right]^{\frac{1}{s}}, \\ \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n \|A_i A_j^*\| \right), \end{cases}$$

and thus we can state that

$$M_1 \leq \begin{cases} \max_{1 \leq k \leq n} |\alpha_k|^2 \sum_{i,j=1}^n \|A_i A_j^*\|; \\ \max_{1 \leq k \leq n} |\alpha_k| \left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\| \right)^s \right)^{\frac{1}{s}}, \\ \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |\alpha_k| \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n \|A_i A_j^*\| \right), \end{cases}$$

and the first part of the theorem is proved.



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By Hölder's inequality we can also have that (for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$)

$$M_p \leq \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \times \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} ; \\ \left(\sum_{i=1}^n |\alpha_i|^t \right)^{\frac{1}{t}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}} , \\ \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \right\} , \end{cases}$$

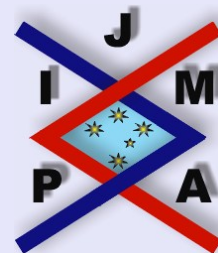
and the second part of (2.1) is proved.

Finally, we may state that

$$M_\infty \leq \sum_{k=1}^n |\alpha_k| \times \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{i=1}^n \max_{1 \leq j \leq n} \{ \|A_i A_j^*\| \} \\ \left(\sum_{i=1}^n |\alpha_i|^m \right)^{\frac{1}{m}} \left[\sum_{i=1}^n \left(\max_{1 \leq j \leq n} \{ \|A_i A_j^*\| \} \right)^l \right]^{\frac{1}{l}} \\ \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1; \\ \sum_{i=1}^n |\alpha_i| \max_{1 \leq i, j \leq n} \{ \|A_i A_j^*\| \} , \end{cases}$$

giving the last part of (2.1). □

Remark 1. It is obvious that out of (2.1) one can obtain various particular inequalities. For instance, the choice $t = 2$, $p = 2$ (therefore $u = q = 2$) in the



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B -branch of (2.2) gives:

$$(2.4) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \left(\sum_{i,j=1}^n \|A_i A_j^*\|^2 \right)^{\frac{1}{2}}$$

$$= \sum_{i=1}^n |\alpha_i|^2 \left(\sum_{i=1}^n \|A_i\|^4 + \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\|^2 \right)^{\frac{1}{2}}.$$

If we consider now the usual Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$(2.5) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \sum_{i=1}^n \|A_i\|^2,$$

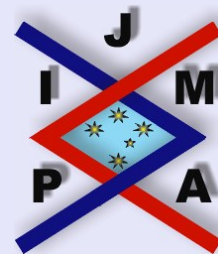
and observe that

$$\left(\sum_{i,j=1}^n \|A_i A_j^*\|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i,j=1}^n \|A_i\|^2 \|A_j^*\|^2 \right)^{\frac{1}{2}} = \sum_{i=1}^n \|A_i\|^2,$$

then we can conclude that (2.4) is a refinement of the (CBS) inequality (2.5).

Corollary 2.2. Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$ so that $A_i A_j^* = 0$ with $i \neq j$. Then

$$(2.6) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \begin{cases} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{cases}$$



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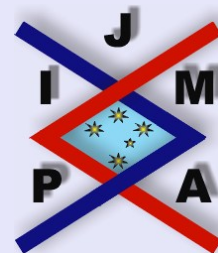
where

$$\tilde{A} := \begin{cases} \max_{1 \leq k \leq n} |\alpha_k|^2 \sum_{i=1}^n \|A_i\|^2; \\ \max_{1 \leq k \leq n} |\alpha_k| \left(\sum_{i=1}^n |\alpha_i|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^n \|A_i\|^{2s} \right)^{\frac{1}{s}}, \\ \quad \text{where } r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |\alpha_k| \sum_{i=1}^n |\alpha_i| \max_{1 \leq i \leq n} \{ \|A_i\|^2 \}, \end{cases}$$

$$\tilde{B} := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n \|A_i\|^2; \\ v \left(\sum_{i=1}^n |\alpha_i|^t \right)^{\frac{1}{t}} \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \|A_i\|^{2u} \right]^{\frac{1}{u}}, \\ \quad \text{where } t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \sum_{i=1}^n |\alpha_i| \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} \{ \|A_i\|^2 \}, \end{cases}$$

where $p > 1$ and

$$\tilde{C} := \begin{cases} \max_{1 \leq i \leq n} |\alpha_i| \sum_{k=1}^n |\alpha_k| \sum_{i=1}^n \|A_i\|^2; \\ \left(\sum_{i=1}^n |\alpha_i|^m \right)^{\frac{1}{m}} \sum_{k=1}^n |\alpha_k| \left(\sum_{i=1}^n \|A_i\|^{2l} \right)^{\frac{1}{l}}, \\ \quad \text{where } m, l > 1, \frac{1}{m} + \frac{1}{l} = 1; \\ \left(\sum_{k=1}^n |\alpha_k| \right)^2 \max_{1 \leq i, j \leq n} \{ \|A_i\|^2 \}. \end{cases}$$



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3. Other Results

A different approach is embodied in the following theorem:

Theorem 3.1. *If $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$, then*

$$(3.1) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \sum_{j=1}^n \|A_i A_j^*\|$$

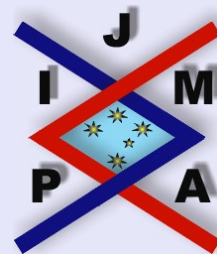
$$\leq \begin{cases} \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^n \|A_i A_j^*\| \right]; \\ \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n \|A_i A_j^*\| \right)^q \right]^{\frac{1}{q}} \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n \|A_i A_j^*\|. \end{cases}$$

Proof. From the proof of Theorem 2.1 we have that

$$\left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| \|A_i A_j^*\|.$$

Using the simple observation that

$$|\alpha_i| |\alpha_j| \leq \frac{1}{2} (|\alpha_i|^2 + |\alpha_j|^2), \quad i, j \in \{1, \dots, n\},$$



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we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| \|A_i A_j^*\| &\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n [|\alpha_i|^2 + |\alpha_j|^2] \|A_i A_j^*\| \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n [|\alpha_i|^2 \|A_i A_j^*\| + |\alpha_j|^2 \|A_i A_j^*\|] \\ &= \sum_{i=1}^n \sum_{j=1}^n |\alpha_i|^2 \|A_i A_j^*\|, \end{aligned}$$

which proves the first inequality in (3.1).

The second part follows by Hölder's inequality and the details are omitted. \square

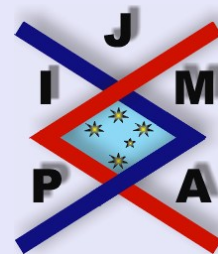
Remark 2. If in (3.1) we choose $\alpha_1 = \dots = \alpha_n = 1$, then we get

$$\left\| \sum_{i=1}^n A_i \right\| \leq \left(\sum_{i=1}^n \|A_i\|^2 + \sum_{1 \leq i \neq j \leq n} \|A_i A_j^*\| \right)^{\frac{1}{2}} \leq \sum_{i=1}^n \|A_i\|,$$

which is a refinement for the generalised triangle inequality.

The following corollary may be stated:

Corollary 3.2. If $A_1, \dots, A_n \in B(H)$ are such that $A_i A_j^* = 0$ for $i \neq j$,



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$i, j \in \{1, \dots, n\}$, then

$$(3.2) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n |\alpha_i|^2 \|A_i\|^2$$

$$\leq \begin{cases} \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \|A_i\|^2; \\ (\sum_{i=1}^n |\alpha_i|^{2p})^{\frac{1}{p}} \left[\sum_{j=1}^n \|A_j\|^{2q} \right]^{\frac{1}{q}} \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^n \|A_i\|^2. \end{cases}$$

Finally, the following result may be stated as well:

Theorem 3.3. If $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$, then

$$(3.3) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n \|A_i A_j^*\|; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{2}{p}} \left(\sum_{i,j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (\sum_{i=1}^n |\alpha_i|)^2 \max_{1 \leq i,j \leq n} \{ \|A_i A_j^*\| \}. \end{cases}$$

Proof. We know that

$$\left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_i| |\alpha_j| \|A_i A_j^*\| =: P.$$



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Firstly, we obviously have that

$$P \leq \max_{1 \leq i, j \leq n} \{|\alpha_i| |\alpha_j|\} \sum_{i, j=1}^n \|A_i A_j^*\| = \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i, j=1}^n \|A_i A_j^*\|.$$

Secondly, by the Hölder inequality for double sums, we obtain

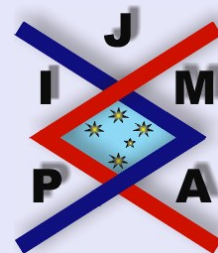
$$\begin{aligned} P &\leq \left[\sum_{i, j=1}^n (|\alpha_i| |\alpha_j|)^p \right]^{\frac{1}{p}} \left(\sum_{i, j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^n |\alpha_i|^p \sum_{j=1}^n |\alpha_j|^p \right)^{\frac{1}{p}} \left(\sum_{i, j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^n |\alpha_i|^p \right)^{\frac{2}{p}} \left(\sum_{i, j=1}^n \|A_i A_j^*\|^q \right)^{\frac{1}{q}}, \end{aligned}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we have

$$\begin{aligned} P &\leq \max_{1 \leq i, j \leq n} \{\|A_i A_j^*\|\} \sum_{i, j=1}^n |\alpha_i| |\alpha_j| \\ &= \left(\sum_{i=1}^n |\alpha_i| \right)^2 \max_{1 \leq i, j \leq n} \{\|A_i A_j^*\|\} \end{aligned}$$

and the theorem is proved. □



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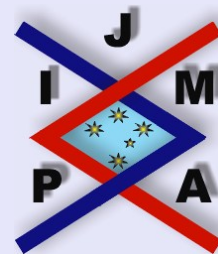
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Corollary 3.4. *If $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$ are such that $A_i A_j^* = 0$ for $i, j \in \{1, \dots, n\}$ with $i \neq j$, then*

$$(3.4) \quad \left\| \sum_{i=1}^n \alpha_i A_i \right\|^2 \leq \begin{cases} \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i=1}^n \|A_i\|^2; \\ (\sum_{i=1}^n |\alpha_i|^p)^{\frac{2}{p}} (\sum_{i=1}^n \|A_i\|^{2q})^{\frac{1}{q}}, \\ \quad \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ (\sum_{i=1}^n |\alpha_i|)^2 \max_{1 \leq i \leq n} \{\|A_i\|^2\}. \end{cases}$$



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4. Vector Inequalities

As pointed out in our previous paper [1], the operator inequalities obtained above may provide various vector inequalities of interest.

If by $M(\alpha, \mathbf{A})$ we denote any of the bounds provided by (2.1), (2.4), (3.1) or (3.3) for the quantity $\|\sum_{i=1}^n \alpha_i A_i\|^2$, then we may state the following general fact:

Under the assumptions of Theorem 2.1, we have:

$$(4.1) \quad \left\| \sum_{i=1}^n \alpha_i A_i x \right\|^2 \leq \|x\|^2 M(\alpha, \mathbf{A}).$$

for any $x \in H$ and

$$(4.2) \quad \left| \sum_{i=1}^n \alpha_i \langle A_i x, y \rangle \right|^2 \leq \|x\|^2 \|y\|^2 M(\alpha, \mathbf{A}).$$

for any $x, y \in H$, respectively.

The proof follows by the Schwarz inequality in the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, see for instance [1], and the details are omitted.

Now, we consider the non zero vectors $y_1, \dots, y_n \in H$. Define the operators [1]

$$A_i : H \rightarrow H, \quad A_i x = \frac{\langle x, y_i \rangle}{\|y_i\|} \cdot y_i, \quad i \in \{1, \dots, n\}.$$



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Since

$$\|A_i\| = \|y_i\|, \quad i \in \{1, \dots, n\}$$

then A_i are bounded linear operators in H . Also, since

$$\langle A_i x, x \rangle = \frac{|\langle x, y_i \rangle|^2}{\|y_i\|} \geq 0, \quad x \in H, \quad i \in \{1, \dots, n\}$$

and

$$\begin{aligned} \langle A_i x, z \rangle &= \frac{\langle x, y_i \rangle \langle y_i, z \rangle}{\|y_i\|}, \\ \langle x, A_i z \rangle &= \frac{\langle x, y_i \rangle \langle y_i, z \rangle}{\|y_i\|}, \end{aligned}$$

giving

$$\langle A_i x, z \rangle = \langle x, A_i z \rangle, \quad x, z \in H, \quad i \in \{1, \dots, n\},$$

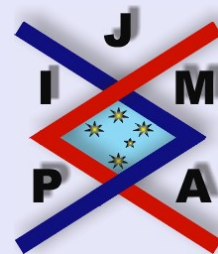
we may conclude that A_i ($i = 1, \dots, n$) are positive self-adjoint operators on H .

Since, for any $x \in H$, one has

$$\|(A_i A_j)(x)\| = \frac{|\langle x, y_j \rangle| |\langle y_j, y_i \rangle|}{\|y_j\|}, \quad i, j \in \{1, \dots, n\},$$

then we deduce that

$$\|A_i A_j\| = |(y_i, y_j)|; \quad i, j \in \{1, \dots, n\}.$$



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If $(y_i)_{i=1, \dots, n}$ is an orthonormal family on H , then $\|A_i\| = 1$ and $A_i A_j = 0$ for $i, j \in \{1, \dots, n\}, i \neq j$.

Now, utilising, for instance, the inequalities in Theorem 3.1 we may state that:

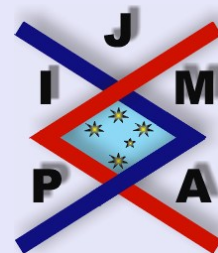
$$\begin{aligned}
 (4.3) \quad & \left\| \sum_{i=1}^n \alpha_i \frac{\langle x, y_i \rangle}{\|y_i\|} y_i \right\|^2 \\
 & \leq \|x\|^2 \sum_{i=1}^n |\alpha_i|^2 \sum_{j=1}^n |\langle y_i, y_j \rangle| \\
 & \leq \|x\|^2 \times \begin{cases} \sum_{i=1}^n |\alpha_i|^2 \max_{1 \leq i \leq n} \left[\sum_{j=1}^n |\langle y_i, y_j \rangle| \right]; \\ \left(\sum_{i=1}^n |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n |\langle y_i, y_j \rangle| \right)^q \right]^{\frac{1}{q}} \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ v \max_{1 \leq i \leq n} |\alpha_i|^2 \sum_{i,j=1}^n |\langle y_i, y_j \rangle|. \end{cases}
 \end{aligned}$$

for any $x, y_1, \dots, y_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$.

The proof follows on choosing $A_i = \frac{\langle \cdot, y_i \rangle}{\|y_i\|} y_i$ in Theorem 3.1 and taking into account that $\|A_i\| = \|y_i\|$,

$$\|A_i A_j^*\| = |\langle y_i, y_j \rangle|, \quad i, j \in \{1, \dots, n\}.$$

We omit the details.



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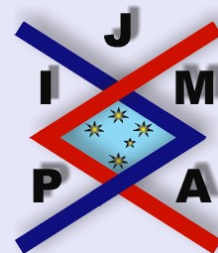
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The choice $\alpha_i = \|y_i\|$ ($i = 1, \dots, n$) will produce some interesting bounds for the norm of the Fourier series

$$\left\| \sum_{i=1}^n \langle x, y_i \rangle y_i \right\|.$$

Notice that the vectors y_i ($i = 1, \dots, n$) are not necessarily orthonormal.

Similar inequalities may be stated if one uses the other two main theorems. For the sake of brevity, they will not be stated here.



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