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ON A NEW GENERALISATION OF OSTROWSKI'S INEQUALITY

B.G. PACHPATTE

57 SHRI NIKETAN COLONY
NEAR ABHINAY TALKIES
AURANGABAD 431 001
(MAHARASHTRA) INDIA.
bgpachpatte@hotmail.com

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ABSTRACT. The aim of this note is to establish a new integral inequality involving two functions and their derivatives. Our result for particular cases yields the well known Ostrowski inequality and its generalization given by Milovanović and Pečarić.

Key words and phrases: Ostrowski's inequality, Functions and their derivatives, Differentiable functions, Empty sum, Taylor's formula.

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1. INTRODUCTION

In 1938, Ostrowski [4] (see [3, p. 468]) proved the following integral inequality.

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty, \tag{1.1}$$

for all $x \in [a,b]$.

In 1976, Milovanović and Pečarić [2] (see [3, p. 469]) proved the following generalization of Ostrowski's inequality.

Let $f : [a,b] \to \mathbb{R}$ be an $n-$times differentiable function, $n \ge 1$ and such that $\left\|f^{(n)}\right\|_\infty = \sup_{t\in(a,b)} \left|f^{(n)}(t)\right| < \infty$. Then

$$\left| \frac{1}{n} \left(f(x) + \sum_{k=1}^{n-1} \frac{n-k}{k!} \right.\right. \tag{1.2}$$

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$$\times\ \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\Bigg)\Bigg|$$
$$\leq \frac{\left\|f^{(n)}\right\|_\infty}{n(n+1)!}\cdot\frac{(x-a)^{n+1}-(b-x)^{n+1}}{b-a},$$

for all $x \in [a,b]$.

The main purpose of this note is to establish a new generalization of Ostrowski's integral inequality involving two functions and their derivatives by using fairly elementary analysis. Our result in special cases yield the inequalities given in (1.1) and (1.2). For some other extensions, generalizations and similar results, see [3] and the references cited therein.

2. Main Result

In what follows, $\mathbb{R}$ and n denote the set of real numbers and a positive integer respectively. The nth derivative of a function $f:[a,b]\to\mathbb{R}$ is denoted by $f^{(n)}(t)$, $t\in[a,b]$. For $n-$times differentiable functions $f,g:[a,b]\to\mathbb{R}$, we use the following notation to simplify the details of presentation:

$$F_k(x) = \frac{n-k}{k!}\cdot\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a},$$
$$G_k(x) = \frac{n-k}{k!}\cdot\frac{g^{(k-1)}(a)(x-a)^k - g^{(k-1)}(b)(x-b)^k}{b-a},$$
$$I_k = \frac{1}{k!}\int_a^b f^{(k)}(y)(x-y)^k\,dy,\qquad I_0=\int_a^b f(y)\,dy,$$
$$J_k = \frac{1}{k!}\int_a^b g^{(k)}(y)(x-y)^k\,dy,\qquad J_0=\int_a^b g(y)\,dy,$$

for $1\leq k\leq n-1$. We use the usual convention that an empty sum is taken to be zero.

Our main result is given in the following theorem.

Theorem 2.1. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions on $[a,b]$ and $n-$times differentiable on (a,b), and whose derivatives $f^{(n)},g^{(n)}:(a,b)\to\mathbb{R}$ are bounded on (a,b), i.e.,*

$$\left\|f^{(n)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(n)}(t)\right|<\infty,\qquad \left\|g^{(n)}\right\|_\infty = \sup_{t\in(a,b)}\left|g^{(n)}(t)\right|<\infty.$$

Then

$$\text{(2.1)}\quad \Bigg|f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)I_0+f(x)J_0\right]$$
$$-\frac{1}{2(b-a)}\left[g(x)\sum_{k=1}^{n-1}I_k + f(x)\sum_{k=1}^{n-1}J_k\right]\Bigg|$$
$$\leq \frac{1}{2(n+1)!}\left[|g(x)|\left\|f^{(n)}\right\|_\infty + |f(x)|\left\|g^{(n)}\right\|_\infty\right]\left[\frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a}\right],$$

for all $x\in[a,b]$.

Proof. Let $x \in [a,b]$, $y \in (a,b)$. With the stipulation on f, g and using Taylor's formula with the Lagrange form of the remainder (see [2, p. 156]) we have

$$f(x) = f(y) + \sum_{k=1}^{n-1} f^{(k)}(y)(x-y)^k + \frac{1}{n!} f^{(n)}(\xi)(x-y)^n, \tag{2.2}$$

$$g(x) = g(y) + \sum_{k=1}^{n-1} g^{(k)}(y)(x-y)^k + \frac{1}{n!} g^{(n)}(\sigma)(x-y)^n, \tag{2.3}$$

where $\xi = y + \alpha(x-y)$ $(0 < \alpha < 1)$ and $\sigma = y + \beta(x-y)$ $(0 < \beta < 1)$. From the definitions of I_k, J_k and integration by parts (see [2]) we have the relations

$$I_0 + \sum_{k=1}^{n-1} I_k = nI_0 - (b-a)\sum_{k=1}^{n-1} F_k(x), \tag{2.4}$$

$$J_0 + \sum_{k=1}^{n-1} J_k = nJ_0 - (b-a)\sum_{k=1}^{n-1} G_k(x). \tag{2.5}$$

Multiplying (2.2) and (2.3) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting, we have

$$\begin{aligned} f(x)g(x) &= \frac{1}{2}g(x)f(y) + \frac{1}{2}f(x)g(y) \\ &\quad + \frac{1}{2}g(x)\sum_{k=1}^{n-1}\frac{1}{k!}f^{(k)}(y)(x-y)^k + \frac{1}{2}f(x)\sum_{k=1}^{n-1}\frac{1}{k!}g^{(k)}(y)(x-y)^k \\ &\quad + \frac{1}{2}\cdot\frac{1}{n!}g(x)f^{(n)}(\xi)(x-y)^n + \frac{1}{2}\cdot\frac{1}{n!}f(x)g^{(n)}(\sigma)(x-y)^n. \end{aligned} \tag{2.6}$$

Integrating (2.6) with respect to y on (a,b) and rewriting, we obtain

$$\begin{aligned} f(x)g(x) &= \frac{1}{2(b-a)}[g(x)I_0 + f(x)J_0] + \frac{1}{2(b-a)}\left[g(x)\sum_{k=1}^{n-1}I_k + f(x)\sum_{k=1}^{n-1}J_k\right] \\ &\quad + \frac{1}{2(b-a)}\cdot\frac{1}{n!}\left[g(x)\int_a^b f^{(n)}(\xi)(x-y)^n\,dy \right. \\ &\qquad \left. + f(x)\int_a^b g^{(n)}(\sigma)(x-y)^n\,dy\right]. \end{aligned} \tag{2.7}$$

From (2.7) and using the properties of modulus, we have

$$\begin{aligned} &\left| f(x)g(x) - \frac{1}{2(b-a)}[g(x)I_0 + f(x)J_0] \right. \\ &\qquad \left. - \frac{1}{2(b-a)}\left[g(x)\sum_{k=1}^{n-1}I_k + f(x)\sum_{k=1}^{n-1}J_k\right]\right| \end{aligned}$$

$$\leq \frac{1}{2(b-a)} \cdot \frac{1}{n!} \left[|g(x)| \int_a^b \left| f^{(n)}(\xi) \right| |x-y|^n \, dy + |f(x)| \int_a^b \left| g^{(n)}(\sigma) \right| |x-y|^n \, dy \right]$$

$$\leq \frac{1}{2(b-a)} \cdot \frac{1}{n!} \left[|g(x)| \left\| f^{(n)} \right\|_\infty + |f(x)| \left\| g^{(n)} \right\|_\infty \right] \int_a^b |x-y|^n \, dy$$

$$= \frac{1}{2(b-a)} \cdot \frac{1}{(n+1)!} \left[|g(x)| \left\| f^{(n)} \right\|_\infty + |f(x)| \left\| g^{(n)} \right\|_\infty \right] \times \left[\frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a} \right],$$

which is the required inequality in (2.1). The proof is complete. □

Corollary 2.2. *Let* $f.g : [a,b] \to \mathbb{R}$ *be continuous functions on* $[a,b]$ *and differentiable on* (a,b) *and whose derivatives* $f', g' : (a,b) \to \mathbb{R}$ *are bounded on* (a,b) *, i.e.,* $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$, $\|g'\|_\infty = \sup_{t\in(a,b)} |g'(t)| < \infty$*. Then*

$$\left| f(x) g(x) - \frac{1}{2(b-a)} \left[g(x) I_0 + f(x) J_0 \right] \right| \leq \frac{1}{2} \left[|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \right] \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a), \tag{2.8}$$

for all $x \in [a,b]$*.*

We note that in the special cases, if we take

(i) $g(x) = 1$ and hence $g^{(n)}(x) = 0$ in (2.1) and

(ii) $g(x) = 1$ and hence $g'(x) = 0$ in (2.8),

we get the inequalities (1.2) and (1.1) respectively. Further, we note that, here we have used Taylor's formula with the Lagrange form of remainder to prove our result. Instead of this, one can use as in [1] the Taylor formula with integral remainder to establish a variant of Theorem A in [1] in the framework of our main result given above. Here we omit the details.

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