CERTAIN PROPERTIES OF GENERALIZED ORLICZ SPACES

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Abstract: In the context of generalized Orlicz spaces X_Φ, the concepts of inclusion, convergence and separability are studied.

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1. Introduction

In [4], Jain, Persson and Upreti studied the generalized Orlicz space X_Φ which is a unification of two generalizations of the Lebesgue L^p-spaces, namely, the X^p-spaces and the usual Orlicz spaces L_Φ. There the authors formulated the space X_Φ giving it two norms, the Orlicz type norm and the Luxemburg type norm and proved the two norms to be equivalent as is the case in usual Orlicz spaces. It was shown that X_Φ is a Banach function space if X is so and a number of basic inequalities such as Hölder's, Minkowski's and Young's were also proved in the framework of X_Φ spaces.

In the present paper, we carry on this study and target some other concepts in the context of X_Φ spaces, namely, inclusion, convergence and separability.

The paper is organized as follows: In Section 2, we collect certain preliminaries which would ease the reading of the paper. The inclusion property in X_Φ spaces has been studied in Section 3. Also, an imbedding has been proved there. In Sections 4 and 5 respectively, the convergence and separability properties have been discussed.

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2. Preliminaries

Let (Ω, Σ, μ) be a complete σ-finite measure space with $\mu(\Omega) > 0$. We denote by $L^0(\Omega)$, the space of all equivalence classes of measurable real valued functions defined and finite a.e. on Ω. A real normed linear space $X = \{u \in L^0(\Omega) : \|u\|_X < \infty\}$ is called a Banach function space (BFS for short) if in addition to the usual norm axioms, $\|u\|_X$ satisfies the following conditions:

P1. $\|u\|_X$ is defined for every measurable function u on Ω and $u \in X$ if and only if $\|u\|_X < \infty$; $\|u\|_X = 0$ if and only if, $u = 0$ a.e.;

P2. $0 \le u \le v$ a.e. $\Rightarrow \|u\|_X \le \|v\|_X$;

P3. $0 < u_n \uparrow u$ a.e. $\Rightarrow \|u\|_X \uparrow \|u\|_X$;

P4. $\mu(E) < \infty \Rightarrow \|\chi_E\|_X < \infty$;

P5. $\mu(E) < \infty \Rightarrow \int_E u(x)dx \le C_E \|u\|_X$,

where $E \subset \Omega$, χ_E denotes the characteristic function of E and C_E is a constant depending only on E. The concept of BFS was introduced by Luxemburg [9]. A good treatment of such spaces can be found, e.g., in [1]

Examples of Banach function spaces are the classical Lebesgue spaces L^p, $1 \le p \le \infty$, the Orlicz spaces L_Φ, the classical Lorentz spaces $L_{p,q}$, $1 \le p$, $p \le \infty$, the generalized Lorentz spaces Λ_ϕ and the Marcinkiewicz spaces M_ϕ.

Let X be a BFS and $-\infty < p < \infty$, $p \ne 0$. We define the space X^p to be the space of all measurable functions f for which

$$\|f\|_{X^p} := \||f|^p\|_X^{\frac{1}{p}} < \infty\,.$$

For $1 < p < \infty$, X^p is a BFS. Note that for $X = L^1$, the space X^p coincides with L^p spaces. These spaces have been studied and used in [10], [11], [12]. Very recently

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in [2], [3], Hardy inequalities (and also geometric mean inequalities in some cases) have been studied in the context of X^p spaces. For an updated knowledge of various standard Hardy type inequalities, one may refer to the monographs [6], [8] and the references therein.

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if

$$\Phi(s) = \int_0^s \phi(t)dt\,,$$

where $\phi : [0,\infty) \to [0,\infty]$, $\phi(0) = 0$ is an increasing, left continuous function which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function Φ is continuous, convex, increasing and satisfies

$$\Phi(0) = 0\,, \quad \lim_{s\to\infty} \Phi(s) = \infty\,.$$

Moreover, a Young function Φ satisfies the following useful inequalities: for $s \geq 0$, we have

$$\begin{cases} \Phi(\alpha s) < \alpha\Phi(s), & \text{if } 0 \leq \alpha < 1 \\ \Phi(\alpha s) \geq \alpha\Phi(s), & \text{if } \alpha \geq 1\,. \end{cases} \tag{2.1}$$

We call a Young function an N-function if it satisfies the limit conditions

$$\lim_{s\to\infty} \frac{\Phi(s)}{s} = \infty \quad \text{and} \quad \lim_{s\to 0} \frac{\Phi(s)}{s} = 0\,.$$

Let Φ be a Young function generated by the function ϕ, i.e.,

$$\Phi(s) = \int_0^s \phi(t)dt\,.$$

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Then the function Ψ generated by the function ψ, i.e.,

$$\Psi(s) = \int_0^s \psi(t)dt\,,$$

where

$$\psi(s) = \sup_{\phi(t)\le s} t$$

is called the complementary function to Φ. It is known that Ψ is a Young function and that Φ is complementary to Ψ. The pair of complementary Young functions Φ, Ψ satisfies Young's inequality

$$u \cdot v \le \Phi(u) + \Psi(v), \quad u, v \in [0, \infty). \tag{2.2}$$

Equality in (2.2) holds if and only if

$$v = \Phi(u) \quad \text{or} \quad u = \Psi(v). \tag{2.3}$$

A Young function Φ is said to satisfy the Δ_2-condition, written $\Phi \in \Delta_2$, if there exist $k > 0$ and $T \ge 0$ such that

$$\Phi(2t) \le k\Phi(t) \quad \text{for all } \ t \ge T\,.$$

The above mentioned concepts of the Young function, complementary Young function and Δ_2-condition are quite standard and can be found in any standard book on Orlicz spaces. Here we mention the celebrated monographs [5], [7].

The remainder of the concepts are some of the contents of [4] which were developed and studied there and we mention them here briefly.

Let X be a BFS and Φ denote a non-negative function on $[0, \infty)$. The generalized Orlicz class $\widetilde{X}_\Phi$ consists of all functions $u \in L^0(\Omega)$ such that

$$\rho_X(u, \Phi) = \|\Phi(|u|)\|_X < \infty\,.$$

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For the case $\Phi(t) = t^p$, $0 < p < \infty$, $\widetilde{X}_\Phi$ coincides algebraically with the space X^p endowed with the quasi-norm

$$\|u\|_{X^p} = \||u|^p\|_X^{\frac{1}{p}} .$$

Let X be a BFS and Φ, Ψ be a pair of complementary Young functions. The generalized Orlicz space, denoted by X_Φ, is the set of all $u \in L^0(\Omega)$ such that

$$\|u\|_\Phi := \sup_v \||u \cdot v|\|_X , \tag{2.4}$$

where the supremum is taken over all $v \in \widetilde{X}_\Psi$ for which $\rho_X(v; \Psi) \le 1$.

It was proved that for a Young function Φ, $\widetilde{X}_\Phi \subset X_\Phi$ and that X_Φ is a BFS, with the norm (2.4). Further, on the generalized Orlicz space X_Φ, a Luxemburg type norm was defined in the following way

$$\|u\|'_\Phi = \inf\left\{k > 0 : \rho_X\left(\frac{|u|}{k}, \Phi\right) \le 1\right\} . \tag{2.5}$$

It was shown that with the norm (2.5) too, the space X_Φ is a BFS and that the two norms (2.4) and (2.5) are equivalent, i.e., there exists constants $c_1, c_2 > 0$ such that

$$c_1\|u\|'_\Phi \le \|u\|_\Phi \le c_2\|u\|'_\Phi . \tag{2.6}$$

In fact, it was proved that $c_2 = 2$.

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3. Comparison of Generalized Orlicz Spaces

We begin with the following definition:

Definition 3.1. *A BFS is said to satisfy the L-property if for all non-negative functions $f, g \in X$, there exists a constant $0 < a < 1$ such that*

$$\|f + g\|_X \geq a(\|f\|_X + \|g\|_X) .$$

Remark 1. It was proved in [2] that the generalized Orlicz space X_Φ contains the generalized Orlicz class $\widetilde{X}_\Phi$. Towards the converse, we prove the following:

Theorem 3.2. *Let Φ be a Young function, X be a BFS satisfying the L-property and $u \in X_\Phi$ be such that $\|u\|_\Phi \neq 0$. Then $\frac{u}{\|u\|_\Phi} \in \widetilde{X}_\Phi$.*

Proof. Let $u \in X_\Phi$. Using the modified arguments used in [7, Lemma 3.7.2], it can be shown that

$$\|u \cdot v\|_X \leq \begin{cases} \|u\|_\Phi \, ; & \text{for } \ \rho_X(v;\Psi) \leq 1, \\ \|u\|_\Phi \rho_X(v;\Psi) \, ; & \text{for } \ \rho_X(v;\Psi) > 1 \, . \end{cases} \tag{3.1}$$

Let $E \subset \Omega$ be such that $\mu(E) < \infty$. First assume that $u \in X_\Phi(\Omega)$ is bounded and that $u(x) = 0$ for $x \in \Omega \setminus E$. Put

$$v(x) = \phi\left(\frac{1}{\|u\|_\Phi}|u(x)|\right) .$$

The monotonicity of Φ and Ψ gives that the functions $\Phi\left(\frac{1}{\|u\|_\Phi}|u(x)|\right)$ and $\Psi(|v(x)|)$ are also bounded. Consequently, property (P2) of X yields that $\left\|\Phi\left(\frac{1}{\|u\|_\Phi}|u(x)|\right)\right\|_X <$

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∞ and $\|\Psi(|v(x)|)\|_X < \infty$ which by using (2.2) gives:

$$\left\| \frac{u \cdot v}{\|u\|_\Phi} \right\|_X \leq \left\| \Phi\left(\frac{|u|}{\|u\|_\Phi} \right) + \Psi(|v|) \right\|_X$$
$$\leq \left\| \Phi\left(\frac{|u|}{\|u\|_\Phi} \right) \right\|_X + \|\Psi(|v|)\|_X$$
$$< \infty .$$

On the other hand, using the L-property of X and (2.3), we get that for some $a > 0$

$$\left\| \frac{u \cdot v}{\|u\|_\Phi} \right\|_X = \left\| \Phi\left(\frac{|u|}{\|u\|_\Phi} \right) + \Psi(|v|) \right\|_X$$
$$\geq a \left[\left\| \Phi\left(\frac{|u|}{\|u\|_\Phi} \right) \right\|_X + \|\Psi(|v|)\|_X \right] . \tag{3.2}$$

Applying (3.1) for $\frac{u}{\|u\|_\Phi}$, v, we find that

$$\max(\rho_X(v,\Psi), 1) \geq \left\| \frac{u \cdot v}{\|u\|_\Phi} \right\|_X$$

and therefore, by (3.2), we get that

$$\max(\rho_X(v,\Psi), 1) \geq a \left[\left\| \Phi\left(\frac{|u|}{\|u\|_\Phi} \right) \right\|_X + \|\Psi(|v|)\|_X \right] .$$

Now, if $\rho_X(v,\Psi) > 1$, then the above estimate gives

$$\left\| \Phi\left(\frac{|u|}{\|u\|_\Phi} \right) \right\|_X \leq \rho_X(v,\Psi) \left(\frac{1}{a} - 1 \right)$$

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and if $\rho_X(v, \Psi) \le 1$, then

$$\left\| \Phi\left(\frac{|u|}{\|u\|_\Phi}\right)\right\|_X + \rho_X(v, \Psi) \le \frac{1}{a}.$$

In any case

$$\left\| \Phi\left(\frac{|u|}{\|u\|_\Phi}\right)\right\|_X < \infty$$

and the assertion is proved for bounded u. For general u, we can follow the modified idea of [10, Lemma 3.7.2]. □

Remark 2. In view of the above theorem, for $u \in X_\Phi$, there exists $c > 0$ such that $cu \in \widetilde{X}_\Phi$. In other words, the space X_Φ is the linear hull of the generalized Orlicz class $\widetilde{X}_\Phi$ with the assumption on X that it satisfies the L-property.

We prove the following useful result:

Proposition 3.3. *Let Φ be a Young function satisfying the Δ_2-condition (with $T = 0$ if $\mu(\Omega) = \infty$) and X be a BFS satisfying the L-property. Then $X_\Phi = \widetilde{X}_\Phi$.*

Proof. Let $u \in X_\Phi$, $\|u\|_\Phi \neq 0$. By Theorem 3.2, we have

$$w = \frac{1}{\|u\|_\Phi} \cdot u \in \widetilde{X}_\Phi.$$

Since $\widetilde{X}_\Phi(\Omega)$ is a linear set, we have

$$\|u\|_\Phi \cdot w = u \in \widetilde{X}_\Phi,$$

i.e.,

$$X_\Phi \subset \widetilde{X}_\Phi.$$

The reverse inclusion is obtained in view of Remark 1 and the assertion follows. □

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Let Φ_1 and Φ_2 be two Young functions. We write $\Phi_2 \prec \Phi_1$ if there exists constants $c > 0$, $T \geq 0$ such that

$$\Phi_2(t) \leq \Phi_1(ct), \quad t \geq T .$$

Now, we prove the following inclusion relation:

Theorem 3.4. *Let X be a BFS satisfying the L-property and Φ_1, Φ_2 be two Young functions such that $\Phi_2 \prec \Phi_1$ and $\mu(\Omega) < \infty$. Then the inclusion*

$$X_{\Phi_1} \subset X_{\Phi_2}$$

holds.

Proof. Since $\Phi_2 \prec \Phi_1$, there exists constants, $c > 0$, $T \geq 0$ such that

$$\Phi_2(t) \leq \Phi_1(ct), \quad t \geq T . \tag{3.3}$$

Let $u \in X_{\Phi_1}$. Then in view of Theorem 3.2, there exists $k > 0$ such that $ku \in \widetilde{X}_{\Phi_1}$, i.e., $\rho_X(ku; \Phi_1) < \infty$. Denote

$$\Omega_1 = \left\{ x \in \Omega; |u(x)| < \frac{cT}{k} \right\} .$$

Then for $x \in \Omega \setminus \Omega_1$, $|u(x)| \geq \frac{cT}{k}$, i.e.,

$$\frac{k}{c}|u(x)| \geq T$$

so that the inequalities (3.3) with t replaced by $\frac{k}{c}|u(x)|$ gives

$$\Phi_2\left(\frac{k}{c}|u(x)|\right) \leq \Phi_1(k|u(x)|)$$

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which implies that

$$\begin{aligned}
\left\|\Phi_2\left(\frac{k}{c}|u(x)|\right)\right\|_X &= \left\|\Phi_2\left(\frac{k}{c}|u(x)|\right)\chi_{\Omega_1} + \Phi_2\left(\frac{k}{c}|u(x)|\right)\chi_{\Omega\setminus\Omega_1}\right\|_X \\
&\leq \left\|\Phi_2\left(\frac{k}{c}|u(x)|\right)\chi_{\Omega_1}\right\|_X + \left\|\Phi_2\left(\frac{k}{c}|u(x)|\right)\chi_{\Omega\setminus\Omega_1}\right\|_X \\
&\leq \Phi_2(T)\|\chi_{\Omega_1}\|_X + \|\Phi_1(k|u(x)|)\chi_{\Omega\setminus\Omega_1}\|_X \\
&= \Phi_2(T)\|\chi_{\Omega_1}\|_X + \rho_X(ku;\Phi_1) \\
&< \infty\,.
\end{aligned}$$

Consequently, $\frac{k}{c}u \in \widetilde{X}_{\Phi_2} \subset X_{\Phi_2}$, i.e., $\frac{k}{c}u \in X_{\Phi_2}$. But since X_{Φ_2} is in particular a vector space we find that $u \in X_{\Phi_2}$ and we are done. □

The above theorem states that $\Phi_2 \prec \Phi_1$ is a sufficient condition for the algebraic inclusion $X_{\Phi_1} \subset X_{\Phi_2}$. The next theorem proves that the condition, in fact, is sufficient for the continuous imbedding $X_{\Phi_1} \hookrightarrow X_{\Phi_2}$.

Theorem 3.5. *Let X be a BFS satisfying the L-property and Φ_1, Φ_2 be two Young functions such that $\Phi_2 \prec \Phi_1$ and $\mu(\Omega) < \infty$. Then the inequality*

$$\|u\|_{\Phi_2} \leq k\|u\|_{\Phi_1}$$

holds for some constant $k > 0$ and for all $u \in X_{\Phi_1}$.

Proof. Let Ψ_1 and Ψ_2 be the complementary functions respectively to Φ_1 and Φ_2. Then $\Phi_2 \prec \Phi_1$ implies that $\Psi_1 \prec \Psi_2$, i.e., there exists constants $c_1, T_1 > 0$ such that

$$\Psi_1(t) \leq \Psi_2(c_1 t) \quad \text{for } \ t \geq T_1$$

or equivalently

$$\Psi_1\left(\frac{t}{c_1}\right) \leq \Psi_2(t) \quad \text{for } \ t \geq c_1 T_1\,.$$

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Further, if $t \le c_1 T_1$, then the monotonicity of Ψ gives

$$\Psi_1\left(\frac{t}{c_1}\right) \le \Psi_1(T_1).$$

The last two estimates give that for all $t > 0$

$$\Psi_1\left(\frac{t}{c_1}\right) \le \Psi_1(T_1) + \Psi_2(t)\,. \tag{3.4}$$

By the property (P4) of BFS, $\|\chi_\Omega\|_X < \infty$. Denote $\alpha = (\Psi_1(T_1)\|\chi_\Omega\|_X + 1)^{-1}$ and $k = \frac{c_1}{\alpha}$. Clearly $0 < \alpha < 1$. We know that for a Young function Φ and $0 < \beta < 1$,

$$\Phi(\beta t) \le \beta\Phi(t), \quad t > 0\,. \tag{3.5}$$

Now, let $v \in \widetilde{X}_{\Psi_2}$ be such that $\rho_X(v;\Psi_2) \le 1$. Then, using (3.5) for $\beta = \alpha$ and $t = \frac{|v(x)|}{c_1}$ and (3.4), we obtain that

$$\begin{aligned}
\rho_X\left(\frac{v}{k};\Psi_1\right) &= \left\|\Psi_1\left(\frac{\alpha|v(x)|}{c_1}\right)\right\|_X \\
&\le \alpha\left\|\Psi_1\left(\frac{|v(x)|}{c_1}\right)\right\|_X \\
&\le \alpha\|\Psi_1(T_1) + \Psi_2(|v(x)|)\|_X \\
&\le \alpha(\Psi_1(T_1)\|\chi_\Omega\|_X + \|\psi_2(|v(x)|)\|_X) \\
&= \alpha(\Psi_1(T_1)\|\chi_\Omega\|_X + \rho_X(v;\Psi_2)) \\
&\le \alpha\alpha^{-1} = 1\,.
\end{aligned}$$

Thus we have shown that $\rho_X(v;\Psi_2) \le 1$ implies $\rho_X\left(\frac{v}{k};\Psi_1\right) \le 1$ and consequently

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using the definition of the generalized Orlicz norm, we obtain

$$\begin{aligned}
\|u\|_{\Phi_2} &= \sup_{\rho(v;\Psi_2)\le 1} \|(|u(x)v(x)|)\|_X \\
&= k \sup_{\rho(v;\Psi_2)\le 1} \left\| \left(\left| u(x)\frac{v(x)}{k} \right| \right) \right\|_X \\
&\le k \sup_{\rho\left(\frac{v}{k};\Psi_1\right)\le 1} \left\| \left(\left| u(x)\frac{v(x)}{k} \right| \right) \right\|_X \\
&= k \sup_{\rho(w;\Psi_1)\le 1} \||u(x)w(x)|\|_X \\
&= k \cdot \|u\|_{\Phi_1}
\end{aligned}$$

and the assertion is proved. $\square$

Remark 3. If Φ_1 and Φ_2 are equivalent Young functions (i.e., $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$) then the norms $\|\cdot\|_{\Phi_1}$ and $\|\cdot\|_{\Phi_2}$ are equivalent.

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4. Convergence

Following the concepts in the Orlicz space $L_\Phi(\Omega)$, we introduce the following definitions.

Definition 4.1. *A sequence $\{u_n\}$ of functions in X_Φ is said to converge to $u \in X_\Phi$, written $u_n \to u$, if*

$$\lim_{n\to\infty} \|u_n - u\|_\Phi = 0\,.$$

Definition 4.2. *A sequence $\{u_n\}$ of functions in X_Φ is said to converge in Φ-mean to $u \in X_\Phi$ if*

$$\lim_{n\to\infty} \rho_X(u_n - u; \Phi) = \lim_{n\to\infty} \|\Phi(|u_n - u|)\|_X = 0\,.$$

We proceed to prove that the two convergences above are equivalent. In the sequel, the following remark will be used.

Remark 4. Let Φ and Ψ be a pair of complementary Young functions. Then in view of Young's inequality (2.2), we obtain for $u \in \widetilde{X}_\Phi$, $v \in \widetilde{X}_\Psi$

$$\begin{aligned} \||uv|\|_X &\le \|\Phi(|u|)\|_X + \|\Psi(|v|)\|_X \\ &= \rho_X(u;\Phi) + \rho_X(v;\Psi) \end{aligned}$$

so that

$$\|u\|_\Phi \le \rho_X(u;\Phi) + 1\,.$$

Now, we prove the following:

Lemma 4.3. *Let Φ be a Young function satisfying the Δ_2-condition (with $T = 0$ if $\mu(\Omega) = \infty$) and r be the number given by*

$$r = \begin{cases} 2 & \text{if } \mu(\Omega) = \infty, \\ \Phi(T)\|\chi_\Omega\|_X + 2 & \text{if } \mu(\Omega) < \infty\,. \end{cases} \tag{4.1}$$

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If there exists an $m \in \mathbb{N}$ *such that*

$$\rho_X(u;\Phi) \le k^{-m}\,, \tag{4.2}$$

where k *is the constant in the* Δ_2*-condition, then*

$$\|u\|_\Phi \le 2^{-m} r\,.$$

Proof. Let $m \in \mathbb{N}$ be fixed. Consider first the case when $\mu(\Omega) < \infty$ and denote

$$\Omega_1 = \{x \in \Omega : 2^m |u(x)| \le T\}\,.$$

Then for $x \in \Omega_1$, we get

$$\Phi(2^m|u(x)|) \le \Phi(T) \tag{4.3}$$

and for $x \in \Omega \setminus \Omega_1$, by repeated applications of the Δ_2-condition, we obtain

$$\Phi(2^m|u(x)|) \le k^m \Phi(|u(x)|)\,. \tag{4.4}$$

Consequently, we have using (4.3) and (4.4)

$$\begin{aligned}
\|\Phi(2^m|u(x)|)\|_X &= \|\Phi(2^m(|u(x)|))\chi_{\Omega_1} + \Phi(2^m|u(x)|)\chi_{\Omega\setminus\Omega_1}\|_X \\
&\le \|\Phi(2^m(|u(x)|))\chi_{\Omega_1}\|_X + \|\Phi(2^m|u(x)|)\chi_{\Omega\setminus\Omega_1}\|_X \\
&\le \Phi(T)\|\chi_{\Omega_1}\|_X + k^m\|\Phi(|u(x)|)\|_X \\
&\le \Phi(T)\|\chi_{\Omega}\|_X + k^m \rho_X(u;\Phi) \\
&\le \Phi(T)\|\chi_{\Omega}\|_X + 1 \\
&= r - 1\,.
\end{aligned}$$

In the case $\mu(\Omega) = \infty$ we take $\Omega_1 = \phi$ and then (4.4) directly gives

$$\|\Phi(2^m|u(x)|)\|_X \le 1 \le r - 1$$

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since $r = 2$ for $\mu(\Omega) = \infty$. Thus in both cases we have

$$\|\Phi(2^m|u(x)|)\|_X \leq r - 1$$

which further, in view of Remark 4 gives

$$\|2^m u(x)\|_\Phi \leq r$$

or

$$\|u\|_\Phi \leq 2^{-m} r$$

and we are done. □

Let us recall the following result from [4]:

Lemma 4.4. *Let $u \in X_\Phi$. Then*

$$\rho_X(u; \Phi) \leq \|u\|'_\Phi \quad \text{if } \|u\|'_\Phi \leq 1$$

and

$$\rho_X(u; \Phi) \geq \|u\|'_\Phi \quad \text{if } \|u\|'_\Phi > 1,$$

where $\|u\|'_\Phi$ denotes the Luxemburg type norm on the space X_Φ given by (2.5).

Now, we are ready to prove the equivalence of the two convergence concepts defined earlier in this section.

Theorem 4.5. *Let Φ be a Young function satisfying the Δ_2-condition. Let $\{u_n\}$ be a sequence of functions in X_Φ. Then u_n converges to u in X_Φ if and only if u_n converges in Φ-mean to u in X_Φ.*

Proof. First assume that u_n converges in Φ-mean to u. We shall now prove that $u_n \to u$. Given $\varepsilon > 0$, we can choose $m \in \mathbb{N}$ such that $\varepsilon > 2^{-m} r$, where r is as

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given by (4.1). Now, since u_n converges in Φ-mean to u, for this m, we can find an M such that

$$\rho_X(u_n - u; \Phi) \leq k^{-m} \quad \text{for} \ \ n \geq M$$

which by Lemma 4.3 implies that

$$\|u_n - u\|_\Phi \leq 2^{-m} r < \varepsilon \quad \text{for} \ \ n \geq M$$

and we get that $u_n \to u$.

Conversely, first note that the two norms $\|\cdot\|_\Phi$ and $\|\cdot\|'_\Phi$ on the space X_Φ are equivalent and assume, in particular, that the constants of equivalence are c_1, c_2, i.e., (2.6) holds.

Now, let $u_n, u \in X_\Phi$ so that

$$\|u_n - u\|_\Phi \leq c_1 \, .$$

Then (2.6) gives

$$\|u_n - u\|'_\Phi \leq 1$$

which, in view of Lemma 4.4 and again (2.6), gives that

$$\begin{aligned} \rho_X(u_n - u; \Phi) &\leq \|u_n - u\|'_\Phi \\ &\leq \frac{1}{c_1} \|u_n - u\|_\Phi \, . \end{aligned}$$

The Φ-mean convergence now, immediately follows from the convergence in X_Φ. $\square$

Remark 5. The fact that the Φ-mean convergence implies norm convergence does not require the use of Δ_2-conditions. It is required only in the reverse implication.

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5. Separability

Remark 6. It is known, e.g., see [7, Theorem 3.13.1], that the Orlicz space $L_\Phi(\Omega)$ is separable if Φ satisfies the Δ_2-condition (with $T = 0$ if $\mu(\Omega) = 0$). In order to obtain the separability conditions for the generalized Orlicz space X_Φ, we can depict the same proof with obvious modifications except at a point where the Lebesgue dominated convergence theorem has been used.

In the framework of general BFS, the following version of the Lebesgue dominated convergence theorem is known, see e.g. [1, Proposition 3.6].

Definition 5.1. *A function f in a Banach function space X is said to have an absolutely continuous norm in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ satisfying $E_n \to \phi$ μ-a.e.*

Proposition A. *A function f in a Banach function space X has an absolutely continuous norm iff the following condition holds; whenever f_n $\{n = 1, 2, \dots\}$ and g are μ-measurable functions satisfying $|f_n| \leq |f|$ for all n and $f_n \to g$ μ-a.e., then $\|f_n - g\|_X \to 0$.*

Now, in view of Remark 6 and Proposition A we have the following result.

Theorem 5.2. *Let X be a BFS having an absolutely continuous norm and Φ be a Young function satisfying the Δ_2-condition (with $T = 0$ if $\mu(\Omega) = 0$). Then the generalized Orlicz space X_Φ is separable.*

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