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A GEOMETRICAL PROOF OF A NEW INEQUALITY FOR THE GAMMA FUNCTION

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ABSTRACT. Using the inclusions between the unit balls for the p-norms, we obtain a new inequality for the gamma function.

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Since the gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

is one of the most important functions in Mathematics, there exists an extensive literature on its inequalities (see [1], [2]).

Our aim here is to present and prove the inequalities

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1 \quad x \in [0,1], \quad n \in \mathbb{N}.$$

As we will show the above inequalities follow immediately from a key geometrical argument. From now on for any $r > 0$, $p, n \geq 1$ we will consider the notation:

$$D^{n,r}_{\|\cdot\|_p} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,/ \|(x_1, \ldots, x_n)\|_p < r\}$$

for the n-ball of radius r for the p-norm $\|(x_1, \ldots, x_n)\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$.

To this end, we need to prove the following:

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Lemma 1. *For all n in $\mathbb{N}$, $p \geq 1$ and $r > 0$ we have:*

$$\text{(1)} \qquad \textit{Volume } \left(D^{n,r}_{\|\cdot\|_p}\right) = 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} r^n.$$

Proof. For $n = 1$, $D^{1,r}_{\|\cdot\|_p}$ is the interval $(-r, r)$, whose measure is $2r$, i.e.,

$$2r = 2\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} r$$

and (1) holds. By induction, let us assume that (1) holds for $n-1$. Then we note that $|x_1|^p + \cdots + |x_n|^p < r^p$ is equivalent to $|x_1|^p + \cdots + |x_{n-1}|^p < r^p - |x_n|^p$ and by virtue of the induction hypothesis we have

$$\begin{aligned}
\text{Volume } \left(D^{n,r}_{\|\cdot\|_p}\right) &= \int_{D^{n,r}_{\|\cdot\|_p}} dx_1 \dots dx_n \\
&= 2\int_0^r \left(\int_{D^{n-1,(r^p-|x_n|^p)^{1/p}}_{\|\cdot\|_p}} dx_1 \dots dx_{n-1} \right) dx_n \\
&= 2\int_0^r 2^{n-1} \frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)} (r^p - x_n^p)^{\frac{n-1}{p}} dx_n \\
&= 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)} r^n \int_0^1 (1-z^p)^{\frac{n-1}{p}} dz,
\end{aligned}$$

where $z = x_n/r$.

If we consider $F(a, b, c, z)$ the first hypergeometric function (see [3]), then

$$\int (1-z^p)^{\frac{n-1}{p}} dz = zF\left(\frac{1}{p}, -\frac{n-1}{p}, 1+\frac{1}{p}, z^n\right)$$

and by well-known properties of the hypergeometric function we deduce:

$$\begin{aligned}
\text{Volume } \left(D^{n,r}_{\|\cdot\|_p}\right) &= 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)} r^n F\left(\frac{1}{p}, -\frac{n-1}{p}, 1+\frac{1}{p}, 1\right) \\
&= 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^{n-1}}{\Gamma\left(1+\frac{n-1}{p}\right)} r^n \frac{\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1+\frac{n-1}{p}\right)}{\Gamma\left(1+\frac{n}{p}\right)} \\
&= 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^{n}}{\Gamma\left(1+\frac{n}{p}\right)} r^n.
\end{aligned}$$

□

Therefore we have

Theorem 2. *For all* $n \in \mathbb{N}$ *and* x *in* $(0,1)$ *we have*

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

Proof. For all n in $\mathbb{N}$ and $p \geq 1$, from the inclusions

$$D^{n,1}_{\|\cdot\|_1} \subseteq D^{n,1}_{\|\cdot\|_p} \subseteq D^{n,1}_{\|\cdot\|_\infty},$$

we deduce

$$\text{Volume}\left(D^{n,1}_{\|\cdot\|_1}\right) \leq \text{Volume}\left(D^{n,1}_{\|\cdot\|_p}\right) \leq \text{Volume}\left(D^{n,1}_{\|\cdot\|_\infty}\right),$$

so by Lemma 1:

$$2^n \frac{\Gamma(2)^n}{\Gamma(n+1)} \leq 2^n \frac{\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \leq 2^n$$

and with $1/p = x$, bearing in mind that $\Gamma(2) = 1$, $\Gamma(n+1) = n!$,

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

□

From this it follows immediately that the function $\Gamma(1+x)^n/\Gamma(1+nx)$ is strictly decreasing on $(0,1]$.

REFERENCES

[1] H. ALZER, On some inequalities for the gamma and psi functions, *Math of Comp.*, **66**(217) (1997), 373–389.

[2] H. ALZER, Sharp bounds for the ratio of $q-$Gamma functions, *Math Nachr.*, **222** (2001), 5–14.

[3] E.W. WEISSTEIN, *Hypergeometric function*, [ONLINE: `http://mathworld.wolfram.com/HypergeometricFunction.html`].