

# Journal of Inequalities in Pure and Applied Mathematics

## NOTE ON FENG QI'S INTEGRAL INEQUALITY

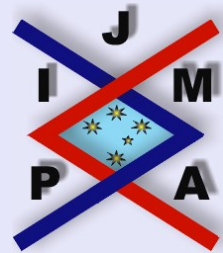
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In the paper [5] Feng Qi proved the following proposition.

**Proposition 1.** *Suppose that  $f \in C^n([a, b])$  satisfies  $f^{(i)}(a) \geq 0$  and  $f^{(n)}(x) \geq n!$  for  $x \in [a, b]$ , where  $0 \leq i \leq n - 1$  and  $n \in \mathbb{N}$ , then*

$$(1) \quad \int_a^b [f(x)]^{n+2} dx \geq \left( \int_a^b f(x) dx \right)^{n+1}.$$

This motivated him to propose an open problem.

**Problem 1.** *Under what conditions does the inequality*

$$(2) \quad \int_a^b [f(x)]^t dx \geq \left( \int_a^b f(x) dx \right)^{t-1}$$

*hold for  $t > 1$ ?*

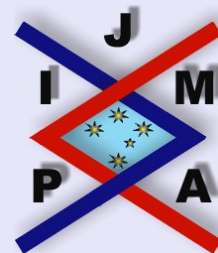
In the joint paper [6], K.W. Yu and F. Qi obtained one answer to the above problem: inequality (2) is valid for all  $f \in C([a, b])$  such that  $\int_a^b f(x) dx \geq (b - a)^{t-1}$  for given  $t > 1$ . Many authors considered different generalizations of Problem 1 (cf. [1, 2, 3, 4]).

We will prove the following answer to Problem 1 which will imply a generalization of Proposition 1.

**Theorem 2.** *Suppose that  $f \in C^1([a, b])$  satisfies  $f(a) \geq 0$  and  $f'(x) \geq (t - 2)(x - a)^{t-3}$  for  $x \in [a, b]$  and  $t \geq 3$ . Then*

$$(3) \quad \int_a^b [f(x)]^t dx \geq \left( \int_a^b f(x) dx \right)^{t-1}$$

*holds. The equality holds only if  $a = b$  or  $f(x) = x - a$  and  $t = 3$ .*



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*Proof.* Let  $f$  be a function satisfying the conditions of Theorem 2. Function  $f$  is increasing because  $f'(x) > 0$  for  $x \in (a, b]$ , so from  $f(\xi) \leq f(x)$  for  $\xi \in [a, x]$  we obtain

$$(4) \quad f(x)(x - a) \geq \int_a^x f(\xi)d\xi, \quad \text{for all } x \in [a, b].$$

Now we define

$$F(x) \triangleq \int_a^x [f(\xi)]^t d\xi - \left( \int_a^x f(\xi)d\xi \right)^{t-1}.$$

Then  $F(a) = 0$  and  $F'(x) = f(x)G(x)$ , where

$$G(x) = [f(x)]^{t-1} - (t - 1) \left( \int_a^x f(\xi)d\xi \right)^{t-2}.$$

Clearly,  $G(a) = [f(a)]^{t-1} \geq 0$  and

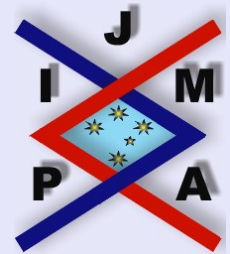
$$G'(x) = (t - 1)f(x) \left( [f(x)]^{t-3} f'(x) - (t - 2) \left( \int_a^x f(\xi)d\xi \right)^{t-3} \right).$$

From the conditions of Theorem 2 and inequality (4) we have

$$(5) \quad [f(x)]^{t-3} f'(x) \geq (t - 2)(f(x)(x - a))^{t-3} \geq (t - 2) \left( \int_a^x f(\xi)d\xi \right)^{t-3}.$$

Thus  $G'(x) \geq 0$ , so with  $G(a) \geq 0$  we get  $G(x) \geq 0$ . From  $F(a) = 0$  and  $F'(x) = f(x)G(x) \geq 0$  it follows that  $F(x) \geq 0$  for all  $x \in [a, b]$ , particularly

$$F(b) = \int_a^b [f(\xi)]^t d\xi - \left( \int_a^b f(\xi)d\xi \right)^{t-1} \geq 0.$$



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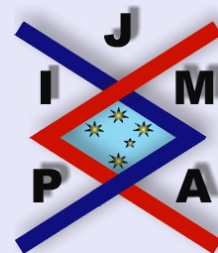
The equality in (3) holds only if  $F'(x) = 0$  for all  $x \in [a, b]$  which is equivalent to  $f(a) = 0$  and  $G'(x) = 0$  and according to (5), if  $t > 3$ , this is valid only for  $f(a) = 0$ ,  $f'(x) = (t - 2)(x - a)^{t-3}$  and  $f$  constant on  $[a, b]$ . But the last two conditions cannot hold simultaneously if  $b \neq a$ . The other possibility for equality to hold is if  $f(a) = 0$  and  $t = 3$ . In that case (5) implies that  $f'(x) = 1$  on  $[a, b]$  so  $f(x) = x - a$ .  $\square$

**Corollary 3.** *Suppose that  $f \in C^1([a, b])$  satisfies  $f(a) \geq 0$  and  $f'(x) \geq n(x - a)^{n-1}$  for  $x \in [a, b]$  and a positive integer  $n$ , then*

$$\int_a^b [f(x)]^{n+2} dx \geq \left( \int_a^b f(x) dx \right)^{n+1}.$$

*Proof.* Set  $t = n + 2$  in Theorem 2.  $\square$

**Remark 1.** *Now we show that Proposition 1 follows from Corollary 3. Let the function  $f$  satisfy the conditions of Proposition 1. Since  $f^{(n)}(x) \geq n!$ , successively integrating  $n - 1$  times over  $[a, x]$  we get  $f'(x) \geq n(x - a)^{n-1}$ ,  $x \in [a, b]$ . Therefore the conditions of Corollary 3 are fulfilled.*



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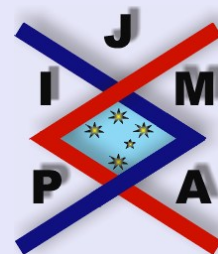
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