Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 1, Article 17, 2004

SOME GEOMETRIC INEQUALITIES FOR THE HOLMES-THOMPSON DEFINITIONS OF VOLUME AND SURFACE AREA IN MINKOWSKI SPACES

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Received 16 June, 2003; accepted 13 January, 2004
Communicated by C.P. Niculescu

ABSTRACT. Let ϵ_d be the volume of the d-dimensional standard Euclidean unit ball. In standard Euclidean space the ratio of the surface area of the unit ball to the volume is equal to the dimension of the space. In Minkowski space (finite dimensional Banach space) where the volume has been normalized according to the Holmes-Thompson definition the ratio is known to lie between $\frac{d\epsilon_d}{2\epsilon_{d-1}}$ and $\frac{d^2\epsilon_d}{2\epsilon_{d-1}}$. We show that when $d = 2$ the lower bound is 2 and equality is achieved if and only if Minkowski space is affinely equivalent to Euclidean, i.e., the unit ball is an ellipse. Stronger criteria involving the inner and outer radii is also obtained for the 2-dimension spaces. In the higher dimensions we discuss the relationship of the Petty's conjecture to the case for equality in the lower limit.

Key words and phrases: Convex body, Isoperimetrix, Mixed volume, Projection body, the Holmes-Thompson definitions of volume and surface area.

2000 *Mathematics Subject Classification.* 52A20, 46B20.

1. INTRODUCTION

In their paper [4] Holmes and Thompson investigated the ratio of

$$\omega(B) = \frac{\epsilon_{d-1}}{d\epsilon_d} \cdot \frac{\mu_B(\partial B)}{\mu_B(B)},$$

where $\epsilon_d = \pi^{d/2}\Gamma(d/2+1)^{-1}$ is the volume of a d-dimensional Euclidean unit ball and $\mu_B(B)$, $\mu_B(\partial B)$ are volume and surface area, respectively, of the unit ball in the d-dimensional Minkowski space for the “Holmes-Thompson definitions" (this will be defined later). They established certain bounds on ω which state that if B is a d-dimensional Minkowski unit ball, then

$$\frac{1}{2} \leq \omega(B) \leq \frac{d}{2}$$

ISSN (electronic): 1443-5756

I would like to thank Professor M. Gage for his valuable comments and suggestions.
081-03

with equality on the right if B is a cube or an 'octahedron'. They raised the question, "What is the lower bound for $\omega(B)$ in R^d?" This problem was solved for the case $d = 2$ in the paper [7]. It was obtained that if B is the unit disc in a two-dimensional Minkowski space, then

$$2 \leq \frac{\mu_B(\partial B)}{\mu_B(B)} \leq \pi$$

with equality on the left if and only if B is an ellipse and equality on the right if and only if B is a parallelogram. Thus, there does not exist another Minkowski plane besides the Euclidean one for which ratio of the length of the unit 'circle' to the area of the unit disc equals 2.

In this paper we prove that for the unit balls possessing a certain property this ratio is greater than d, with equality if and only if B is an ellipsoid and further this property is implied by the Petty's conjectured projection inequality for the unit balls.

There will be also proved some isoperimetric inequalities for the Holmes-Thompson definitions of volume and surface area.

We recommend seeing the interesting book by A.C. Thompson "Minkowski Geometry" for a thorough discussion on this topic.

2. SOME BACKGROUND MATERIAL AND NOTATION

In this section we collect the facts we will need from the theory of convex bodies.

A Minkowski space is a pair $(X, \|\cdot\|)$ in which X is finite dimension and $\|\cdot\|$ is a norm. We will assume $d = \dim X$. The unit ball in $(X, \|\cdot\|)$ is the set

$$B := \{x \in X : \|x\| \leq 1\}.$$

The unit sphere in $(X, \|\cdot\|)$ is the boundary of the unit ball, which is denoted by ∂B. Thus,

$$\partial B := \{x \in X : \|x\| = 1\}.$$

If K is a convex set in X, the polar reciprocal K° of K is defined by

$$K^\circ := \{f \in X^* : f(x) \leq 1 \text{ for all } x \in K\}.$$

The dual ball is the polar reciprocal of B and is also the unit ball in the induced metric on X^*.

Recall that a convex body is a non-empty, closed, bounded convex set.

If K_1 and K_2 are the convex bodies in X, and $\alpha_i \geq 0$, $1 \leq i \leq 2$, then the Minkowski sum of these convex bodies is defined as

$$\alpha_1 K_1 + \alpha_2 K_2 := \{x : x = \alpha_1 x_1 + \alpha_2 x_2,\ x_i \in K_i\}.$$

It is easy to show that the Minkowski sum of convex bodies is itself a convex body.

We shall suppose that X also possesses the standard Euclidean structure and that λ is the Lebesgue measure induced by that structure. We refer to this measure as volume (area) and denote it as $\lambda(\cdot)$. The volume λ gives rise to a dual volume λ^* on the convex subset of X^*, and they coincide in R^d.

Recall that $\lambda(\alpha K) = \alpha^d \lambda(K)$ and $\lambda(\partial(\alpha K)) = \alpha^{d-1}\lambda(\partial K)$, for $\alpha \geq 0$.

Definition 2.1. The function h_K defined by

$$h_K(f) := \sup\{f(x) :\ x \in K\}$$

is called the support function of K.

Note that $h_{\alpha K} = \alpha h_K$, for $\alpha \geq 0$. If K is symmetric, then h_K is even function, and in this case $h_K(f) = \sup\{|f(x)| : x \in K\}$. In R^d we define $f(x)$ as the usual inner product of f and x.

Every support function is sublinear (convex) and conversely every sublinear function is the support function of some convex set (see [12, p. 52]).

Definition 2.2. If K is a convex body with 0 as interior point, then for each $x \neq 0$ in X the radial function $\rho_K(x)$ is defined to be that positive number such that $\rho_K(x)x \in \partial K$.

The support function of the convex body K is the inverse of radial function of K°. In other words $\rho_{K^\circ}(f) = (h_K(f))^{-1}$ and $\rho_K(x) = (h_{K^\circ}(x))^{-1}$.

One of the fundamental theorem of convex bodies states that if K is a symmetric convex body in X, then

$$\lambda(K)\lambda^*(K^\circ) \leq \epsilon_d^2,$$

where ϵ_d is the volume of a d-dimensional Euclidean ball. Moreover, equality occurs if and only if K is an ellipsoid. It is called the Blaschke-Santalo Theorem (see [12, p. 52]).

The best lower bound is known only for convex bodies which are zonoids (see [12, p. 52]). That is

$$\frac{4^d}{d!} \leq \lambda(K)\lambda^*(K^\circ),$$

with equality if and only if K is a parallelotope. It is called Mahler-Reisner Theorem.

Recall that zonoids are the closure of zonotopes with respect to the Hausdorff metric, and zonotopes are finite Minkowski sum of the symmetric line segments. When $d = 2$ all symmetric convex bodies are zonoids (see Gardner's book more about zonoids).

The Euclidean structure on X induces on each $(d-1)$-dimensional subspace (hyperplane) a Lebesgue measure and we call this measure area denoting by $s(\cdot)$. If the surface ∂K of a convex body K does not have a smooth boundary, then the set of points which ∂K is not differentiable is at most countable and has measure 0. We will denote the Euclidean unit vectors in X by u and in X^* by $\hat{f}$.

Definition 2.3. The mixed volume $V(K[d-1], L)$ of the convex bodies K and L in X is defined by

$$V(K[d-1], L) = d^{-1} \lim_{\varepsilon \to 0} \varepsilon^{-1}\{\lambda(K + \varepsilon L) - \lambda(K)\} = d^{-1} \int_{\partial K} h_L(\hat{f}_x) ds(x), \tag{2.1}$$

where $ds(\cdot)$ denotes the Euclidean surface area element of ∂K.

$V(K, \dots, K) = V(K[d])$ is the standard Euclidean volume of $\lambda(K)$. The mixed volume $V(K[d-1], L)$ measures the surface area in some sense and satisfies

$$V(\alpha K[d-1],\ L) = \alpha^{d-1} V(K[d-1],\ L), \text{ for } \alpha \geq 0.$$

See Thompson's book ([12, p. 56]) for those and the other properties of mixed volumes.

Theorem 2.1 (Minkowski inequality for mixed volumes)**.** *(see* [10, p. 317] *or* [12, p. 57]*). If K_1 and K_2 are convex bodies in X, then*

$$V^d(K_1[d-1],\ K_2) \geq \lambda(K_1)^{d-1}\lambda(K_2)$$

with equality if and only if K_1 and K_2 are homothetic.

If $K_2 = B$ is the unit ball in Euclidean space, then this inequality becomes the standard Isoperimetric Inequality.

Definition 2.4. The projection body ΠK of a convex body K in X is defined as the body whose support function is given by

$$h_{\Pi K}(u) = \lim_{\varepsilon \to 0} \frac{\lambda(K + \varepsilon[u]) - \lambda(K)}{\varepsilon},$$

where $[u]$ denotes the line segment joining $-\frac{u}{2}$ to $\frac{u}{2}$.

Note that $\Pi K = \Pi(-K)$ and $\Pi K \subseteq X^*$. The function $h_{\Pi K}$ is the area of the orthogonal projection of K onto a hyperplane perpendicular to u. A projection body is a centered zonoid. If K_1 and K_2 are centered convex bodies in X, and if ΠK_1 and ΠK_2 are equal, then K_1 and K_2 are coincide.

For a convex body K in X and $u \in S^{d-1}$ we denote by $\lambda_{d-1}(K \mid u^{\perp})$ the $(d-1)$ dimensional volume of the projection of K onto a hyperplane orthogonal to u.

Theorem 2.2. *(see* [13]*). A convex body $K \in X$ is a zonoid if and only if*

$$V(K,\ L_1[d-1]) \leq V(K,\ L_2[d-1])$$

for all L_1, $L_2 \in X$ which fulfill $\lambda_{d-1}(L_1 \mid u^{\perp}) \leq \lambda_{d-1}(L_2 \mid u^{\perp})$ for all $u \in S^{d-1}$.

Theorem 2.3. *(see* [3, p. 321] *or* [6]*). If K is a convex body in X, then*

$$\binom{2d}{d} d^{-d} \leq \lambda^{d-1}(K)\lambda((\Pi K)^{\circ}) \leq (\epsilon_d/\epsilon_{d-1})^d$$

with equality on the right side if and only if K is an ellipsoid, and with equality on the left side if and only if K is a simplex.

The right side of this inequality is called the Petty projection inequality, and the left side was established by Zhang.

The k-dimensional convex volume of a convex body lying in a k- dimensional hyperplane Y is a multiple of the standard translation invariant Lebesgue measure, i.e.,

$$\mu = \sigma_B(Y)\lambda.$$

Choosing the 'correct' multiple, which can depend on orientation, is not as easy as it might seem. Also, these two measures μ and λ must agree in the standard Euclidean space.

The Holmes-Thompson d-dimensional volume is defined by

$$\mu_B(K) = \frac{\lambda(K)\lambda^*(B^{\circ})}{\epsilon_d},$$

i.e.,

$$\sigma_B(X) = \frac{\lambda^*(B^{\circ})}{\epsilon_d}$$

and for a k-flat P containing a convex body L

$$\mu_B(L) = \frac{\lambda(L)\lambda^*((P \cap B)^{\circ})}{\epsilon_k}.$$

(See Thompson's book and see also Alvarez-Duran's paper for connections with symplectic volume). This definition coincides with the standard notion of volume if the space is Euclidean. From this point on, the word volume will stand for the Holmes-Thompson volume.

The Holmes-Thompson volume has the following properties:

(1) $\mu_B(B) = \mu_{B^{\circ}}(B^{\circ})$.
(2) $\mu_B(B) \leq \epsilon_d$, is from Blaschke-Santalo Inequality.

The definition can be extended to measure the $(d-1)$-dimension surface volume of a convex body using

$$\mu_B(\partial K) = \int_{\partial K} \sigma_B(\hat{f}_x) ds(\hat{f}_x), \tag{2.2}$$

where ds is standard Lebesgue surface measure and $\hat{f}_x \in X^*$ is zero on the tangent hyperplane at x.

If ∂K does not have a smooth boundary, then the set of points on the boundary of K at which there is not a unique tangent hyperplane has measure zero.

Expanding (2.2) and using Fubini's Theorem one can show that if A and B are two unit balls in X, then

$$\mu_B(\partial A) = \mu_{A^\circ}(\partial B^\circ)$$

and in particular $\mu_B(\partial B) = \mu_{B^\circ}(\partial B^\circ)$.

We can relate the Holmes-Thompson $(d-1)$-dimensional surface volume to the Minkowski mixed volume $V(K[d-1],\ L)$ as follows:

$\sigma_B(\hat{f})$ is a convex function (see Thompson's book), and therefore is the support function of some convex body I_B. Hence equation (2.2) shows that

$$\mu_B(\partial K) = dV(K[d-1],\ I_B), \tag{2.3}$$

where I_B is that convex body whose support function is σ_B.

Note that the ratio $\mu_B(\partial I_B)$ to $\lambda(I_B)$ is equal d, i.e.,

$$\mu_B(\partial I_B) = d\lambda(I_B). \tag{2.4}$$

It turns out (see Thompson's book) that if B is the unit ball in X and I_B is the convex body defined as above, then

$$I_B = \frac{\Pi(B^\circ)}{\epsilon_{d-1}}. \tag{2.5}$$

Thus, I_B is a centered zonoid.

Minkowski Inequality for mixed volume shows that in a Minkowski space $(X,\ B)$, among all convex bodies with volume $\lambda(I_B)$ those with minimum surface volume are the translates of I_B. Likewise, among convex bodies with the Minkowski surface volume $\mu_B(\partial I_B)$ those with maximum volume are the translates of I_B (see [12, p. 144]).

If volume is some other fixed constant, then the convex bodies with minimal surface volume are the translates of a suitable multiple of I_B. The same applies, dually, for the convex bodies of maximum volume for a given surface volume.

The homogenity properties normalize (2.4) by replacing I_B by $\hat{I}_B = \frac{I_B}{\sigma_B}$ so that

$$\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$$

as in the Euclidean case. The convex body $\hat{I}_B$ is called isoperimetrix.

The relation between the Holmes-Thompson surface volume and mixed volume becomes

$$\mu_B(\partial K) = d\sigma_B V(K[d-1],\ \hat{I}_B).$$

3. THE UNIT BALL AND THE ISOPERIMETRIX

We can summarize the relationship between the unit ball and the isoperimetrix. First by definition

$$\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B).$$

Second setting $K = B^\circ$ in Petty projection inequality and using (2.5) for the dual of I_B, we obtain

$$\mu_{B^\circ}(\hat{I}_B^\circ) \leq \mu_{B^\circ}(B^\circ) \tag{3.1}$$

with equality if and only if B is an ellipsoid.

Proposition 3.1. i) *If* $\hat{I}_B \subseteq B$ *then* B *is an ellipsoid and* $\hat{I}_B = B$.

ii) $\mu_{\hat{I}_B}(B) \leq \mu_B(B)$ *and* $\mu_{\hat{I}_B}(\hat{I}_B) \leq \mu_B(\hat{I}_B)$.

Proof. i) If $\hat{I}_B \subseteq B$ then $B^\circ \subseteq \hat{I}_B^\circ$. Thus, $\lambda^*(B^\circ) \leq \lambda^*(\hat{I}_B^\circ)$, which is a contradiction of (3.1).

ii) Multiplying both sides to $\lambda(B)/\epsilon_d$ ($\lambda(\hat{I}_B)/\epsilon_d$) in (3.1), we obtain those inequalities. □

From the above arguments it follows that if $\hat{I}_B = \alpha B$, then $\alpha \geq 1$ and equality holds if and only if B is an ellipsoid.

It is also interesting to know the relationship between $\mu_B(B)$ and $\mu_B(\hat{I}_B)$, which we will apply in the next section. In a two-dimensional space it is not difficult to establish this relationship.

Proposition 3.2. *If (X, B) is a two-dimensional Minkowski space, then*

$$\mu_B(B) \leq \mu_B(\hat{I}_B)$$

with equality if and only if B is an ellipse.

Proof. Recall that in a two-dimensional Minkowski space $\lambda^*(B^\circ) = \lambda(I_B)$, since I_B is the rotation of B°. Then from the Blaschke-Santalo Inequality we obtain

$$\lambda(B) \leq \frac{\pi^2}{\lambda^{*2}(B^\circ)}\lambda^*(B^\circ) = \frac{\pi^2}{\lambda^{*2}(B^\circ)}\lambda(I_B) = \lambda(\hat{I}_B).$$

Thus,

$$\mu_B(B) \leq \mu_B(\hat{I}_B).$$

Obviously, equality holds if and only if B is an ellipse. □

4. THE RATIO OF THE SURFACE AREA TO THE VOLUME FOR THE UNIT BALL AND PETTY'S CONJECTURED PROJECTION INEQUALITY

Petty's conjectured projection inequality (see [8, p. 136]) states that if K is a convex body in X, then

$$\epsilon_d^{-2}\lambda(\Pi K)\lambda^{1-d}(K) \geq \left(\frac{\epsilon_{d-1}}{\epsilon_d}\right)^d \tag{4.1}$$

with equality if and only if K is an ellipsoid.

In his paper [5] Lutwak described this conjecture as "possibly the major open problem in the area of affine isoperimetric inequalities" and gave an 'equivalent' non-technical version of this conjecture. It is also known that this conjecture is true in a two-dimensional Minkowski space (see Schneider [9]).

Setting $K = B^\circ$ (assume $X = R^d$) we can rewrite (4.1) as

$$\epsilon_d^{d-2}\lambda(\Pi B^\circ) \geq \epsilon_{d-1}^d\lambda^{d-1}(B^\circ).$$

Using (2.5), we have

$$\lambda^{d-1}(B^\circ) \leq \epsilon_d^{d-2}\lambda(I_B). \tag{4.2}$$

Multiplying both sides to $\lambda(B^\circ)$, we obtain

$$\mu_B(\hat{I}_B) \geq \epsilon_d. \tag{4.3}$$

Inequalities (4.2) and (4.3) are also Petty's conjectured projection inequality for the unit balls, and these hold with equality when $d = 2$.

In (4.2) using the Blaschke-Santalo Inequality, we get

$$\lambda(B)\lambda^d(B^\circ) \leq \epsilon_d^2\lambda^{d-1}(B^\circ) \leq \epsilon_d^d\lambda(I_B).$$

Thus, we have the next inequality

$$\mu_B(B) \le \mu_B(\hat{I}_B) \tag{4.4}$$

with equality if and only if B is an ellipsoid.

We have obtained that if Petty's conjectured projection inequality for the unit balls holds, then (4.4) is true.

In the previous section we showed that this inequality is valid for the two-dimensional spaces.

If we multiply both sides of (4.2) to $\lambda^{d-1}(B)$ and apply the Minkowski mixed volumes inequality, then

$$\frac{\lambda^{d-1}(B)\lambda^{d-1}(B^\circ)}{\epsilon_d^{d-1}} \le \epsilon_d^{-1}\lambda^{d-1}(B)\lambda(I_B) \le \epsilon_d^{-1}V^d(B[d-1],\ I_B).$$

Using (2.3) for $K = B$, we have

$$\mu_B^d(\partial B) \ge d^d \epsilon_d \mu_B^{d-1}(B) \tag{4.5}$$

with equality if and only if B is an ellipsoid.

We can also rewrite (4.5) as

$$\left(\frac{\mu_B(\partial B)}{\varpi_d}\right)^d \ge \left(\frac{\mu_B(B)}{\epsilon_d}\right)^{d-1}, \tag{4.6}$$

where $\varpi_d = d\epsilon_d$ is the surface area of the unit ball in the Euclidean space.

Inequality (4.6) is the isoperimetric inequality for the Holmes-Thompson definition of volume and surface area, and it is also well known that this inequality is true when $d = 2$.

Theorem 4.1. *If B is the unit ball in a d-dimensional Minkowski space such that $\mu_B(B) \le \mu_B(\hat{I}_B)$, then*

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \ge d$$

with equality if and only if B is an ellipsoid.

Proof. $\mu_B(B) \le \mu_B(\hat{I}_B)$ can be written as

$$\lambda(B)\lambda^d(B^\circ) \le \epsilon_d^d \lambda(I_B).$$

Multiplying both sides to $\frac{\lambda^{d-1}(B)}{\epsilon_d^d}$ and applying Minkowski Inequality for the mixed volumes, we obtain

$$\frac{\lambda^d(B)\lambda^d(B^\circ)}{\epsilon_d^d} \le \lambda^{d-1}(B)\lambda(I_B) \le V^d(B[d-1],\ I_B) = \frac{\mu_B^d(\partial B)}{d^d}.$$

Thus,

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \ge d$$

and equality holds if and only if B is an ellipsoid. □

Corollary 4.2. *Let B be the unit ball in a $d-$dimensional Minkowski space. If Petty's conjectured projection inequality is true for the unit ball, then*

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \ge d$$

with equality if and only if B is an ellipsoid.

Proof. We have been seen that if Petty's conjectured projection inequality is true, then $\mu_B(B) \le \mu_B(\hat{I}_B)$. Hence the result follows from Theorem 4.1. □

Conjecture 4.3. *If B is the unit ball and $\hat{I}_B$ is the isoperimetrix defined as above in a Minkowski space, then*

$$\mu_B(B) \leq \mu_B(\hat{I}_B)$$

with equality if and only if B is an ellipsoid.

It has been shown that this conjecture is true in a two-dimensional Minkowski space.

Definition 4.1. If K is a convex body in X, the inner radius of K, $r(K)$ is defined by

$$r(K) := \max\{\alpha : \exists x \in X \ with \ \alpha \hat{I}_B \subseteq K + x\},$$

and the outer radius of K, $R(K)$ is defined by

$$R(K) := \min\{\alpha : \exists x \in X \ with \ \alpha \hat{I}_B \supseteq K + x\}.$$

Lemma 4.4. *If $r(B)$ is the inner radius of the unit ball of B, then*

$$r(B) \leq 1$$

with equality if and only if B is an ellipsoid.

Proof. We know by (3.1) that $\lambda(\hat{I}_B^\circ) \leq \lambda(B^\circ)$. Using the fact that $B^\circ \subseteq \frac{1}{r}\hat{I}_B^\circ$, we obtain the result. □

Lemma 4.5. *If $d \geq 3$ and $R(B)$ is the outer radius of the unit ball of B in a d-dimensional Minkowski space $(X,\ B)$, then*

$$R(B) \geq \frac{\epsilon_{d-1}}{d\epsilon_d} \binom{2d}{d}^{\frac{1}{d}}.$$

Proof. Setting $K = B^\circ$ in Zhang's inequality and using (2.5) for the dual of I_B we obtain that

$$\lambda(\hat{I}_B^\circ) \geq \lambda(B^\circ) \left(\frac{\epsilon_{d-1}}{\epsilon_d}\right)^d \binom{2d}{d} d^{-d}.$$

The result follows from the fact that $R^d \lambda(B^\circ) \geq \lambda(\hat{I}_B^\circ)$. □

For two-dimensional spaces, it was shown in [7] that $R(B) \geq \frac{3}{\pi}$, with equality if and only if B is an affine regular hexagon.

Remark 4.6. From $R(B) = 1$, it does not follow that B is an ellipsoid.

For two-dimensional Minkowski spaces, stronger result was also obtained. Namely, it was proved that if $r(B)$ and $R(B)$ are the inner and outer radii of the unit disc of B, respectively, in a two-dimensional Minkowski space, then

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \geq r + \frac{1}{r}$$

and

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \geq R + \frac{1}{R},$$

with equality if and only if B is an ellipse (see, [7]).

In a higher dimension, we can also obtain a stronger result when $R(B) \leq 1$, i.e., $B \subseteq \hat{I}_B$. Since I_B is maximizing and minimizing the volume and surface area, respectively, we have

$$\frac{\mu_B(\partial K)^d}{\mu_B(K)^{d-1}} \geq \frac{\mu_B(\partial \hat{I}_B)^d}{\mu_B(\hat{I}_B)^{d-1}} = d^d \mu_B(\hat{I}_B).$$

But $\mu_B(\hat{I}_B) \geq \frac{1}{R^d}\mu_B(B)$.

Hence

$$\frac{\mu_B(\partial B)}{\mu_B(B)} \geq \frac{d}{R}$$

with equality if and only if B is an ellipsoid.

Proposition 4.7. *If B is the unit ball in a d-dimensional Minkowski space such that $\mu_B(\partial B) \geq d\epsilon_d$, then*

(i)) $$\frac{\mu_B(\partial B)}{\mu_B(B)} \geq d,$$

(ii)) $$\left(\frac{\mu_B(\partial B)}{d\epsilon_d}\right)^d \geq \left(\frac{\mu_B(B)}{\epsilon_d}\right)^{d-1}.$$

Proof. Since $\mu_B(B) \leq \epsilon_d$ we obtain both inequalities. □

There exist examples such that $\mu_B(\partial B) < d\epsilon_d$ (see Thompson [11]).

Theorem 4.8. *Let (X, B) be a d- dimensional Minkowski space and $\mu_B(\partial B) \leq d\epsilon_d$, then*

$$\mu_{\hat{I}_B}^{d-1}(B)\mu_B(\hat{I}_B) \leq \epsilon_d^d$$

with equality if and only if B is an ellipsoid.

Proof. Using (2.3), we can rewrite $\mu_B(\partial B) \leq d\epsilon_d$ as

$$V^d(B[d-1],\ I_B) \leq \epsilon_d^d.$$

From the Minkowski Inequality we obtain

(4.7) $$\lambda^{d-1}(B)\lambda(I_B) \leq \epsilon_d^d.$$

We know from the Petty projection inequality that

(4.8) $$\lambda^{d-1}(B^\circ)\lambda(I_B^\circ) \leq \epsilon_d^d.$$

Multiplying (4.7) and (4.8) we get

$$\lambda^{d-1}(B)\lambda^{d-1}(B^\circ)\lambda(I_B)\lambda(I_B^\circ) \leq \epsilon_d^{2d}.$$

The left side of this inequality can be also written as

$$\mu_B^{d-2}(B)\mu_{\hat{I}_B}(B)\mu_B(\hat{I}_B) \leq \epsilon_d^d.$$

Recalling that $\mu_{\hat{I}_B}(B) \leq \mu_B(B) \leq \epsilon_d$, we obtain the desired result. One can see that equality holds if and only B is an ellipsoid. □

Proposition 4.9. *If B is the unit ball in a d-dimensional Minkowski space and if $\lambda_{d-1}(B|u^\perp) \leq \lambda(\hat{I}_B|u^\perp)$ for all $u \in S^{d-1}$, then*

$$\mu_B(B) \leq \mu_B(\hat{I}_B).$$

Proof. Since $\hat{I}_B$ is a zonoid, setting $K = L_2 = \hat{I}_B$ and $L_1 = B$ in Theorem 2.2 we have

$$V^d(B[d-1],\ \hat{I}_B) \leq \lambda^d(\hat{I}_B).$$

Now we can obtain the result from the Minkowski Inequality for the mixed volumes. □

Proposition 4.10. *If B is the unit ball in a d-dimensional Minkowski space such that B is a zonoid, then*

$$\mu_B(\partial B) \geq \frac{4^d}{\epsilon_d(d-1)!}.$$

Proof. Since B is a zoonoid by Mahler-Reizner Inequality we have

$$\mu_B(B) \geq \frac{4^d}{\epsilon_d d!}.$$

Assuming that the conjecture is true, the result follows from Theorem 4.1. □

When $d = 3$, the smallest value of $\mu_B(\partial B)$ that has been found so far is $\frac{36}{\pi}$ in the case when B is either the rhombic-dodecahedron or its dual (see [4] or Section 6.5 in Thompson's book).

Problem 4.11. If B is the unit ball in a d-dimensional Minkowski space such that $\mu_B(\partial B) < d\epsilon_d$, then is this still true

$$\left(\frac{\mu_B(B)}{d\epsilon_{d\partial}}\right)^d \geq \left(\frac{\mu_B(B)}{\epsilon_d}\right)^{d-1}?$$

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