Journal of Inequalities in Pure and Applied Mathematics

ON LANDAU TYPE INEQUALITIES FOR FUNCTIONS WITH HÖLDER CONTINUOUS DERIVATIVES

LJ. MARANGUNIĆ AND J. PEČARIĆ

Department of Applied Mathematics
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, Zagreb, Croatia.
EMail: ljubo.marangunic@fer.hr

Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, Zagreb
Croatia.
EMail: pecaric@element.hr

ISSN (electronic): 1443-5756
079-04

volume 5, issue 3, article 72, 2004.

Received 08 March, 2004; accepted 11 April, 2004.

Communicated by: N. Elezović

Abstract

An inequality of Landau type for functions whose derivatives satisfy Hölder's condition is studied.

2000 Mathematics Subject Classification: 26D15
Key words: Landau inequality, Hölder continuity

Contents

1. Introduction

S.S. Dragomir and C.I. Preda have proved the following theorem (see [1]):

Theorem A. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ locally absolutely continuous function on I. If $f \in L_\infty(I)$ and the derivative $f' : I \to \mathbb{R}$ satisfies Hölder's condition*

$$|f'(t) - f'(s)| \leq H \cdot |t - s|^\alpha \quad \textit{for any} \quad t, s \in I, \tag{1.1}$$

where $H > 0$ and $\alpha \in (0, 1]$ are given, then $f' \in L_\infty(I)$ and one has the inequalities:

$$||f'|| \leq \begin{cases} \left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot ||f||^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} \\ \qquad \textit{if} \quad m(I) \geq 2^{\frac{\alpha+2}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}; \\ \frac{4 \cdot ||f||}{m(I)} + \frac{H}{2^\alpha(\alpha+1)}[m(I)]^\alpha \\ \qquad \textit{if} \quad 0 < m(I) \leq 2^{\frac{\alpha+2}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}, \end{cases} \tag{1.2}$$

where $||\cdot||$ is the ∞-norm on the interval I, and $m(I)$ is the length of I.

In our paper we shall give an improvement of this theorem.

2. Main Results

Theorem 2.1. *Let I be an interval and $f : I \to \mathbb{R}$ function on I satisfying conditions of Theorem A. Then $f' \in L_\infty(I)$ and the following inequlities hold:*

$$(2.1)\qquad ||f'|| \le \begin{cases} \left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot ||f||^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} \\ \qquad \text{if} \quad m(I) \ge 2^{\frac{1}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}; \\ \frac{2||f||}{m(I)} + \frac{H}{\alpha+1}[m(I)]^\alpha \\ \qquad \text{if} \quad 0 < m(I) \le 2^{\frac{1}{\alpha+1}} \left(\frac{||f||}{H}\right)^{\frac{1}{\alpha+1}} \left(1+\frac{1}{\alpha}\right)^{\frac{1}{\alpha+1}}, \end{cases}$$

where $||\cdot||$ is the ∞-norm on the interval I, and $m(I)$ is the length of I.

In our proof and in the subsequent discussion we use three lemmas.

Lemma 2.2. *Let $a, b \in \mathbb{R}$, $a < b$, $\alpha \in (0, 1]$. Then the following inequality holds:*

$$(2.2)\qquad (b-x)^{\alpha+1} + (x-a)^{\alpha+1} \le (b-a)^{\alpha+1}, \quad \forall x \in [a, b].$$

Proof. Consider the function $y : [a, b] \to \mathbb{R}$ given by:

$$y(x) = (b-x)^{\alpha+1} + (x-a)^{\alpha+1}.$$

We observe that the unique solution of the equation

$$y'(x) = (\alpha+1)\left[(x-a)^\alpha - (b-x)^\alpha\right] = 0$$

is $x_0 = \frac{a+b}{2} \in [a,b]$. The function $y'(x)$ is decreasing on (a, x_0) and increasing on (x_0, b). Thus, the maximal values for $y(x)$ are attained on the boundary of $[a,b] : y(a) = y(b) = (b-a)^{\alpha+1}$, which proves the lemma. □

A generalization of the following lemma is proved in [1]:

Lemma 2.3. *Let $A, B > 0$ and $\alpha \in (0,1]$. Consider the function $g_\alpha : (0,\infty) \to \mathbb{R}$ given by:*

$$g_\alpha(\lambda) = \frac{A}{\lambda} + B \cdot \lambda^\alpha. \tag{2.3}$$

Define $\lambda_0 := \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0,\infty)$. Then for $\lambda_1 \in (0,\infty)$ we have the bound

$$\inf_{\lambda \in (0,\lambda_1]} g_\alpha(\lambda) = \begin{cases} \frac{A}{\lambda_1} + B \cdot \lambda_1^\alpha & \text{if} \quad 0 < \lambda_1 < \lambda_0 \\ (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}} \cdot A^{\frac{\alpha}{\alpha+1}} \cdot B^{\frac{1}{\alpha+1}} & \text{if} \quad \lambda_1 \geq \lambda_0. \end{cases} \tag{2.4}$$

Proof. We have:

$$g'_\alpha(\lambda) = -\frac{A}{\lambda^2} + \alpha \cdot B \cdot \lambda^{\alpha-1}.$$

The unique solution of the equation $g'_\alpha(\lambda) = 0$, $\lambda \in (0,\infty)$, is $\lambda_0 = \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0,\infty)$. The function $g_\alpha(\lambda)$ is decreasing on $(0, \lambda_0)$ and increasing on (λ_0, ∞). The global minimum for $g_\alpha(\lambda)$ on $(0,\infty)$ is:

$$g_\alpha(\lambda_0) = A\left(\frac{\alpha B}{A}\right)^{\frac{1}{\alpha+1}} + B\left(\frac{A}{\alpha B}\right)^{\frac{\alpha}{\alpha+1}} = (\alpha+1)\alpha^{-\frac{\alpha}{\alpha+1}} \cdot A^{\frac{\alpha}{\alpha+1}} \cdot B^{\frac{1}{\alpha+1}}, \tag{2.5}$$

which proves (2.4). □

Lemma 2.4. *Let* $A, B > 0$ *and* $\alpha \in (0,1]$. *Consider the functions* $g_\alpha : (0,\infty) \to \mathbb{R}$ *and* $h_\alpha : (0,\infty) \to \mathbb{R}$ *defined by:*

$$\begin{cases} g_\alpha(\lambda) = \frac{A}{\lambda} + B \cdot \lambda^\alpha \\ h_\alpha(\lambda) = \frac{2A}{\lambda} + \frac{B}{2^\alpha}\lambda^\alpha. \end{cases} \tag{2.6}$$

Define $\lambda_0 := \left(\frac{A}{\alpha B}\right)^{\frac{1}{\alpha+1}} \in (0,\infty)$. *Then for* $\lambda_1 \in (0,\infty)$ *we have:*

$$\begin{cases} \inf\limits_{\lambda\in(0,\lambda_1]} g_\alpha(\lambda) < \inf\limits_{\lambda\in(0,\lambda_1]} h_\alpha(\lambda) & \text{if} \quad 0 < \lambda_1 < 2\lambda_0 \\ \inf\limits_{\lambda\in(0,\lambda_1]} g_\alpha(\lambda) = \inf\limits_{\lambda\in(0,\lambda_1]} h_\alpha(\lambda) & \text{if} \quad \lambda_1 \geq 2\lambda_0. \end{cases} \tag{2.7}$$

Proof. In Lemma 2.3, we found that the global minimum for $g_\alpha(\lambda)$ is obtained for $\lambda = \lambda_0$. Similarly we find that the global minimum for $h_\alpha(\lambda)$ is obtained for $\lambda = 2\lambda_0$, and its value is equal to the minimal value of $g_\alpha(\lambda)$, i.e. $h_\alpha(2\lambda_0) = g_\alpha(\lambda_0)$.

The only solution of equation $g_\alpha(\lambda) = h_\alpha(\lambda)$, $\lambda \in (0,\infty)$, is:

$$\lambda_S = \left[\frac{A}{B(1-2^{-\alpha})}\right]^{\frac{1}{\alpha+1}},$$

and we can easily check that $\lambda_0 < \lambda_S < 2\lambda_0$. Thus, for $\lambda_1 < \lambda_0$ we have $g_\alpha(\lambda_1) < h_\alpha(\lambda_1)$ and $\inf\limits_{\lambda\in(0,\lambda_1]} g_\alpha(\lambda) < \inf\limits_{\lambda\in(0,\lambda_1]} h_\alpha(\lambda)$, and the rest of the proof is obvious. □

Proof of Theorem 2.1. Now we start proving our theorem using the identity:

$$f(x) = f(a) + (x-a)f'(a) + \int_a^x [f'(s) - f'(a)]ds; \qquad a, x \in I \tag{2.8}$$

or, by changing x with a and a with x:

$$f(a) = f(x) + (a-x)f'(x) + \int_x^a [f'(s) - f'(x)]ds; \qquad a, x \in I. \tag{2.9}$$

Analogously, we have for $b \in I$:

$$f(b) = f(x) + (b-x)f'(x) + \int_x^b [f'(s) - f'(x)]ds; \qquad b, x \in I. \tag{2.10}$$

From (2.9) and (2.10) we obtain:

$$\begin{aligned} f(b) - f(a) = (b-a)f'(x) &+ \int_x^b [f'(s) - f'(x)]ds \\ &+ \int_a^x [f'(s) - f'(x)]ds; \quad a, b, x \in I \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} f'(x) = \frac{f(b) - f(a)}{b-a} &- \frac{1}{b-a}\int_x^b [f'(s) - f'(x)]ds \\ &- \frac{1}{b-a}\int_a^x [f'(s) - f'(x)]ds. \end{aligned} \tag{2.12}$$

Assuming that $b > a$ we have the inequality:

$$|f'(x)| \leq \frac{|f(b) - f(a)|}{b - a} + \frac{1}{b - a}\left|\int_x^b |f'(s) - f'(x)|ds\right| + \frac{1}{b - a}\left|\int_a^x |f'(s) - f'(x)|ds\right|. \tag{2.13}$$

Since f' is of $\alpha - H$ Hölder type, then:

$$\begin{aligned} \left|\int_x^b |f'(s) - f'(x)|ds\right| &\leq H \cdot \left|\int_x^b |s - x|^\alpha ds\right| \\ &= H\int_x^b (s - x)^\alpha ds \\ &= \frac{H}{\alpha + 1}(b - x)^{\alpha+1}; \qquad b, x \in I,\ b > x \end{aligned} \tag{2.14}$$

$$\begin{aligned} \left|\int_a^x |f'(s) - f'(x)|ds\right| &\leq H \cdot \left|\int_a^x |s - x|^\alpha ds\right| \\ &= H\int_a^x (x - s)^\alpha ds \\ &= \frac{H}{\alpha + 1}(x - a)^{\alpha+1}; \qquad a, x \in I,\ a < x. \end{aligned} \tag{2.15}$$

From (2.13), (2.14) and (2.15) we deduce:

$$|f'(x)| \leq \frac{|f(b) - f(a)|}{b - a} \tag{2.16}$$

$$+\frac{H}{(b-a)(\alpha+1)}[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}];$$
$$a,b,x\in I,\ a<x<b.$$

Since $f\in L_\infty(I)$ then $|f(b)-f(a)|\leq 2\cdot||f||$. Using Lemma 2.2 we obviously get that:

$$|f'(x)|\leq\frac{2||f||}{b-a}+\frac{H}{\alpha+1}(b-a)^\alpha;\qquad a,b,x\in I,\ a<x<b. \tag{2.17}$$

Denote $b-a=\lambda$. Since $a,b\in I$, $b>a$, we have $\lambda\in(0,m(I))$, and we can analyze the right-hand side of the inequality (2.17) as a function of variable λ. Thus we obtain:

$$|f'(x)|\leq\frac{2||f||}{\lambda}+\frac{H}{\alpha+1}\lambda^\alpha=g_\alpha(\lambda) \tag{2.18}$$

for $x\in I$ and for every $\lambda\in(0,m(I))$.

Taking the infimum over $\lambda\in(0,m(I))$ in (2.18), we get:

$$|f'(x)|\leq\inf_{\lambda\in(0,m(I))}g_\alpha(\lambda). \tag{2.19}$$

If we take the supremum over $x\in I$ in (2.19) we conclude that

$$\sup_{x\in I}|f'(x)|=||f'||\leq\inf_{\lambda\in(0,m(I))}g_\alpha(\lambda). \tag{2.20}$$

Making use of Lemma 2.3 we obtain the desired result (2.1). □

Remark 2.1. *Denote* $\lambda_0 = \left[2\left(1+\frac{1}{\alpha}\right)\frac{||f||}{H}\right]^{\frac{1}{\alpha+1}}$. *Comparing the results of Theorem A and Theorem 2.1 we can see that in the case of* $m(I) \geq 2\lambda_0$ *the estimated values for* $||f'||$ *in both theorems coincide. If* $0 < m(I) < 2\lambda_0$ *the estimated value for* $||f'||$ *given by (2.1) is better than the one given by (1.2). Namely, using Lemma 2.4 we have:*

$$\frac{2||f||}{m(I)}+\frac{H}{\alpha+1}[m(I)]^{\alpha} < \frac{4||f||}{m(I)}+\frac{H}{2^{\alpha}(\alpha+1)}[m(I)]^{\alpha}; \quad m(I) \in (0, \lambda_0] \tag{2.21}$$

and

$$\left[2\left(1+\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{\alpha+1}} \cdot ||f||^{\frac{\alpha}{\alpha+1}} \cdot H^{\frac{1}{\alpha+1}} < \frac{4||f||}{m(I)}+\frac{H}{2^{\alpha}(\alpha+1)}[m(I)]^{\alpha}; \quad m(I) \in [\lambda_0, 2\lambda_0). \tag{2.22}$$

Remark 2.2. *Let the conditions of Theorem 2.1 be fulfilled. Then a simple consequence of (2.11) is the following inequality:*

$$|(b-a)f'(x) - f(b) + f(a)| \leq \frac{H}{\alpha+1}\left[(b-x)^{\alpha+1} + (x-a)^{\alpha+1}\right]; \quad a, b, x \in I,\ a < x < b.$$

This result is an extension of the result obtained by V.G. Avakumović and S. Aljančić in [2] (see also [3]).

References

[1] S.S. DRAGOMIR AND C.J. PREDA, Some Landau type inequalities for functions whose derivatives are Hölder continuous, *RGMIA Res. Rep. Coll.*, **6**(2) (2003), Article 3. ONLINE [http://rgmia.vu.edu.au/v6n2.html].

[2] V.G. AVAKUMOVIĆ AND S. ALJANČIĆ, Sur la meilleure limite de la dérivée d'une function assujetie à des conditions supplementaires, *Acad. Serbe Sci. Publ. Inst. Math.*, **3** (1950), 235–242.

[3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.