Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 4, Article 112, 2004

DIFFERENTIAL ESTIMATE FOR n-ARY FORMS ON CLOSED ORTHANTS

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Received 09 April, 2004; accepted 03 December, 2004
Communicated by A. Lupaş

ABSTRACT. We prove a differential inequality for real forms of arbitrary degree, the problem being considered on closed orthants $H \subset \mathbb{R}^n$. A sufficient positivity criterion is derived. Our results allow computer implementation and contain enough information to imply the fundamental $AM - GM$ inequality.

Key words and phrases: Homogeneous polynomial, Symmetric polynomial, Estimate, Positivity.

2000 *Mathematics Subject Classification.* 26D15, 26D05.

1. INTRODUCTION AND NOTATIONS

Positivity criteria for real n-ary d-forms (d-homogeneous polynomials $f : \mathbb{R}^n \to \mathbb{R}$) are of practical significance, but effective results exist for lower degree only.

- For quadratic forms, Sylvester's criterion characterizes strict positivity on $\mathbb{R}^n \setminus \{0_n\}$.
- For symmetric cubics ($d = 3$), positivity on the first orthant $\mathbb{R}^n_+$ is reduced in [2] to a finite number of tests (see Theorem 1.1 below), which are the same for all cubics.
- For symmetric quartics ($d = 4$) on $\mathbb{R}^n_+$ or $\mathbb{R}^n$, and for symmetric quintics ($d = 5$) on $\mathbb{R}^n_+$, positivity is expressed in [11] in terms of finite test-sets depending on the symmetric form. For quartics, explicit discriminants and effective related algorithms (Maple worksheets) are derived in [12].

All mentioned results provide equivalent conditions and allow computer implementation. For arbitrary degree, it is of interest to find "reasonable" sufficient conditions for positivity on closed orthants in $\mathbb{R}^n$. In our entire discussion we require that $n \in \mathbb{N}$, $n \geq 2$.

For every $k \in \{1, \ldots, n\}$, write $0_k := (0, \ldots, 0) \in \mathbb{R}^k$, $1_k := (1, \ldots, 1) \in \mathbb{R}^k$, and set $\epsilon_k := (1_k, 0_{n-k}) \in \mathbb{R}^n$, $\bar{\epsilon}_k := k^{-1}\epsilon_k$. For $x \in \mathbb{R}^n$, it is convenient to write x_k for its kth component. Therefore, we avoid denoting vectors with symbols with lower indexes (upper indexes will be allowed) and $0_k,\ 1_k,\ \epsilon_k,\ \bar{\epsilon}_k$ are *the only exceptions* to this rule.

ISSN (electronic): 1443-5756

076-04

For every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, set

$$\operatorname{supp}(x) := \{j \in \{1, \dots, n\} \mid x_j \neq 0\}, \quad \|x\| := \sum_{j=1}^{n} |x_j|.$$

We need the following theorem, which is known in the context of even symmetric sextics (for the original statement see [2, Th. 3.7, p. 567]).

Theorem 1.1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a symmetric cubic. Then*

$$g \geq 0 \text{ on } \mathbb{R}^n_+ \iff g(\epsilon_k) \geq 0 \text{ for every } k \in \{1, \dots, n\}.$$

2. Main Results

Let us consider an arbitrary d-form $f : \mathbb{R}^n \to \mathbb{R}$, with $d \in \mathbb{N}^*$.

For ease of exposition, let us define the order relation "$\ll$" on $\mathbb{R}^n$ by

$$\begin{aligned} u \ll v &\overset{\text{def}}{\iff} u_j \in \{0, v_j\} \text{ for every } j \in \{1, \dots, n\} \\ &\iff u_j = v_j \text{ for every } j \in \operatorname{supp}(u). \end{aligned}$$

Remark 2.1. The following properties of "$\ll$" are immediate.

1): $0_n \ll u$ for every $u \in \mathbb{R}^n$.
2): $u \ll 1_n \iff u \in \{0, 1\}^n$.
3): For every $u \in \mathbb{R}^n$, the set $\{x \in \mathbb{R}^n \mid x \ll u\}$ is finite.
4): If H is a closed orthant in $\mathbb{R}^n$ and if $u \ll v \in H$, then $u \in H$.
5): If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal linear isomorphism or a permutation of coordinates or a composition of finitely many such operators, then

$$u \ll v \iff Tu \ll Tv.$$

The following result provides a lower estimate[1] for f in terms of its d-th differential $f^{(d)}$. As we shall see, its symmetric variant (Theorem 2.3) contains enough information to imply the fundamental $AM - GM$ inequality.

Theorem 2.2. *Let $H \subset \mathbb{R}^n$ be a closed orthant. There exist $u^1, \dots, u^d \in \{-1, 0, 1\}^n \cap H$, such that $0_n \neq u^1 \ll \cdots \ll u^d$ and*

$$f(x) \geq \frac{\|x\|^d}{d!} \cdot \frac{f^{(d)}(0_n)(u^1, \dots, u^d)}{\|u^1\| \cdots \|u^d\|} \text{ for every } x \in H. \tag{2.1}$$

In particular, if $f^{(d)}$ satisfies the inequality

$$f^{(d)}(0_n)(u^1, \dots, u^d) \geq 0 \tag{2.2}$$

for all $0_n \neq u^1 \ll \cdots \ll u^d \in \{-1, 0, 1\}^n \cap H$, then $f \geq 0$ on H.

Theorem 2.3. *Assume f to be symmetric, with $d \geq 4$. Then there exist in $\{1, \dots, n\}$ integers $k_1 \geq \cdots \geq k_{d-3} \geq k$, such that*

$$f(x) \geq \frac{6\|x\|^d}{d!} f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}) \tag{2.3}$$

$$= \frac{\|x\|^d}{d!} f^{(d)}(0_n)(\bar{\epsilon}_{k_1}, \dots, \bar{\epsilon}_{k_{d-3}}, \bar{\epsilon}_k, \bar{\epsilon}_k, \bar{\epsilon}_k) \text{ for every } x \in \mathbb{R}^n_+. \tag{2.4}$$

In particular, we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c)*, where:*

(a): $f^{(d)}(0_n)(\epsilon_{k_1}, \dots, \epsilon_{k_d}) \geq 0$ *for all* $k_1 \geq \cdots \geq k_d$*,*

[1] Replacing f by $-f$ leads to the corresponding upper estimate.

(b): $f^{(d-3)}(\epsilon_k)(\epsilon_{k_1}, \ldots, \epsilon_{k_{d-3}}) \geq 0$ *for all* $k_1 \geq \cdots \geq k_{d-3} \geq k$,
(c): $f \geq 0$ *on* $\mathbb{R}^n_+$.

Computer implementation of Theorems 2.2 and 2.3 is possible. The presence of $f^{(d)}$ in the above statements poses no serious computation problem, since we have the identity

$$f^{(d)}(0_n)(u^1, \ldots, u^d) = \sum_{J \subset \{1,\ldots,d\}} (-1)^{\mathrm{card}(J)} f\left(-\sum_{j\in J} u^j\right) \text{ for all } u^1, \ldots, u^d \in \mathbb{R}^n.$$

3. PROOFS

3.1. **The Non-symmetric Case.** We first need the following lemma:

Lemma 3.1. *There exist* $u \in \{0,1\}^n$ *and* $y \in \mathbb{R}^n_+$*, such that* $\|y\| = 1$, $\mathrm{supp}(y) = \mathrm{supp}(u)$*, and*

$$f(x) \geq \frac{f'(y)(u)}{d\|u\|} \cdot \|x\|^d \text{ for every } x \in \mathbb{R}^n_+. \tag{3.1}$$

If f *is symmetric, we can find* y *such that* $y_1 \geq \cdots \geq y_n$*. In this case, we have* $u = \epsilon_k$ *for some* $k \in \{1, \ldots, n\}$.

Proof. Let us first observe that inequality (3.1) is d-homogeneous in x. Consider the compact set $K := \{x \in \mathbb{R}^n_+ | \|x\| = 1\}$ and choose $y \in K$, such that $f(y) = \min(f|_K)$. Assume for simplicity that $y_1 \geq \cdots \geq y_n$ (if f is symmetric, we can find y with this property). It follows that $\mathrm{supp}(y) = \{1, \ldots, k\}$ for some $k \leq n$. We claim that

$$kdf(y) = f'(y)(\epsilon_k). \tag{3.2}$$

By Euler's theorem on homogeneous functions we get

$$df(y) = \sum_{j=1}^{k} y_j \frac{\partial f}{\partial x_j}(y) = f'(y)(y). \tag{3.3}$$

We need to consider two cases:

i) If $k = 1$, then $y = \epsilon_1$ and (3.3) obviously reduces to (3.2).
ii) If $k \geq 2$, then $y' := (y_1, \ldots, y_k)$ is a global minimum for the restriction of the map $]0, \infty[^k \ni z \mapsto f(z, 0_{n-k}) \in \mathbb{R}$ to the subset $\{z \in]0, \infty[^k \mid \|z\| = 1\}$. Thus, applying the method of Lagrange multipliers shows that

$$\frac{\partial f}{\partial x_1}(y) = \frac{\partial f}{\partial x_2}(y) = \cdots = \frac{\partial f}{\partial x_k}(y) = \lambda \tag{3.4}$$

for some $\lambda \in \mathbb{R}$. Now (3.3) and (3.4) yield $\lambda = df(y)$. Using this in (3.4) leads to

$$f'(y)(\epsilon_k) = \sum_{j=1}^{k} \frac{\partial f}{\partial x_j}(y) = k\lambda = kdf(y).$$

Our claim is proved. For every $x \in \mathbb{R}^n_+ \setminus \{0_n\}$ we have $\|x\|^{-1}x \in K$, and so

$$\frac{f(x)}{\|x\|^d} = f(\|x\|^{-1}x) \geq f(y) = \frac{f'(y)(\epsilon_k)}{kd} = \frac{f'(y)(\epsilon_k)}{d\|\epsilon_k\|},$$

which proves (3.1) for $u = \epsilon_k$. □

Proof of Theorem 2.2.
Step 1. We first consider the particular case $H = \mathbb{R}^n_+$. Let us show by induction that for every $i \in \{1, \dots, d\}$, there exist $y^i \in \mathbb{R}^n_+$ and $u^1, \dots, u^i \in \{0,1\}^n$, such that $\|y^i\| = 1$, $\operatorname{supp}(y^i) = \operatorname{supp}(u^i)$, $u^i \ll \cdots \ll u^1$, and

$$f(x) \geq \frac{(d-i)!\|x\|^d}{d!\,\|u^1\| \cdots \|u^i\|} \cdot f^{(i)}(y^i)(u^1, \dots, u^i) \text{ for every } x \in \mathbb{R}^n_+. \tag{3.5}$$

For $i = 1$ this is clear, by Lemma 3.1. Assuming the statement to hold for some $i < d$, we will prove it for $i+1$. For simplicity, we shall assume that $y^i_1 \geq \cdots \geq y^i_n$ (if f is symmetric, we can find such y^i at each step of our induction). Consequently, we have $\operatorname{supp}(y^i) = \{1, \dots, k\}$ and $u^i = \epsilon_k$ for some $k \in \{1, \dots, n\}$. Let us observe that the map $\mathbb{R}^k \ni z \mapsto f^{(i)}(z, 0_{n-k})(u^1, \dots, u^i) \in \mathbb{R}$ is a $(d-i)$-form. According to Lemma 3.1, there exist $y^{i+1} = (\zeta, 0_{n-k}) \in \mathbb{R}^k_+ \times \mathbb{R}^{n-k}_+$ and $u^{i+1} = (v, 0_{n-k}) \in \{0,1\}^n$, such that $\|y^{i+1}\| = 1$, $\operatorname{supp}(y^{i+1}) = \operatorname{supp}(u^{i+1})$, and

$$f^{(i)}(z, 0_{n-k})(u^1, \dots, u^i) \geq \frac{f^{(i+1)}(y^{i+1})(u^1, \dots, u^i, u^{i+1})}{(d-i)\|u^{i+1}\|} \cdot \|z\|^{d-i} \text{ for every } z \in \mathbb{R}^k_+.$$

For $z = (y^i_1, \dots, y^i_k)$ we have $(z, 0_{n-k}) = y^i$, $\|z\| = \|y^i\| = 1$, and consequently

$$f^{(i)}(y^i)(u^1, \dots, u^i) \geq \frac{f^{(i+1)}(y^{i+1})(u^1, \dots, u^{i+1})}{(d-i)\|u^{i+1}\|}. \tag{3.6}$$

We also have $0_n \neq u^{i+1} \ll \epsilon_k = u^i$. Now combining (3.5) with (3.6) completes our induction, as well as the proof for $H = \mathbb{R}^n_+$, since $f^{(d)}$ is constant, $f^{(d)}(y^d) = f^{(d)}(0_n)$.
Step 2. We next turn to the general case. Clearly, we can find $\delta_1, \dots, \delta_n \in \{-1, 1\}$, such that the linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$, $Tx = (\delta_1 x_1, \dots, \delta_n x_n)$, maps $\mathbb{R}^n_+$ onto H, that is, $T(\mathbb{R}^n_+) = H$. Applying the conclusion of Step 1 to the d-form $f \circ T$ yields the existence of d vectors $v^1, \dots, v^d \in \mathbb{R}^n_+$, such that $0_n \neq v^d \ll \cdots \ll v^1 \ll 1_n$ and

$$f(Tx) \geq \frac{\|x\|^d}{d!} \cdot \frac{(f \circ T)^{(d)}(0_n)(v^1, \dots, v^d)}{\|v^1\| \cdots \|v^d\|} \text{ for every } x \in \mathbb{R}^n_+.$$

Let us observe that $\|Tx\| = \|x\|$ for every $x \in \mathbb{R}^n$, and that

$$(f \circ T)^{(d)}(0_n)(v^1, \dots, v^d) = f^{(d)}(0_n)(Tv^1, \dots, Tv^d),$$
$$0_n \neq Tv^1 \ll \cdots \ll Tv^d \ll T1_n \in \{-1, 1\}^n.$$

It follows that the vectors $u^i := Tv^i$ are all in $\{-1, 0, 1\}^n \cap H$, and that (2.1) holds. $\square$

3.2. **The Symmetric Case.** The following needed lemma is a slight generalization of Theorem 1.1.

Lemma 3.2. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a symmetric polynomial with $\deg(g) \leq 3$. Then for every $\sigma > 0$ we have*

$$\min\{g(x) \mid x \in \mathbb{R}^n_+,\ \|x\| = \sigma\} = \min_{1 \leq k \leq n} g(\sigma \bar{\epsilon}_k).$$

Proof. Fix $\sigma > 0$ and set $\alpha := \min_{1 \leq k \leq n} g(\sigma \bar{\epsilon}_k)$, $K := \{x \in \mathbb{R}^n_+ \mid \|x\| = \sigma\}$. Hence, $\alpha = g(\sigma \bar{\epsilon}_p)$ for some $p \in \{1, \dots, n\}$. We have for g the decomposition $g = \sum_{i=0}^3 g_i$, with g_i symmetric i-form for every $i \in \{0, 1, 2, 3\}$. Now define the symmetric cubic

$$h : \mathbb{R}^n \to \mathbb{R}, \quad h = \sum_{i=0}^{3} S^{3-i} g_i - \alpha S^3,$$

where $S(x) := \sigma^{-1}\sum_{j=1}^{n} x_j$. Obviously, $h|_K = g|_K - \alpha$. By Theorem 1.1, we have $h \geq 0$ on $\mathbb{R}_+^n$, and so $g|_K \geq \alpha$. Since $\sigma\bar{\epsilon}_p \in K$ and $g(\sigma\bar{\epsilon}_p) = \alpha$, we get $\alpha = \min_{x\in K} g(x)$. □

Proof of Theorem 2.3. As in the proof of Theorem 2.2 (Step 1), by the same induction based on Lemma 3.1, we get $y \in \mathbb{R}_+^n$ and $u^1 = \epsilon_{k_1}, \ldots, u^{d-3} = \epsilon_{k_{d-3}}$, such that $\|y\| = 1$, $k_1 \geq \cdots \geq k_{d-3}$, $\operatorname{supp}(y) = \operatorname{supp}(\epsilon_{k_{d-3}}) = \{1, \ldots, k_{d-3}\}$, and

$$f(x) \geq \frac{6\|x\|^d}{d!} \cdot f^{(d-3)}(y)(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}}) \text{ for every } x \in \mathbb{R}_+^n. \tag{3.7}$$

Note that the above inequality corresponds to (3.5) for $i = d-3$. Since $\|y\| = 1$, $\operatorname{supp}(y) = \{1, \ldots, k_{d-3}\}$, and the polynomial map

$$\mathbb{R}^{k_{d-3}} \ni z \mapsto f^{(d-3)}(z, 0_{n-k_{d-3}})(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}}) \in \mathbb{R}$$

is a symmetric 3-form, applying Lemma 3.2 shows that

$$f^{(d-3)}(y)(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}}) \geq f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}}) \tag{3.8}$$

for some $k \leq k_{d-3}$. Now combining (3.7) with (3.8) yields (2.3). As the map

$$g : \mathbb{R}^k \to \mathbb{R}, \quad g(z) = f^{(d-3)}(z, 0_{n-k})(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}})$$

is a 3-form, we have $6g(z) = g'''(z)(z,z,z) = g'''(0_k)(z,z,z)$ for every $z \in \mathbb{R}^k$. This gives

$$\begin{aligned} f^{(d-3)}(\bar{\epsilon}_k)(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}}) &= g(1_k) \\ &= \frac{g'''(0_k)(1_k, 1_k, 1_k)}{6} \\ &= \frac{f^{(d)}(0_n)(\bar{\epsilon}_{k_1}, \ldots, \bar{\epsilon}_{k_{d-3}}, \bar{\epsilon}_k, \bar{\epsilon}_k, \bar{\epsilon}_k)}{6}, \end{aligned}$$

which proves (2.4). □

Example 3.1. Proof of the fundamental $AM - GM$ inequality by verifying condition (a) from Theorem 2.3.

Proof. Let $n \in \mathbb{N}^*$ and $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \sum_{i=1}^n x_i^n - n\prod_{i=1}^n x_i$. We shall prove that $f \geq 0$ on $\mathbb{R}_+^n$. Since Theorem 1.1 shows this for $n \leq 3$, assume that $n \geq 4$. For all $u^1, \ldots, u^n \in \mathbb{R}^n$, we have

$$f^{(n)}(0_n)(u^1, \ldots, u^n) = \sum_{i_1,\cdots,i_n=1}^{n} \frac{\partial^n f}{\partial x_{i_1}\cdots\partial x_{i_n}}(0_n)\, p_{i_1}(u^1)\cdots p_{i_n}(u^n), \tag{3.9}$$

where $p_1, \ldots, p_n : \mathbb{R}^n \to \mathbb{R}$ are the standard linear projections. For ease of exposition, let us define the map

$$v : \mathbb{R}^n \to \{1, \ldots, n\}, \quad v(x) = \operatorname{card}(\{x_1, \ldots, x_n\}),$$

and consider the set $A := \{1, \ldots, n\}^n$. For every $i = (i_1, \ldots, i_n) \in A$, set

$$\frac{\partial^n f}{\partial x^i} := \frac{\partial^n f}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}} \equiv \begin{cases} n!, & v(i) = 1 \\ -n, & v(i) = n \\ 0, & 1 < v(i) < n \end{cases} \tag{3.10}$$

the last equality being easily checked. Now fix $k_1 \geq \cdots \geq k_n$ in $\{1, \ldots, n\}$, and let

$$B := \{i \in A \,|\, i_1 \leq k_1, i_2 \leq k_2, \ldots, i_n \leq k_n\}.$$

By (3.9) and (3.10) we get

$$f^{(n)}(0_n)(\epsilon_{k_1},\dots,\epsilon_{k_n}) = \sum_{i\in B} \frac{\partial^n f}{\partial x^i}(0_n) = n!\cdot \operatorname{card}(B_1) - n\cdot\operatorname{card}(B_n), \tag{3.11}$$

where $B_1 := \{i \in B \,|\, v(i) = 1\}$ and $B_n := \{i \in B \,|\, v(i) = n\}$. Since obviously

$$B_1 = \{\epsilon_n, 2\epsilon_n, \dots, k_n\epsilon_n\}, \quad B_n \subset \{i \in A \,|\, i_n \le k_n,\ v(i) = n\} =: E,$$

we have $\operatorname{card}(B_1) = k_n$ and $\operatorname{card}(B_n) \le \operatorname{card}(E) = k_n(n-1)!$. To prove the last equality, let us observe that every element $i \in E$ can be obtained by selecting $i_n \in \{1,\dots,k_n\}$ (there are k_n possibilities), and then choosing pairwise distinct $i_1,\dots,i_{n-1} \in \{1,\dots,n\}\setminus\{i_n\}$ (that is, a permutation of this set). By (3.11) we get

$$f^{(n)}(0_n)(\epsilon_{k_1},\dots,\epsilon_{k_n}) \ge 0.$$

The conclusion follows by Theorem 2.3. □

For Pólya's general result on strictly positive forms, we refer the reader to [3]. Bounds for the exponent from Pólya's theorem are given in [5, 8]. Various symmetric inequalities can be found especially in [3, 6], but also in [1, 4, 7, 9]. Some general results on symmetric inequalities can be found in [10] and [11].

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