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ON THE MAXIMUM MODULUS OF POLYNOMIALS. II

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ABSTRACT. Let $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having no zeros in the open unit disc, and suppose that $\max_{|z|=1} |f(z)| = 1$. How small can $\max_{|z|=\rho} |f(z)|$ be for any $\rho \in [0\,,\,1)$? This problem was considered and solved by Rivlin [4]. There are reasons to consider the same problem under the additional assumption that $f'(0) = 0$. This was initiated by Govil [2] and followed up by the present author [3]. The exact answer is known when the degree n is even. Here, we make some observations about the case where n is odd.

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1. INTRODUCTION

For any entire function f let

$$M(f;\rho) := \max_{|z|=\rho} |f(z)| \qquad (0 \le \rho < \infty)\,,$$

and denote by $\mathcal{P}_n$ the class of all polynomials of degree at most n. If $f \in \mathcal{P}_n$, then, applying the maximum modulus principle to the polynomial

$$f^{\sim}(z) := z^n \overline{f(1/\overline{z})}\,,$$

we see that

$$M(f;r) = r^n M(f^{\sim}; r^{-1}) \ge r^n M(f^{\sim};1) = r^n M(f;1) \qquad (0 \le r < 1)\,, \tag{1.1}$$

where equality holds if and only if $f(z) := cz^n$, $c \in \mathbb{C}$, $c \neq 0$. For the same reason

$$M(f;R) = R^n M(f^{\sim}; R^{-1}) \le R^n M(f^{\sim};1) = R^n M(f;1) \qquad (R \ge 1)\,. \tag{1.2}$$

Rivlin [6] proved that if $f \in \mathcal{P}_n$ and $f(z) \neq 0$ for $|z| < 1$, then

$$M(f;r) \ge M(f;1)\left(\frac{1+r}{2}\right)^n \qquad (0 \le r < 1), \tag{1.3}$$

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where equality holds if and only if $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ has a zero of multiplicity n on the unit circle, that is, if and only if $c_0 \neq 0$ and $|c_1| = |p'(0)| = n|c_0|$.

The preceding inequality was generalized by Govil [2] as follows.

Theorem A. *Let $f \in \mathcal{P}_n$. Furthermore let $f(z) \neq 0$ for $|z| < 1$. Then,*

$$M(f; r_1) \geq M(f; r_2) \left(\frac{1 + r_1}{1 + r_2}\right)^n \qquad (0 \leq r_1 < r_2 \leq 1)\,. \tag{1.4}$$

Here again equality holds for polynomials of the form $f(z) := c\,(1 + \mathrm{e}^{\mathrm{i}\gamma} z)^n$, where $c \in \mathbb{C},\ c \neq 0,\ \gamma \in \mathbb{R}$.

The next result which is also due to Govil [2] gives a refinement of (1.4) under the additional assumption that $f'(0) = 0$.

Theorem B. *Let $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ for $|z| < 1$, and let $c_1 = f'(0) = 0$. Then for $0 \leq r_1 < r_2 \leq 1$, we have*

$$M(f; r_1) \geq M(f; r_2) \left(\frac{1 + r_1}{1 + r_2}\right)^n \left\{1 - \frac{(1-r_2)(r_2-r_1)n}{4} \left(\frac{1+r_1}{1+r_2}\right)^{n-1}\right\}^{-1}. \tag{1.5}$$

Improving upon Theorem B, we proved (see [3] or [5, Theorem 12.4.10]) the following result.

Theorem C. *Let $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ for $|z| < 1$, and let $\lambda := c_1/(nc_0)$. Then*

$$M(f; r_1) \geq M(f; r_2) \left(\frac{1 + 2|\lambda| r_1 + r_1^2}{1 + 2|\lambda| r_2 + r_2^2}\right)^{\frac{n}{2}} \qquad (0 \leq r_1 < r_2 \leq 1)\,. \tag{1.6}$$

Note. It may be noted that $0 \leq |\lambda| \leq 1$.

If n is even, then for any $r_2 \in (0\,,\,1]$, and any $r_1 \in [0\,,\,r_2)$, equality holds in (1.6) for

$$f(z) := c(1 + 2|\lambda| \mathrm{e}^{\mathrm{i}\gamma} z + \mathrm{e}^{2\mathrm{i}\gamma} z^2)^{n/2}, \quad c \in \mathbb{C},\ c \neq 0,\ |\lambda| \leq 1,\ \gamma \in \mathbb{R}.$$

By an argument different from the one used to prove Theorem C, we obtained in [4] the following refinement of (1.6).

Theorem D. *Let $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ for $|z| < 1$, and let $\lambda := c_1/(nc_0)$. Then, for any $\gamma \in \mathbb{R}$, we have*

$$|f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})| \geq |f(r_2\,\mathrm{e}^{\mathrm{i}\gamma})| \left(\frac{1 + 2|\lambda| r_1 + r_1^2}{1 + 2|\lambda| r_2 + r_2^2}\right)^{\frac{n}{2}} \qquad (0 \leq r_1 < r_2 \leq 1)\,. \tag{1.7}$$

Again, (1.7) is not sharp for odd n. The proof of (1.7) is based on the observation that for $0 \leq r < 1$, we have

$$r\,\Re \frac{f'(r)}{f(r)} = n - \Re \frac{n}{1 - r\,\varphi(r)} \leq n - \frac{n}{1 + r\,|\varphi(r)|}\,,$$

where

$$\varphi(z) := \frac{f'(z)}{zf'(z) - nf(z)}$$

is analytic in the closed unit disc, and $\max_{|z|=1} |\varphi(z)| \leq 1$. Since $\varphi(0) = -\lambda$, a familiar generalization of Schwarz's lemma [7, p. 212] implies that $|\varphi(r)| \leq (r + \lambda)/(\lambda r + 1)$ for $0 \leq r < 1$, and so if $0 \leq r_1 < r_2 \leq 1$, then

$$|f(r_2)| = |f(r_1)|\,\exp\left(\int_{r_1}^{r_2} \Re \frac{f'(r)}{f(r)}\,\mathrm{d}r\right) \leq |f(r_1)| \left(\frac{1 + 2|\lambda| r_2 + r_2^2}{1 + 2|\lambda| r_1 + r_1^2}\right)^{\frac{n}{2}}\,,$$

which readily leads us to (1.7).

It is intriguing that this reasoning works fine for any even n, and so does the one that was used to prove Theorem C, but somehow both lack the sophistication needed to settle the case where n is odd. We know that when n is even, the polynomials which minimize $|f(r_1)|/|f(r_2)|$ have two zeros of multiplicity $n/2$ each. However, $n/2 \notin \mathbb{N}$ when n is odd, and so the form of the extremals must be different in the case where n is even.

Q.I. Rahman, who co-authored [4], had communicated with James Clunie about Theorem D years earlier, and had asked him for his thoughts about possible extremals when n is odd and c_1 is 0. In other words, what kind of a polynomial f of odd degree n would minimize $|f(r)|/|f(1)|$ if

$$f(z) := \prod_{\nu=1}^{n}(1+\zeta_\nu\, z) \qquad \left(|\zeta_1| \le 1, \ldots, |\zeta_n| \le 1\,;\ \sum_{\nu=1}^{n} \zeta_\nu = 0\right) ?$$

Generally, one uses a variational argument in such a situation. In a written note, Clunie remarked that, in the case of odd degree polynomials, the condition $\sum_{\nu=1}^{n} \zeta_\nu = 0$ is much more difficult to work with than it is in the case of even degree polynomials, and proposed to check if

$$\text{(1.8)} \qquad \frac{|f(r)|}{|f(1)|} \ge \frac{1+r^3}{2} \quad \text{for } \ 0 \le r \le 1 \ \text{ if } \ n=3 \ \text{ and } \ f'(0)=0$$

and

$$\frac{|f(r)|}{|f(1)|} \ge \frac{1+r^3}{2}\,\frac{1+r^2}{2} \quad \text{for } \ 0 \le r \le 1 \ \text{ if } \ n=5 \ \text{ and } \ f'(0)=0\,.$$

He added that *if above held, it would seem reasonable to conjecture that if* $n = 2m+1,\ m \in \mathbb{N}$, *and* $f'(0) = 0$*, then*

$$\text{(1.9)} \qquad \frac{|f(r)|}{|f(1)|} \ge \frac{1+r^3}{2}\left(\frac{1+r^2}{2}\right)^{m-1} \quad \text{for } \ 0 \le r \le 1\,.$$

We shall see that (1.8) does not hold at least for $r = 0$. The same can be said about (1.9).

2. STATEMENT OF RESULTS

Let $\lambda \in \mathbb{C}$, $|\lambda| \le 1$. We shall denote by $\mathcal{P}_{n,\lambda}$ the class of all polynomials of the form $f(z) := \sum_{\nu=0}^{n} c_\nu\, z^\nu$, not vanishing in the open unit disc, such that $c_1/(nc_0) = \lambda$. Thus, if f belongs to $\mathcal{P}_{n,\lambda}$, then

$$f(z) := c_0 \prod_{\nu=1}^{n}(1+\zeta_\nu\, z) \qquad \left(|\zeta_1| \le 1, \ldots, |\zeta_n| \le 1\,;\ \sum_{\nu=1}^{n} \zeta_\nu = n\,\lambda\right) .$$

Let us take any two numbers r_1 and r_2 in $[0,1]$ such that $r_1 < r_2$. Then by (1.7), for any real γ, we have

$$\frac{|f(r_2\,\mathrm{e}^{\mathrm{i}\gamma})|}{|f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})|} \le \left(\frac{1+2|\lambda|r_2+r_2^2}{1+2|\lambda|r_1+r_1^2}\right)^{\frac{n}{2}} \qquad (0 \le r_1 < r_2 \le 1)\,.$$

In addition, we know that the upper bound for $|f(r_2\,\mathrm{e}^{\mathrm{i}\gamma})|/|f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})|$ given by the preceding inequality is attained if the degree n is even, and that it is attained for a polynomial which has exactly two distinct zeros, *each of multiplicity* $n/2$ *and of modulus* 1. When it comes to the case where n is odd, this bound is not sharp. What then is the best possible upper bound for $|f(r_2\,\mathrm{e}^{\mathrm{i}\gamma})/|f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})|$ when n is odd; is the bound attained? If the bound is attained, can we say something about the extremals? We shall first show that

$$\text{(2.1)} \qquad \Omega_{r_1,r_2,\gamma} := \sup\left\{\frac{|f(r_2\,\mathrm{e}^{\mathrm{i}\gamma})|}{|f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})|} : f \in \mathcal{P}_{n,\lambda}\right\}$$

is attained. For this it is enough to prove that for any $c \neq 0$ the polynomials

$$\left\{f \in \mathcal{P}_{n,\lambda} : f(r_1\,\mathrm{e}^{\mathrm{i}\gamma}) = c\right\}$$

form a *normal family of functions*, say $\mathcal{F}_c$ (for the definition of a normal family see [1, pp. 210–211]). In order to prove that $\mathcal{F}_c$ is normal, let $f(z) := a_0 \prod_{\nu=1}^{n}(1+\zeta_\nu z)$, where $|\zeta_1| \leq 1, \ldots, |\zeta_n| \leq 1$. Then $|f(z)| \leq |a_0|\,2^n$ for $|z| = 1$ whereas $|c| = |f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})| \geq |a_0|\,(1-r_1)^n$. Hence

$$\max_{|z|=1} |f(z)| \leq \frac{2^n}{(1-r_1)^n}\,|c|\,,$$

and so, by (1.2), we have

$$\max_{|z|=R>1} |f(z)| \leq \frac{2^n}{(1-r_1)^n}\,|c|\,R^n \qquad (f \in \mathcal{F}_c)\ . \tag{2.2}$$

Since any compact subset of $\mathbb{C}$ is contained in $|z| < R$ for some large enough R, inequality (2.2) implies that the polynomials in $\mathcal{F}_c$ are uniformly bounded on every compact set. By a well-known result, for which we refer the reader to [1, p. 216], the family $\mathcal{F}_c$ is *normal*. Hence $\Omega_{r_1,r_2,\gamma}$, defined in (2.1), is attained. This implies that

$$\omega_{r_1,r_2,\gamma} := \inf\left\{\frac{|f(r_1\,\mathrm{e}^{\mathrm{i}\gamma})|}{|f(r_2\,\mathrm{e}^{\mathrm{i}\gamma})|} : f \in \mathcal{P}_{n,\lambda}\right\} \tag{2.3}$$

is also attained.

Given $r_1 < r_2$ in $[0\,,\,1]$ and a real number γ, let $\mathcal{E} = \mathcal{E}(n;\, r_1,\, r_2;\, \gamma)$ denote the set of all polynomials $f \in \mathcal{P}_{n,\lambda}$ for which the infimum $\omega_{r_1,r_2,\gamma}$ defined in (2.3) is attained. Does a polynomial $f \in \mathcal{P}_{n,\lambda}$ necessarily have all its zeros on the unit circle? We already know that the answer to this question is “yes" for even n, we have yet to find out if the same holds when n is odd. The following result contains the answer.

Theorem 2.1. *For $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ let $\mathcal{P}_{n,\lambda}$ denote the class of all polynomials of the form $f(z) := \sum_{\nu=0}^{n} c_\nu\, z^\nu$, not vanishing in the open unit disc, such that $c_1/(nc_0) = \lambda$. Given $r_1 < r_2$ in $[0\,,\,1]$ and a real number γ, let $\mathcal{E} = \mathcal{E}(n;\, r_1,\, r_2;\, \gamma)$ denote the set of all polynomials $f \in \mathcal{P}_{n,\lambda}$ for which the infimum $\omega_{r_1,r_2,\gamma}$ defined in (2.3) is attained. Then, any $g \in \mathcal{E}$ must have at least $n-1$ zeros on the unit circle.*

The theoretical possibility that a polynomial $g \in \mathcal{E}$ may not have all its n zeros on the unit circle can indeed occur in the case where n is odd. This is illustrated by our next result.

Theorem 2.2. Let $f(z) := \sum_{\nu=0}^{3} c_\nu z^\nu \neq 0$ for $|z| < 1$, and let $c_1 = 0$. Then, for any real γ, we have

$$\frac{|f(0)|}{|f(\rho\,\mathrm{e}^{\mathrm{i}\gamma})|} \geq \frac{4}{4+4\rho^2+\rho^4} \qquad (0 < \rho \leq 1)\,. \tag{2.4}$$

For any given $\rho \in (0\,,\,1]$ equality holds in (2.4) for constant multiples of the polynomial

$$f_\rho(z) := \left(1 - \frac{\rho + \mathrm{i}\sqrt{4-\rho^2}}{4}\,z\,\mathrm{e}^{-\mathrm{i}\gamma}\right)\left(1 - \frac{\rho - \mathrm{i}\sqrt{4-\rho^2}}{4}\,z\,\mathrm{e}^{-\mathrm{i}\gamma}\right)\left(1 + \frac{\rho}{2}\,z\right)\ .$$

Remark 2.3. Inequality (2.4) says in particular that (1.8) does not hold for $r = 0$. In (1.8) it is presumed that the lower bound is attained by a polynomial that has all its zeros on the unit circle. Surprisingly, it turns out to be false.

The following result is a consequence of Theorem 2.2. It is obtained by choosing γ such that $\left|f\left(\rho\,\mathrm{e}^{\mathrm{i}\gamma}\right)\right| = \max_{|z|=\rho} |f(z)|$.

Corollary 2.4. Let $f(z) := \sum_{\nu=0}^{3} c_\nu z^\nu \neq 0$ for $|z| < 1$, and let $c_1 = 0$. Then

$$|f(0)| \geq \frac{4}{4 + 4\rho^2 + \rho^4} \max_{|z|=\rho} |f(z)| \qquad (0 < \rho \leq 1)\,. \tag{2.5}$$

The estimate is sharp for each $\rho \in (0\,,\,1]$.

3. AN AUXILIARY RESULT

Lemma 3.1. For any given $a \in [0\,,\,1/2]$, $b := \sqrt{1-a^2}$ and $\beta \in \mathbb{R}$, let

$$f_{a,\beta}(z) := \left(1 + (a+ib)z\mathrm{e}^{\mathrm{i}\beta}\right)\left(1 + (a-ib)z\mathrm{e}^{\mathrm{i}\beta}\right)\left(1 - 2az\mathrm{e}^{\mathrm{i}\beta}\right)\,.$$

Then, for any $\rho \in [0\,,\,1]$ and any real θ, we have

$$\left|f_{a,\beta}\left(\rho e^{i\theta}\right)\right| \leq \left|f_{a,\beta}\left(-\rho\mathrm{e}^{-\mathrm{i}\beta}\right)\right| = 1 + (1-4a^2)\rho^2 + 2a\rho^3\,.$$

Proof. It is enough to prove the result for $\beta = 0$. The case $a = 1/2$ being trivial, let $a \in (0, 1/2)$. We have

$$\begin{aligned}
\left|f_{a,0}\left(\rho\mathrm{e}^{\mathrm{i}\theta}\right)\right|^2 &= \left|\left(1 + a\rho\mathrm{e}^{\mathrm{i}\theta}\right)^2 + b^2\rho^2\mathrm{e}^{2\mathrm{i}\theta}\right|^2\left(1 - 4a\rho\cos\theta + 4a^2\rho^2\right)\\
&= \left|1 + 2a\rho\mathrm{e}^{\mathrm{i}\theta} + \rho^2\mathrm{e}^{2\mathrm{i}\theta}\right|^2\left(1 - 4a\rho\cos\theta + 4a^2\rho^2\right)\\
&= \left|\mathrm{e}^{-\mathrm{i}\theta} + 2a\rho + \rho^2\mathrm{e}^{\mathrm{i}\theta}\right|^2\left(1 - 4a\rho\cos\theta + 4a^2\rho^2\right)\\
&= \left|\left(1+\rho^2\right)\cos\theta + 2a\rho + \mathrm{i}\,\left(-1+\rho^2\right)\sin\theta\right|^2\\
&\qquad\times\left(1 - 4a\rho\cos\theta + 4a^2\rho^2\right)\\
&= \left\{1 - 2\rho^2 + 4a^2\rho^2 + \rho^4 + \left(4a\rho + 4a\rho^3\right)\cos\theta + 4\rho^2\cos^2\theta\right\}\\
&\qquad\times\left(1 - 4a\rho\cos\theta + 4a^2\rho^2\right)\\
&= \left\{1 - \left(1-4a^2\right)\rho^2\right\}^2 + 4a^2\rho^6 + 4a\rho^3\left(3 - \rho^2 + 4a^2\rho^2\right)\cos\theta\\
&\qquad + 4\left(1-4a^2\right)\rho^2\cos^2\theta - 16a\rho^3\cos^3\theta.
\end{aligned}$$

So, $\left|f_{a,0}\left(\rho\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leq |f_{a,0}(-\rho)|$ for all real θ if and only if

$$a\rho(3 - \rho^2 + 4a^2\rho^2)(1+\cos\theta) - (1-4a^2)(1-\cos^2\theta) - 4a\rho(1+\cos^3\theta) \leq 0\,,$$

that is, if and only if

$$a\rho(3 - \rho^2 + 4a^2\rho^2) - (1-4a^2)(1-\cos\theta) - 4a\rho(1 - \cos\theta + \cos^2\theta) \leq 0\,.$$

To prove this latter inequality, we may replace $\cos\theta$ by t, set

$$A(t) := a\rho\left(3 - \rho^2 + 4a^2\rho^2\right) - 1 + 4a^2 - 4a\rho + \left(1 - 4a^2 + 4a\rho\right)t - 4a\rho t^2$$

and show that $A(t) \leq 0$ for $-1 \leq t \leq 1$. First we note that

$$A(-1) \leq A(1) = \{-1 - (1-4a^2)\rho^2\}\,a\rho < 0\,,$$

and so, we may restrict ourselves to the open interval $(-1, 1)$.

Clearly, $A'(t)$ vanishes if and only if $t = (1 - 4a^2 + 4a\rho)/(8a\rho)$ which is inadmissible for $\rho \leq (1-4a^2)/(4a)$. So, if $\rho \leq (1-4a^2)/(4a)$, then $A'(t)$ is positive for all $t \in (-1, 1)$ since $A'(0)$ is; and $A(t) \leq A(1) \leq 0$.

Now, let $\rho > (1-4a^2)/(4a)$. Since $A''(t) = -8a\rho < 0$, the function A must have a local maximum at $t = (1-4a^2+4a\rho)/(8a\rho)$. However,

$$\begin{aligned} A\left(\frac{1-4a^2+4a\rho}{8a\rho}\right) &= a\rho(3-\rho^2+4a^2\rho^2) - 1 + 4a^2 - 4a\rho \\ &\qquad + \frac{(1-4a^2+4a\rho)^2}{8a\rho} - \frac{(1-4a^2+4a\rho)^2}{16a\rho} \\ &= -\{a\rho + (1+a\rho^3)(1-4a^2)\} \\ &\qquad + \frac{(1-4a^2)^2 + 16a^2\rho^2 + 8a\rho(1-4a^2)}{16a\rho} \\ &= -(1+a\rho^3)(1-4a^2) + \frac{(1-4a^2)^2}{16a\rho} + \frac{1}{2}(1-4a^2) \\ &= \left\{-\left(\frac{1}{2}+a\rho^3\right) + \frac{1-4a^2}{16a\rho}\right\}(1-4a^2) \\ &< -\left(\frac{1}{4}+a\rho^3\right)(1-4a^2) \quad \text{since} \quad \rho > \frac{1-4a^2}{4a} \\ &< 0\,. \end{aligned}$$

□

4. PROOFS OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. Let $g(z) := c_0 \prod_{\nu=1}^{n}(1+\zeta_\nu z)$. Suppose, if possible, that $|\zeta_j| < 1$ and $|\zeta_k| < 1$, where $1 \le j < k \le n$. Now, consider the function

$$\psi(w) := \frac{\{1+(\zeta_j - w)r_1\,\mathrm{e}^{\mathrm{i}\gamma}\}\{1+(\zeta_k + w)r_1\,\mathrm{e}^{\mathrm{i}\gamma}\}}{\{1+(\zeta_j - w)r_2\,\mathrm{e}^{\mathrm{i}\gamma}\}\{1+(\zeta_k + w)r_2\,\mathrm{e}^{\mathrm{i}\gamma}\}}\,,$$

which is analytic and different from zero in the disc $|w| < 2\delta$ for all small $\delta > 0$. Hence, its minimum modulus in $|w| < \delta$ cannot be attained at $w = 0$. This means that if g_w is obtained from g by changing ζ_j to $\zeta_j - w$ and ζ_k to $\zeta_k + w$, then, for all small $\delta > 0$, we can find w of modulus δ such that

$$\left|\frac{g_w\left(r_1\,\mathrm{e}^{\mathrm{i}\gamma}\right)}{g_w\left(r_2\,\mathrm{e}^{\mathrm{i}\gamma}\right)}\right| < \left|\frac{g\left(r_1\,\mathrm{e}^{\mathrm{i}\gamma}\right)}{g\left(r_2\,\mathrm{e}^{\mathrm{i}\gamma}\right)}\right|\,.$$

This is a contradiction since $g_w \in \mathcal{P}_{n,\lambda}$ for $|w| < \min\{1-|\zeta_j|\,,\,1-|\zeta_k|\}$. □

Proof of Theorem 2.2. We wish to minimize the quantity $|f(0)|/|f(\rho\mathrm{e}^{\mathrm{i}\gamma})|$ over the class $\mathcal{P}_{3,0}$ of all polynomials of the form

$$f(z) := c_0 \prod_{\nu=1}^{3}(1+\zeta_\nu z) \qquad \left(|\zeta_1| \le 1,\ |\zeta_2| \le 1,\ |\zeta_3| \le 1,\ \sum_{\nu=1}^{3}\zeta_\nu = 0\right).$$

Given $\rho \in (0\,,\,1]$ and $\gamma \in \mathbb{R}$, let

$$m_{\rho,\gamma} := \inf\left\{\frac{|f(0)|}{|f(\rho\mathrm{e}^{\mathrm{i}\gamma})|} : f \in \mathcal{P}_{3,0}\right\}.$$

As we have already explained, $m_{\rho,\gamma}$ is attained, i.e., there exists a cubic $f^* \in \mathcal{P}_{3,0}$ such that

$$\frac{|f^*(0)|}{|f^*(\rho\mathrm{e}^{\mathrm{i}\gamma})|} = m_{\rho,\gamma}\,.$$

In fact, there is at least one such cubic f^* with $f^*(0) = 1$. By Theorem 2.1, the cubic f^* must have *at least* two zeros on the unit circle. In other words, if $f^*(z) := \prod_{\nu=1}^{3}(1 + \zeta_\nu\, z)$, then at most one of the numbers $\zeta_1,\ \zeta_2$, and ζ_3 can lie in the open unit disc. Thus, only two possibilities need to be considered, namely (i) $|\zeta_1| = |\zeta_2| = |\zeta_3| = 1$, and (ii) $|\zeta_1| = |\zeta_2| = 1,\ 0 < |\zeta_3| < 1$.

Case (i). Since $\zeta_1 + \zeta_2 + \zeta_3 = 0$, the extremal f^* could only be of the form $f^*(z) := 1 + z^3 e^{3i\beta},\ \beta \in [0, 2\pi/3]$, and then we would clearly have

$$\frac{|f^*(0)|}{|f^*(\rho e^{i\gamma})|} \geq \frac{1}{1+\rho^3} \qquad (0 < \rho \leq 1,\ \gamma \in \mathbb{R})\ . \tag{4.1}$$

Case (ii). This time, because of the condition $\zeta_1 + \zeta_2 + \zeta_3 = 0$, the extremal f^* could only be of the form

$$f^*(z) := \left\{1 + (a+ib)ze^{i\beta}\right\}\left\{1 + (a-ib)ze^{i\beta}\right\}(1 - 2aze^{i\beta})\,,$$

where $0 < a < 1/2,\ b = \sqrt{1-a^2}$ and $\beta \in \mathbb{R}$. Then, for any real γ and any $\rho \in (0\,,\,1]$, we would, by Lemma 3.1, have

$$\frac{|f^*(0)|}{|f^*(\rho e^{i\gamma})|} \geq \min_{0<a<1/2} \frac{1}{1 + (1-4a^2)\rho^2 + 2a\rho^3} = \frac{4}{4+4\rho^2+\rho^4}\,. \tag{4.2}$$

Comparing (4.1) and (4.2), we see that if $f \in \mathcal{P}_{3,0}$, then

$$\frac{|f(0)|}{|f(\rho e^{i\gamma})|} \geq \frac{4}{4+4\rho^2+\rho^4} \qquad (0 < \rho \leq 1,\ \gamma \in \mathbb{R})\ ,$$

which proves (2.4). □

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