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## A GENERALIZATION FOR OSTROWSKI'S INEQUALITY IN $\mathbb{R}^2$

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Abstract

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## Abstract

We establish a new Ostrowski's inequality in  $\mathbb{R}^2$  by using an idea of B.G. Pachpatte.

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*Key words:* Ostrowski's inequality, Mean value theorem.

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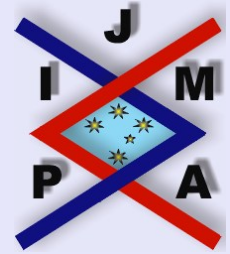
A.M. Ostrowski proved the following inequality (see [1, p. 226–227]):

$$(1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all  $x \in [a, b]$ , where  $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty = \sup_{x \in (a, b)} |f'(x)| < \infty$ .

Recently, by using a fairly elementary analysis, B.G. Pachpatte [2] established the following inequality of type (1) involving two functions and their derivatives.

**Theorem 1.** *Let  $f, g : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions on  $[a, b]$  and differentiable on  $(a, b)$ , whose derivative  $f', g' : (a, b) \rightarrow \mathbb{R}$  are bounded on*



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$(a, b)$ , i.e.,  $\|f'\|_\infty = \sup_{x \in (a,b)} |f'(x)| < \infty$ ,  $\|g'\|_\infty = \sup_{x \in (a,b)} |g'(x)| < \infty$ . Then

$$\left| f(x)g(x) - \frac{1}{2(b-a)} \left[ g(x) \int_a^b f(y)dy + f(x) \int_a^b g(y)dy \right] \right| \leq \frac{1}{2} \{ |g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty \} \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a),$$

for all  $x \in [a, b]$ .

In this paper, by means of B.G. Pachpatte's idea, we prove the following

**Theorem 2.** Let  $D = [a, b] \times [a, b]$ ,  $intD = (a, b) \times (a, b)$ ,  $f, g : D \rightarrow \mathbb{R}$  be continuous functions on  $D$  and differentiable on  $intD$ , whose partial derivatives  $f_x, f_y, g_x, g_y : intD \rightarrow \mathbb{R}$  are bounded on  $intD$ , i.e.,

$$\|f_x\|_\infty = \sup_{(x,y) \in intD} |f_x(x, y)| < \infty, \quad \|f_y\|_\infty = \sup_{(x,y) \in intD} |f_y(x, y)| < \infty,$$

$$\|g_x\|_\infty = \sup_{(x,y) \in intD} |g_x(x, y)| < \infty, \quad \|g_y\|_\infty = \sup_{(x,y) \in intD} |g_y(x, y)| < \infty.$$

Then

$$\left| f(u_1, v_1)g(u_1, v_1) - \frac{1}{2(b-a)^2} \left[ g(u_1, v_1) \iint_D f(u_2, v_2)du_2dv_2 + f(u_1, v_1) \iint_D g(u_2, v_2)du_2dv_2 \right] \right|$$



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$$\begin{aligned} &\leq \frac{1}{2} [|g(u_1, v_1)| \|f_x\|_\infty + |f(u_1, v_1)| \|g_x\|_\infty] \left[ \frac{1}{4} + \frac{(u_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \\ &\quad + \frac{1}{2} [|g(u_1, v_1)| \|f_y\|_\infty + |f(u_1, v_1)| \|g_y\|_\infty] \left[ \frac{1}{4} + \frac{(v_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \end{aligned}$$

for all  $(u_1, v_1) \in D$ .

By taking  $g(x, y) = 1$  in Theorem 2, we get the following Ostrowski like inequality in  $\mathbb{R}^2$ ,

**Corollary 3.**

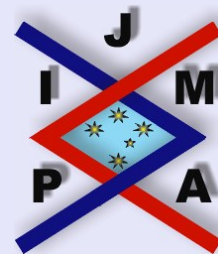
$$\begin{aligned} &\left| f(u_1, v_1) - \frac{1}{(b-a)^2} \iint_D f(u_2, v_2) du_2 dv_2 \right| \\ &\quad \leq \|f_x\|_\infty \left[ \frac{1}{4} + \frac{(u_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \\ &\quad \quad + \|f_y\|_\infty \left[ \frac{1}{4} + \frac{(v_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a). \end{aligned}$$

*Proof of Theorem 2.* By the mean value theorem, there exist  $\xi_1, \eta_1, \xi_2, \eta_2 \in (a, b)$  such that

$$(2) \quad f(u_1, v_1) - f(u_2, v_2) = f_x(\xi_1, v_1)(u_1 - u_2) + f_y(u_2, \eta_1)(v_1 - v_2),$$

and

$$(3) \quad g(u_1, v_1) - g(u_2, v_2) = g_x(\xi_2, v_1)(u_1 - u_2) + g_y(u_2, \eta_2)(v_1 - v_2).$$



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Multiplying both sides of (2) and (3) by  $g(u_1, v_1)$  and  $f(u_1, v_1)$  respectively and adding we get

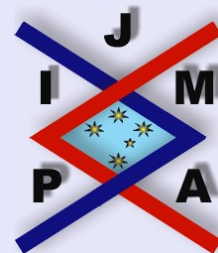
$$\begin{aligned} & 2f(u_1, v_1)g(u_1, v_1) - [f(u_2, v_2)g(u_1, v_1) + f(u_1, v_1)g(u_2, v_2)] \\ &= g(u_1, v_1)[f_x(\xi_1, v_1)(u_1 - u_2) + f_y(u_2, \eta_1)(v_1 - v_2)] \\ & \quad + f(u_1, v_1)[g_x(\xi_2, v_1)(u_1 - u_2) + g_y(u_2, \eta_2)(v_1 - v_2)]. \end{aligned}$$

Integrate both sides with respect to  $u_2, v_2$  over  $D$ . Note that by the proof of the mean value theorem, we know that  $f_x(\xi_1, v_1)$ ,  $f_y(u_2, \eta_1)$ ,  $g_x(\xi_2, v_1)$  and  $g_y(u_2, \eta_2)$  are Riemann-integrable for  $(u_2, v_2) \in D$ . Rewriting we get

$$\begin{aligned} & f(u_1, v_1)g(u_1, v_1) - \frac{1}{2(b-a)^2} \left[ g(u_1, v_1) \iint_D f(u_2, v_2) du_2 dv_2 \right. \\ & \quad \left. + f(u_1, v_1) \iint_D g(u_2, v_2) du_2 dv_2 \right] \\ &= \frac{1}{2(b-a)^2} g(u_1, v_1) \iint_D [f_x(\xi_1, v_1)(u_1 - u_2) + f_y(u_2, \eta_1)(v_1 - v_2)] du_2 dv_2 \\ & \quad + \frac{1}{2(b-a)^2} f(u_1, v_1) \iint_D [g_x(\xi_2, v_1)(u_1 - u_2) + g_y(u_2, \eta_2)(v_1 - v_2)] du_2 dv_2. \end{aligned}$$

So

$$\left| f(u_1, v_1)g(u_1, v_1) - \frac{1}{2(b-a)^2} \left[ g(u_1, v_1) \iint_D f(u_2, v_2) du_2 dv_2 \right. \right. \\ \left. \left. + f(u_1, v_1) \iint_D g(u_2, v_2) du_2 dv_2 \right] \right|$$




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$$\begin{aligned} &\leq \frac{1}{2(b-a)^2} |g(u_1, v_1)| \\ &\quad \times \left[ \|f_x\|_\infty \iint_D |u_1 - u_2| du_2 dv_2 + \|f_y\|_\infty \iint_D |v_1 - v_2| du_2 dv_2 \right] \\ &\quad + \frac{1}{2(b-a)^2} |f(u_1, v_1)| \\ &\quad \times \left[ \|g_x\|_\infty \iint_D |u_1 - u_2| du_2 dv_2 + \|g_y\|_\infty \iint_D |v_1 - v_2| du_2 dv_2 \right]. \end{aligned}$$

Note that

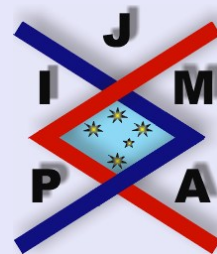
$$\begin{aligned} \iint_D |u_1 - u_2| du_2 dv_2 &= (b-a) \left[ \frac{(u_1 - a)^2 + (u_1 - b)^2}{2} \right] \\ &= \left[ \frac{1}{4} + \frac{(u_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \end{aligned}$$

and

$$\iint_D |v_1 - v_2| du_2 dv_2 = \left[ \frac{1}{4} + \frac{(v_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a).$$

We obtain that

$$\begin{aligned} &\left| f(u_1, v_1)g(u_1, v_1) - \frac{1}{2(b-a)^2} \left[ g(u_1, v_1) \iint_D f(u_2, v_2) du_2 dv_2 \right. \right. \\ &\quad \left. \left. + f(u_1, v_1) \iint_D g(u_2, v_2) du_2 dv_2 \right] \right| \end{aligned}$$



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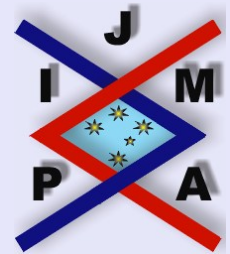
$$\leq \frac{1}{2} [|g(u_1, v_1)| \|f_x\|_\infty + |f(u_1, v_1)| \|g_x\|_\infty] \left[ \frac{1}{4} + \frac{(u_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)$$

$$+ \frac{1}{2} [|g(u_1, v_1)| \|f_y\|_\infty + |f(u_1, v_1)| \|g_y\|_\infty] \left[ \frac{1}{4} + \frac{(v_1 - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a).$$

□

**Remark 1.** Let  $f(x, y) = f(x), g(x, y) = g(x)$  in Theorem 2. Then  $f_y = 0, g_y = 0$ . We obtain Theorem 1.

**Remark 2.** Let  $f(x, y) = f(x), g(x, y) = 1$  in Theorem 2, we recapture the well known Ostrowski inequality.



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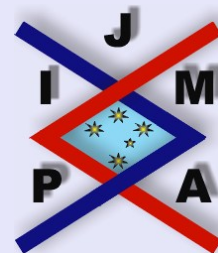
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