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BOUNDS FOR LINEAR RECURRENCES WITH RESTRICTED COEFFICIENTS

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ABSTRACT. This paper derives inequalities for general linear recurrences. Optimal bounds for solutions to the recurrence are obtained when the coefficients of the recursion lie in intervals that include zero. An important aspect of the derived bounds is that they are easily computable. The results bound solutions of triangular matrix equations and coefficients of ratios of power series.

Key words and phrases: Recurrence, Restricted Coefficients, Power Series, Triangular Matrices.

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1. INTRODUCTION

This paper derives bounds for solutions to the linear recurrence

$$b_n = \sum_{k=1}^{n-1} \alpha_{n,k} b_k,\ n \geq 2. \tag{1.1}$$

Throughout, we assume that $b_1 \neq 0$ as $b_1 = 0$ implies that $b_n = 0$ for all $n \geq 2$. Our results bound $\{b_n\}_{n=1}^{\infty}$ in a term-by-term manner with a second order time-homogeneous linear recursion that is readily analyzable.

Our motivation for studying (1.1) lies in applied probability. There it is useful to have a bound for coefficients of a ratio of power series when limited information is available on the constituent series (cf. Kijima [14], Kendall [13], Heathcote [11], Feller [6]). The series comprising the ratio are often probability generating functions. Linear algebra is another setting where (1.1) arises.

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Example 1.1. What is the largest $|b_5|$ possible in (1.1) when $b_1 = -1$ and $\alpha_{n,k} \in [-3, 0]$ for all n and k? In Section 2, we show that $|b_5| \leq 99$ for such situations, and that this value is produced by $\alpha_{n,k}$ having the alternating form

$$\begin{array}{ccccc} & \alpha_{n,1} & \alpha_{n,2} & \alpha_{n,3} & \alpha_{n,4} \\ n=2 & -3 & & & \\ n=3 & 0 & -3 & & \\ n=4 & -3 & 0 & -3 & \\ n=5 & 0 & -3 & 0 & -3 \end{array} \quad . \tag{1.2}$$

Specifically, these $\alpha_{n,k}$ give $b_2 = 3, b_3 = -9, b_4 = 30$, and $b_5 = -99$. We return to this example in Section 2.

Example 1.2. For a fixed $I \subset \Re$, let $\mathcal{F}_I$ be the set of I-power series defined by

$$F_I = \{f : f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \text{ and } a_k \in I \text{ for each } k \geq 1\}. \tag{1.3}$$

Flatto, Lagarias, and Poonen [7] and Solomyak [22] proved independently that if z is a root of a series in $\mathcal{F}_{[0,1]}$, then $|z| \geq 2/(1+\sqrt{5})$. As $z = -2/(1+\sqrt{5})$ is a root of $1+z+z^3+z^5+\cdots$, this bound is tight over $\mathcal{F}_{[0,1]}$. The coefficients of the multiplicative inverse of a series in $\mathcal{F}_{[0,1]}$ cannot increase at a rate larger than the golden ratio.

We will show later that the coefficients of the multiplicative inverse of a power series in $\mathcal{F}_{[0,1]}$ are bounded by the ubiquitous Fibonacci numbers. This gives a "first constant" for the aforementioned rate. Observe that

$$\left(1 + \sum_{n=1}^{\infty} z^{2n-1}\right)^{-1} = 1 - z + z^2 - 2z^3 + 3z^4 - 5z^4 + \cdots , \tag{1.4}$$

the coefficients on the right hand side of (1.4) having the magnitude of the Fibonacci numbers. Hence, the first constant is also good. We return to this setting in Section 4.

Example 1.3. Consider the lower triangular linear system $L\vec{x} = \vec{b}$ where L is the 10×10 matrix with (i, j)th entry

$$L_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 10, & \text{if } i > j \\ 0, & \text{if } i < j \end{cases} , \tag{1.5}$$

and the ith component of $\vec{b}$ is $b_i = i^2$ for $1 \leq i \leq 10$. The exact solution is

$$\vec{x} = \begin{bmatrix} 1 \\ -6 \\ 59 \\ -524 \\ 4725 \\ -42514 \\ 382639 \\ -3443736 \\ 30993641 \\ -278942750 \end{bmatrix} = L^{-1}\vec{b}. \tag{1.6}$$

The condition number of L is 26633841560.0; this essentially drives the rate of growth of x_i in i (cf. Trefethen and Bau [23] for general discussion). Our results will imply that all matrix equations $L\vec{x} = \vec{b}$, with L an $n \times n$ unit lower triangular matrix with $L_{i,j} \in [0, 10]$ for $1 \le i < j \le n$ and $|b_i| \le i^2$, have solutions whose ith component x_i is bounded by (coefficients rounded to three decimal places)

$$|x_i| \le (0.142)\ 10.099^i + 3.538\ (-0.099)^i - 0.400\, i + 0.320,\ 1 \le i \le n. \tag{1.7}$$

The first four values of the right hand side of (1.7) are $1, 14, 145$, and 1472. These show essentially the same order of magnitude as the x_i's; hence the bound is performing reasonably. We return to this example in Section 3.

Recurrences with varying or random coefficients have been studied by many previous authors. A partial survey of such literature contains Viswanath [24] and [25], Viswanath and Trefethen [26], Embree and Trefethen [5], Wright and Trefethen [28], Mallik [16], Popenda [20], Kittapa [15], and Odlyzko [19].

Our methods of proof are based on a careful analysis of sign changes in solutions to (1.1). This differs considerably from past authors, who typically take a more analytic approach. An advantage of our discourse is that it is entirely elementary, discrete, and self-contained. A disadvantage of our arguments lie with laborious bookkeeping.

Study of (1.1) could alternatively be based on linear algebraic or analytic techniques. Some of the applications considered here, namely solutions of linear matrix equations and coefficients of ratios of power series, are indeed classical problems. However, linear algebraic and analytic techniques have yielded disappointing explicit bounds to date. Hence, this paper explores alternative methods.

The rest of this paper proceeds as follows. Section 2 presents the main theorem, some variants of this result, and discussion of the hypotheses and optimality. Sections 3 and 4 consider application of the results to lower triangular linear systems and coefficients of ratios of power series, respectively. Proofs are deferred to Section 5. There, a simple case of our main result is first proven to convey the logic of our sign change analyses.

2. RESULTS

The general form of our main result is the following.

Theorem 2.1. *Suppose that $A \ge 1$ and $0 \le B \le A$ are constants and that $\{D_n\}_{n=2}^{\infty}$ is a nondecreasing sequence of nonnegative real numbers. Suppose that the coefficients in (1.1) are restricted to intervals: $\alpha_{n,1} \in [-D_n, D_n]$ for $n \ge 2$ and $\alpha_{n,k} \in [-A, B]$ for $n \ge 2$ and $2 \le k \le n-1$. Then solutions to (1.1) satisfy $|b_n|/|b_1| \le U_n$ for all $n \ge 1$, where*

$$U_n = \begin{cases} 1, & \text{if } n = 1 \\ D_2, & \text{if } n = 2 \\ AD_2 + D_3, & \text{if } n = 3 \\ AU_{n-1} + (1+B)U_{n-2} + D_n - D_{n-2}, & \text{if } n > 3 \end{cases}. \tag{2.1}$$

Neglecting the bookkeeping complications induced by a general $\{D_n\}$, the difference equation in (2.1) is second-order, time-homogeneous, and linear. In many cases, one can solve (2.1) explicitly for U_n. As such, we view U_n as being "easy to compute". The generality added by a non-decreasing $\{D_n\}$ is relevant in probabilistic settings where generalized renewal equations are common (cf. Feller [6] and Heathcote [11]).

For cases where asymmetric bounds on $\alpha_{n,1}$ are available, we offer the following.

Theorem 2.2. *Suppose that* $A \geq 1$ *and that* $C \geq 0$ *and* $D \geq 0$. *If* $\alpha_{n,1} \in [-C, D]$ *and* $\alpha_{n,k} \in [-A, 0]$ *for all* $n \geq 2$ *and* $2 \leq k \leq n-1$, *then* $|b_n|/|b_1| \leq U_n$ *for all* $n \geq 1$, *where*

$$U_n = \begin{cases} 1, & \text{if } n = 1 \\ \max(C, D), & \text{if } n = 2 \\ A\max(C, D) + \min(C, D), & \text{if } n = 3 \\ AU_{n-1} + U_{n-2}, & \text{if } n > 3 \end{cases}. \tag{2.2}$$

Theorems 2.1 and 2.2 are proven in Section 5. There, we first prove the results in the simple setting where $A = C = \Delta > 1$, $D = 0$, and $b_1 = -1$ to convey the basic ideas of a sign change analysis. In particular, we prove the following Corollary.

Corollary 2.3. *Suppose that* $b_1 = -1$ *and that* $\alpha_{n,k} \in [-\Delta, 0]$ *for all* n, k *where* $\Delta \geq 1$. *Then* $|b_n| \leq U_n$ *for all* $n \geq 1$, *where* $\{U_n\}$ *satisfies*

$$U_n = \begin{cases} \Delta^{n-1}, & \text{if } n \leq 2 \\ \Delta U_{n-1} + U_{n-2}, & \text{if } n \geq 3 \end{cases}. \tag{2.3}$$

Solving (2.3) explicitly for U_n gives

$$U_n = \frac{\Delta}{\sqrt{\Delta^2+4}}\left(r_1^{n-1} - \left(-\frac{1}{r_1}\right)^{n-1}\right), \tag{2.4}$$

for $n \geq 2$, where r_1 is the root

$$r_1 = \frac{\Delta + \sqrt{\Delta^2+4}}{2} \tag{2.5}$$

of the characteristic polynomial associated with (2.3). The other root of the characteristic polynomial in (2.3) is $r_2 = 2^{-1}(\Delta - \sqrt{\Delta^2+4})$. Observe that $|r_1| > |r_2|$ and $r_1 r_2 = -1$.

The flexibility allowed in bounds for $\alpha_{n,1}$ in Theorems 2.1 and 2.2 comes at a bookkeeping price during the proof in Section 5. The benefits of such generality will become apparent in Sections 3 and 4 where we bound solutions of nonhomogeneous (rather than merely homogeneous) matrix equations and the coefficients of power series ratios (rather than merely reciprocals).

This section concludes with some comments on the assumptions and optimality of Theorems 2.1 and 2.2.

Remark 2.4. (Optimality of Theorems 2.1 and 2.2). For a given b_1, $\{D_n\}_{n=2}^{\infty}$, A, and B, the bound in (2.1) cannot be improved upon. To see this, set

$$\alpha_{n,1} = \begin{cases} -D_n & \text{if } n \text{ is odd} \\ D_n & \text{if } n \text{ is even} \end{cases} \tag{2.6}$$

and

$$\alpha_{n,k} = \begin{cases} -A & \text{if } n+k \text{ is odd} \\ B & \text{if } n+k \text{ is even} \end{cases} \tag{2.7}$$

for $n \geq 2$ and $1 < k \leq n-1$. It is easy to verify from (1.1) that $b_n = (-1)^n U_n b_1$ for $n \geq 2$, implying that the bound in Theorem 2.1 is achieved. A similar construction shows that the bound in Theorem 2.2 is also optimal.

For completeness, we also consider situations where $0 \leq A \leq B$. In this case, a straightforward analysis will yield the following bound for solutions to (1.1).

Remark 2.5. Consider the setup in Theorem 2.1 except that $0 \le A \le B$. Then $\{U_n^*\}_{n=1}^{\infty}$ defined by

$$U_n^* = \begin{cases} 1, & \text{if } n = 1 \\ D_2, & \text{if } n = 2 \\ BD_2 + D_3, & \text{if } n = 3 \\ (B+1)U_{n-1}^* + D_n - D_{n-1}, & \text{if } n > 3 \end{cases} \tag{2.8}$$

is a bound satisfying $|b_n|/|b_1| \le U_n^*$ for all $n \ge 1$. This bound is achieved in the case where $\alpha_{n,1} = D_n$ and $\alpha_{n,k} = B$ for $n \ge 2$ and $2 \le k \le n-1$.

The above results provide optimal bounds for $|b_n|$ when $\alpha_{n,k} \in [-A, B]$ except when $0 \le B < A < 1$. As our next remark shows, the condition $A \ge 1$ is essential for optimality.

Remark 2.6. Optimality of Theorem 2.1 may not occur when $A < 1$. To see this, suppose that $B < A < 1$ and consider $\{b_n\}_{n=1}^{\infty}$ satisfying (1.1) with $b_1 = -1$, $\alpha_{2,1} = D_2$, $\alpha_{3,1} = D_3$, $\alpha_{3,2} = B$, $\alpha_{4,1} = -D_4$, $\alpha_{4,2} = -A$, and $\alpha_{4,3} = -A$. Then (1.1) gives $b_2 = -D_2$, $b_3 = -(BD_2 + D_3)$, and

$$\begin{aligned} b_4 &= D_4 + A(BD_2 + D_3) + AD_2 \\ &= (A + AB)D_2 + AD_3 + D_4 \\ &> (A^2 + B)D_2 + AD_3 + D_4, \end{aligned} \tag{2.9}$$

where the strict inequality above follows from $A + AB > A^2 + B$ (which follows from $B < A < 1$). Applying (2.1) now gives

$$\begin{aligned} b_4 &> A(AD_2 + D_3) + (B+1)D_2 + D_4 - D_2 \\ &= AU_3 + (1+B)U_2 + D_4 - D_2 \\ &= U_4. \end{aligned} \tag{2.10}$$

Hence, U_n may not bound $|b_n|$ in this setting.

Example 2.1. In the setting of Example 1.1, the $\{\alpha_{n,k}\}$ producing the maximal $\{|b_n|\}$ are obtained via the argument in Remark 2.4. When $\alpha_{n,k} \in [-3, 0]$ for all n and k, the maximal $|b_n|$'s are produced with $\alpha_{n,k}$ either -3 or 0 in the alternating fashion depicted in the table in Example 1.1.

3. TRIANGULAR LINEAR SYSTEMS WITH RESTRICTED ENTRIES

Theorems 2.1 and 2.2 have applications to systems of linear equations. Consider the lower triangular linear system

$$\begin{bmatrix} l_{1,1} & 0 & \dots & 0 \\ l_{2,1} & l_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \tag{3.1}$$

with $l_{i,i} \ne 0$ for $1 \le i \le n$. Solving this for $\{x_j\}$ gives

$$x_m = \frac{c_m}{l_{m,m}} x_0 - \sum_{k=1}^{m-1} \frac{l_{m,k}}{l_{m,m}} x_k, \; 1 \le m \le n, \tag{3.2}$$

with $x_0 = 1$. Letting $b_{m+1} = x_m$ for $0 \le m \le n$ produces

$$b_{m+1} = \frac{c_m}{l_{m,m}} b_1 - \sum_{k=2}^{m} \frac{l_{m,k-1}}{l_{m,m}} b_k \tag{3.3}$$

which is (1.1) with $\alpha_{m,1} = c_{m-1}/l_{m-1,m-1}$ and $\alpha_{m,k} = -l_{m-1,k-1}/l_{m-1,m-1}$ for $2 \le k \le m-1$. Hence, Theorems 2.1 and 2.2 become the following.

Corollary 3.1. *Consider the linear system in (3.1). Suppose that $0 \le B \le A$ and that D_k is nondecreasing in k. Then*

(i) *If $c_i/l_{i,i} \in [-D_{i+1}, D_{i+1}]$ for $1 \le i \le n$ and $l_{i,j}/l_{i,i} \in [-B, A]$ for $2 \le i \le n$ and $1 \le j \le i$, then $|x_i| \le U_{i+1}$ for $2 \le i \le n$ where $\{U_k\}$ is as in (2.1).*

(ii) *If $c_i/l_{i,i} \in [-C, D]$ for $1 \le i \le n$ and $l_{i,j}/l_{i,i} \in [0, A]$ for $2 \le i \le n$ and $1 \le j \le i$, then $|x_i| \le U_{i+1}$ for $1 \le i \le n$ where $\{U_k\}$ is as in (2.2).*

Example 3.1. Returning to Example 1.3, the bound in (1.7) follows from Part (i) of Corollary 3.1 with $D_i = (i-1)^2$, $A = 10$, and $B = 0$. The difference equation in (2.1) simplifies to

$$U_n = 10U_{n-1} + U_{n-2} + 4n - 8. \tag{3.4}$$

Corollary 3.1 compares favorably to the bounds for matrix equation solutions with coefficients that are restricted to more general intervals in Neumaier [17], Hansen [9] and [8], Hansen and Smith [10], and Kearfott [12]. Here, optimal bounds are obtained regardless of interval widths and dimension; moreover, the computational burden is limited to solving the second-order linear recurrences in (2.1) or (2.2).

If $c_i = 0$ for $i \ge 2$ in (3.1) (this situation is discussed further in Viswanath and Trefethen [26]), then (3.2) is

$$x_m = -\sum_{k=1}^{m-1} \frac{l_{m,k}}{l_{m,m}} x_k, \ 1 \le m \le n, \tag{3.5}$$

with $x_1 = c_1/l_{1,1}$. One can now bound $|x_n|$ via Theorem 2.1 or 2.2.

4. RATIOS OF POWER SERIES

The recurrence equation (1.1) arises when computing coefficients of ratios of formal power series. Equating coefficients in the expansion

$$h_0 + h_1 z + h_2 z^2 + \cdots = \frac{g_0 + g_1 z + g_2 z^2 + \cdots}{f_0 + f_1 z + f_2 z^2 + \cdots} \tag{4.1}$$

(take $f_0 = 1$ and $g_0 = 1$ for simplicity) gives $h_0 = 1$ and

$$h_n = (g_n - f_n)h_0 - \sum_{j=1}^{n-1} f_{n-j} h_j, \ n \ge 1. \tag{4.2}$$

The theorems in Section 2 translate to the following.

Corollary 4.1. *Suppose that $0 \le B \le A$, that $\{D_n\}_{n=2}^{\infty}$ is a nondecreasing sequence of non-negative real numbers, and that $\{f_n\}_{n=0}^{\infty}$, $\{g_n\}_{n=0}^{\infty}$, and $\{h_n\}_{n=0}^{\infty}$ satisfy (4.1) with $f_0 = g_0 = 1$.*

(i) *If $g_n - f_n \in [-D_{n+1}, D_{n+1}]$ for all $n \ge 1$ and $f_n \in [-B, A]$ for all $n \ge 0$, then $|h_n| \le U_{n+1}$ for all $n \ge 0$ where $\{U_n\}_{n=1}^{\infty}$ is as in (2.1).*

(ii) *If $g_n - f_n \in [-C, D]$ for $n \ge 1$ and $f_n \in [0, A]$ for $n \ge 0$, then $|h_n| \le U_{n+1}$ for $n \ge 0$ where $\{U_n\}_{n=1}^{\infty}$ is as in (2.2).*

Merely inverting a power series simplifies the statements in Corollary 4.1. Here, $g_k = 0$ for all $k \geq 1$ and $g_0 = 1$. Using this in (4.2), applying Part (i) of Corollary 3.1 (with $D_n \equiv A$) and Part (ii) of Corollary 3.1 (with $C = A$ and $D = 0$), and solving (2.1) and (2.2) for $\{U_n\}$ gives the following results.

Corollary 4.2. *Suppose that $0 \leq B \leq A$ and that $g_k = 0$ for $k \geq 1$ and $g_0 = 1$. Let*

$$r_1 = \frac{A + \sqrt{A^2 + 4(1+B)}}{2} \tag{4.3}$$

be a root of the characteristic polynomial in (2.1).

(i) *If $f_n \in [-B, A]$ for all $n \geq 0$, then*

$$|h_n| \leq \kappa_1 r_1^n + \kappa_2 \left[\frac{-(B+1)}{r_1}\right]^n \tag{4.4}$$

for all $n \geq 1$ where

$$\kappa_1 = \frac{2(B+1) - A + \sqrt{A^2 + 4(1+B)}}{2(B+1)\sqrt{A^2 + 4(1+B)}}, \tag{4.5}$$

and

$$\kappa_2 = -\frac{2(B+1) - A - \sqrt{A^2 + 4(1+B)}}{2(B+1)\sqrt{A^2 + 4(1+B)}}. \tag{4.6}$$

(ii) *If $f_n \in [0, A]$ for all $n \geq 0$ ($B = 0$), then*

$$|h_n| \leq \frac{A}{\sqrt{A^2+4}} r_1^n - \frac{A}{\sqrt{A^2+4}} \left[\frac{-1}{r_1}\right]^n \tag{4.7}$$

for all $n \geq 1$.

Remark 4.3. Corollary 4.2 (ii) is optimal as the bound is attained for $f(z) = 1 + Az + Az^3 + Az^5 + \cdots$. Regarding the sharpness of Corollary 4.2 (i), set $f(z) = 1 + Az - Bz^2 + Az^3 + \cdots$. If u_n is taken to be the bound on the right hand side of (4.4) then it is not difficult to show that u_n and h_n are similar in magnitude: $u_n/|h_n| \leq 1 + 2(A-B)/A^2$ and

$$\lim_{n\to\infty} \frac{u_n}{|h_n|} = 1 + \frac{(A-B)(\sqrt{A^2 + 4(1+B)} - A)}{AB + 2A + B\sqrt{A^2 + 4(1+B)}}. \tag{4.8}$$

Hence, the rate is again sharp.

Corollaries 4.1 and 4.2 are useful when generating functions or formal power series are utilized such as in enumerative combinatorics and stochastic processes (cf. Wilf [27], Feller [6], Kijima [14]).

The above results provide bounds for the location of the smallest root of a complex valued power series. Power series with restricted coefficients have been studied in the context of determining distributions of zeroes (cf. Flatto *et al.* [7], Solomyak [22], Beaucoup *et al.* [1], [2], and Pinner [21]). Related problems for polynomials have been considered by Odlyzko and Poonen [18], Yamamoto [29], Borwein and Pinner [4], and Borwein and Erdelyi [3]. As mentioned above, Flatto *et al.* [7] and Solomyak [22] independently proved that if z is a root of a series in $\mathcal{F}_{[0,1]}$, then $|z| \geq 2/(1+\sqrt{5})$. The following extension of this result is a consequence of Corollary 4.2.

Corollary 4.4. *If z is a root of a power series in $\mathcal{F}_{[-B,A]}$ with $0 \leq B \leq A$, then*

$$|z| \geq \frac{2}{A + \sqrt{A^2 + 4(1+B)}}. \tag{4.9}$$

Proof. Suppose that $f \in \mathcal{F}_{[-B,A]}$. Apply Part (i) of Corollary 4.2 and note from (4.4) that $f(z)^{-1}$ is finite for $|z| < r_1^{-1}$ (Observe that r_1 is the root of the characteristic polynomial with largest magnitude). If f had a root in $\{z : |z| < r_1^{-1}\}$, say at $z = z_0$, then we would have the contradiction $|f(z_0)|^{-1} = \infty$. □

The result in Corollary 4.4 is again optimal: for given $0 \leq B \leq A$, $f(z) = 1 + Az - Bz^2 + Az^3 - Bz^4 + \cdots$ has a root at $z = -r_1^{-1}$.

5. PROOFS

This section proves Theorem 2.1. As the arguments for Theorem 2.2 are similar, we concentrate on Theorem 2.1 only. While the proof of Theorem 2.1 is self-contained and elementary, it does employ a “sign change analysis” of $\{b_n\}_{n=1}^{\infty}$ which is case-by-case intensive and delicate. Attempts to find a direct analytic argument, by other authors as well as ourselves, have been unsuccessful to date. In particular, standard manipulations with classical inequalities do not yield the sharpness or generality of Theorem 2.1. The rudimentary structure of the problem emerges with the sign change arguments. Moreover, the arguments provide both a convergence rate and explicit “first constant” bound for the rate. Obtaining an explicit first constant, a practical matter needed to apply the bounds, takes considerably more effort in general.

The sign-change arguments below first bound all solutions to (1.1) that have a particular sign configuration; in the notation below, this is $|b_n| \leq |B_n|$ for all $n \geq 1$. A subsequent analysis is needed to bound $|B_n|$ by an accessible quantity; in the notation below, this is $|B_n| \leq U_n$ where U_n is defined in (2.1). We first consider the arguments for Corollary 2.3 as these are reasonably brief and convey the essence of the general analysis.

Arguments for Corollary 2.3. Suppose that $b_1 = -1$ and let $P = \{n \geq 1 : b_n \geq 0\}$ and $N = \{n \geq 1 : b_n < 0\}$ partition the sign configuration of $\{b_n\}_{n=1}^{\infty}$. Now define B_n recursively in n from N and P via $B_1 = -1$ and

$$B_n = \begin{cases} \Delta - \Delta \displaystyle\sum_{\substack{2 \leq r \leq n-1 \\ r \in N}} B_r, & n \in P \\ -\Delta \displaystyle\sum_{\substack{2 \leq r \leq n-1 \\ r \in P}} B_r, & n \in N \end{cases} \tag{5.1}$$

for $n \geq 2$. A simple induction with (5.1) will show that B_n and b_n have the same sign for $n \geq 1$.

We now prove by induction that $|b_n| \leq |B_n|$ for all $n > 1$. First, assume that $n > 1$ and that $n \in P$. Returning to (1.1) and collecting positive and negative terms gives

$$b_n = \alpha_{n,1} b_1 + \sum_{\substack{2 \leq r \leq n-1 \\ r \in P}} \alpha_{n,r} b_r + \sum_{\substack{2 \leq r \leq n-1 \\ r \in N}} \alpha_{n,r} b_r. \tag{5.2}$$

Using $b_1 = -1$, the bound $\alpha_{n,k} \in [-\Delta, 0]$ for all n, k, and neglecting the first summation in (5.2) gives

$$\begin{aligned} b_n &\leq \Delta + \sum_{\substack{2 \leq r \leq n-1 \\ r \in N}} - \Delta b_r \\ &= \Delta + \Delta \sum_{\substack{2 \leq r \leq n-1 \\ r \in N}} |b_r|. \end{aligned} \tag{5.3}$$

Using the inductive hypothesis and the fact that $|b_n| = b_n$ in (5.3) produces

$$\begin{aligned} |b_n| &\le \Delta + \Delta \sum_{\substack{2\le r\le n-1 \\ r\in N}} |B_r| \\ &= \Delta - \Delta \sum_{\substack{2\le r\le n-1 \\ r\in N}} B_r \\ &= B_n \end{aligned} \tag{5.4}$$

after (5.1) is applied. An analogous argument works when $n \in N$.

We now finish the arguments for Corollary 2.3 by inductively showing that $|B_n| \le U_n$ from (5.1). First, it is easy to verify that $|B_i| \le U_i$, for $1 \le i \le 3$ for all possible sign configurations of $\{B_1, B_2, B_3\}$. Now assume that $n \in P$ ($B_n \ge 0$) where $n > 3$. If $n-1 \in P$ $(B_{n-1} \ge 0)$, then $B_n = B_{n-1}$ by (5.1) and $|B_n| = |B_{n-1}| \le U_{n-1} \le U_n$ since U_n is nondecreasing in n (this follows from $\Delta \ge 1$). So we need only consider the case where $n-1 \in N$ ($B_{n-1} < 0$). If $r \in N$ for all $r \le n-1$ ($B_r < 0$ for $1 \le r \le n-1$), then $B_2 = B_3 = \cdots = B_{n-1} = 0$ by (5.1) and we have $B_n = \Delta = U_3 \le U_n$.

Finally, consider the case where a non-negative element in $\{B_1, \ldots, B_{n-2}\}$ exists; that is, $r \in P$ for some $2 \le r \le n-2$. Let r^* be the largest such integer and set $k = n - r^* - 1$. For signs of $\{B_n\}$, we have $B_{n-k-1} \ge 0$ ($B_{n-k-1} \in P$) and $B_j < 0$ for $n-k \le j \le n-1$. Using these in (5.1) gives $B_{n-1} = \cdots = B_{n-k}$. Applying (5.1) yet again produces

$$\begin{aligned} B_n &= \Delta - \Delta \sum_{\substack{2\le r\le n-1 \\ r\in N}} B_r \\ &= \Delta - \Delta \sum_{r=n-k}^{n-1} B_r - \Delta \sum_{\substack{2\le r\le n-k-2 \\ r\in N}} B_r \\ &= B_{n-k-1} - \Delta k B_{n-k}. \end{aligned} \tag{5.5}$$

Applying the induction hypothesis and the triangle inequality in (5.5) produces

$$|B_n| \le U_{n-k-1} + \Delta k U_{n-k}, \tag{5.6}$$

and the difference equation in (2.3) can be used to increase the smallest subscript appearing on the right hand side of (5.6) to $n-k$:

$$|B_n| \le U_{n-k+1} + \Delta(k-1) U_{n-k}. \tag{5.7}$$

Since U_n is nondecreasing in n and $\Delta(k-1) \ge 1$, we may swap the coefficients on U_{n-k+1} and U_{n-k} in (5.7) to obtain

$$|B_n| \le U_{n-k} + \Delta(k-1) U_{n-k+1}. \tag{5.8}$$

Note that (5.8) is (5.6) with k replaced by $k-1$. As the discourse from (5.6) – (5.8) is merely algebraic, we iterate the above arguments to obtain

$$|B_n| \le U_{n-(k-j)-1} + \Delta(k-j) U_{n-(k-j)} \tag{5.9}$$

for each $0 \le j \le k-1$. In particular, taking $j = k-1$ in (5.9) now gives

$$|B_n| \le U_{n-2} + \Delta U_{n-1}. \tag{5.10}$$

Applying (2.3) in (5.10) immediately gives the required bound $|B_n| \le U_n$ and finishes our work. The arguments for the case where $n \in N$ are similar. □

Following the logic of the above arguments, we now present the proof of Theorem 2.1 in its generality.

Proof of Theorem 2.1. We first reduce to the case where $b_1 = -1$ by examining b_n/b_1. Again let $P = \{n \geq 1 : b_n \geq 0\}$ and $N = \{n \geq 1 : b_n < 0\}$ be the sign partition for $\{b_n\}_{n=1}^{\infty}$. This time, define a bounding sequence $\{B_n\}_{n=1}^{\infty}$ for this sign configuration recursively in n via $B_1 = -1$, and for $n \geq 2$ by

$$B_n = \begin{cases} D_n - A \displaystyle\sum_{\substack{2\leq r\leq n-1 \\ r\in N}} B_r + B \displaystyle\sum_{\substack{2\leq r\leq n-1 \\ r\in P}} B_r, & n \in P \\ -D_n - A \displaystyle\sum_{\substack{2\leq r\leq n-1 \\ r\in P}} B_r + B \displaystyle\sum_{\substack{2\leq r\leq n-1 \\ r\in N}} B_r, & n \in N \end{cases}. \tag{5.11}$$

As before, an induction will show that B_n and b_n have the same sign for each $n \geq 1$. This fact will be used repeatedly in the discourse below.

We now justify the majorizing properties of $\{B_n\}$ by inductively showing that $|b_n| \leq |B_n|$ for all $n \geq 1$. First, consider the case where $n \in P$. Now partition positive and negative terms in (1.1) and apply the bounds assumed on the $\alpha_{n,k}$'s in Theorem 2.1 to get

$$b_n \leq -D_n b_1 + B \sum_{\substack{2\leq r\leq n-1 \\ r\in P}} b_r - A \sum_{\substack{2\leq r\leq n-1 \\ r\in N}} b_r. \tag{5.12}$$

Applying $b_1 = -1$ and the induction hypothesis, and then (5.11) gives

$$\begin{aligned} b_n &\leq D_n + B \sum_{\substack{2\leq r\leq n-1 \\ r\in P}} |B_r| + A \sum_{\substack{2\leq r\leq n-1 \\ r\in N}} |B_r| \\ &= D_n + B \sum_{\substack{2\leq r\leq n-1 \\ r\in P}} B_r - A \sum_{\substack{2\leq r\leq n-1 \\ r\in N}} B_r \\ &= B_n. \end{aligned} \tag{5.13}$$

Similar arguments tackle the case where $n \in N$. Equation (5.13) represents the core of our arguments. The remainder of our work lies with devising a useful bound for the B_n's in (5.11).

To complete the proof of Theorem 2.1, it remains to show that $|B_n| \leq U_n$ for all $n \geq 1$. For this it will be convenient to have the following technical lemma which we prove after the arguments for Theorem 2.1 (one can verify non-circularity of discourse).

Lemma 5.1. *Consider the setup in Theorem 2.1 and define $\{E_n\}_{n=1}^{\infty}$ via $E_0 = 1$, $E_1 = A$, $E_2 = A^2 + B$, and $E_j = AE_{j-1} + (1 + B)E_{j-2}$ for $j \geq 3$. Then U_n can be expressed as*

$$U_n = D_n + \sum_{j=2}^{n-1} E_{n-j} D_j, \tag{5.14}$$

for $n \geq 2$, with the inequality

$$U_n - (1 + B)U_{n-1} \geq D_n - D_{n-1} \tag{5.15}$$

holding for $n \geq 3$. Finally, in the case where $n \geq 2$ and $B_j < 0$ for $1 \leq j \leq n - 1$ ($j \in N$ for $1 \leq j \leq n - 1$) and $B_n \geq 0$ ($n \in P$), we have

$$B_n = D_n + \sum_{j=2}^{n-1} A(1 + B)^{n-j-1} D_j. \tag{5.16}$$

We now return to the proof of Theorem 2.1. Assume first that $n \in P$ $(B_n > 0)$. We start inductive verification that $|B_j| \le U_j$ for all $j \ge 1$ by noting that $|B_1| = U_1 = 1$ and $|B_2| = D_2 = U_2$. For B_3, first note that if $B_2 \ge 0$ and $B_3 \ge 0$ ($\{2,3\} \subset P$), then

$$\begin{aligned} |B_3| &= D_3 + BD_2 \\ &\le U_3, \end{aligned} \tag{5.17}$$

where the inequality in (5.17) follows from (2.1), $D_j \ge 0$ for all j, and $B \le A$. In the case where $B_2 < 0$ and $B_3 < 0$ ($\{2,3\} \subset N$), then (5.17) again holds. In the cases where there is one negative and one positive sign amongst $\{B_2, B_3\}$, one can verify that

$$\begin{aligned} |B_3| &= D_3 + AD_2 \\ &\le U_3 \end{aligned} \tag{5.18}$$

by direct application of (2.1).

Now assume that $|B_k| \le U_k$ for $1 \le k \le n-1$. When $n-1 \in P$ ($B_{n-1} \ge 0$), use (5.11) to get

$$B_n = (1+B)B_{n-1} + D_n - D_{n-1}. \tag{5.19}$$

Applying the induction hypothesis that $B_{n-1} \le U_{n-1}$ and (5.15) in (5.19) produces

$$\begin{aligned} B_n &\le (1+B)U_{n-1} + D_n - D_{n-1} \\ &\le U_n \end{aligned} \tag{5.20}$$

as claimed.

It remains to consider the case where $n-1 \in N$. First suppose that $r \in N$ for all $r \le n-1$. From Lemma 5.1, $E_1 = A$ and $E_2 = A^2 + B \ge A(1+B)$ since $A \ge 1$ and $A \ge B$. Using $A \ge B$ and Lemma 5.1 in an induction argument will easily verify the inequality $E_j \ge A(1+B)^{j-1}$ for all $j \ge 1$. Comparing coefficients in (5.16) and (5.14) now yields $|B_n| \le U_n$ as claimed.

Having dealt with the case where the B_j are negative for all $1 \le j \le n-1$, now suppose that there exists a non-negative B_j amongst the first $n-1$ indices. In particular, suppose that $r \in P$ for some $2 \le r \le n-2$ and let r^* denote the largest such integer. Set $k = n - r^* - 1$. For signs of $\{B_n\}$, we have $B_{n-k-1} \ge 0$ ($n-k-1 \in P$), $B_j < 0$ ($j \in N$ for $n-k \le j \le n-1$), and our standing assumption that $B_n \ge 0$ ($n \in P$). Using these facts in (5.11) produces

$$B_n = D_n - A \sum_{n-k \le r \le n-1} B_r + A \sum_{\substack{2 \le r \le n-k-2 \\ r \in N}} B_r + B \sum_{\substack{2 \le r \le n-k-2 \\ r \in P}} B_r + B|B_{n-k-1}|. \tag{5.21}$$

Now combine the definition of B_{n-k-1} in (5.11) with (5.21) to get

$$B_n = D_n + (1+B)|B_{n-k-1}| - D_{n-k-1} - A \sum_{n-k \le r \le n-1} B_r. \tag{5.22}$$

Returning to (5.11) with the fact that $B_j < 0$ for $n-k \le j \le n-1$ identifies the rightmost summation in (5.22):

$$\sum_{n-k \le r \le n-1} B_r = -|B_{n-k}| \sum_{i=0}^{k-1} (1+B)^i + D_{n-k} \sum_{i=0}^{k-2} (1+B)^i - \sum_{i=1}^{k-1} (1+B)^{i-1} D_{n-i}. \tag{5.23}$$

Combining (5.22) and (5.23) expresses B_n explicitly in terms of B_{n-k} and B_{n-k-1}:

$$B_n = D_n + (1+B)|B_{n-k-1}| - D_{n-k-1} + A|B_{n-k}| \sum_{i=0}^{k-1}(1+B)^i - AD_{n-k}\sum_{i=0}^{k-2}(1+B)^i + A\sum_{i=1}^{k-1}(1+B)^{i-1}D_{n-i}. \tag{5.24}$$

The induction hypothesis gives $|B_{n-k-1}| \le U_{n-k-1}$ and $|B_{n-k}| \le U_{n-k}$; using these in (5.24) along with $B_n = |B_n|$ gives the bound

$$|B_n| \le D_n + (1+B)U_{n-k-1} - D_{n-k-1} + AU_{n-k}\sum_{i=0}^{k-1}(1+B)^i - AD_{n-k}\sum_{i=0}^{k-2}(1+B)^i + A\sum_{i=1}^{k-1}(1+B)^{i-1}D_{n-i}. \tag{5.25}$$

Making the substitution $J_i = A\sum_{m=0}^{i}(1+B)^m$ into (5.25) now yields

$$|B_n| \le D_n + (1+B)U_{n-k-1} - D_{n-k-1} + U_{n-k}J_{k-1} - D_{n-k}J_{k-2} + A\sum_{i=1}^{k-1}(1+B)^{i-1}D_{n-i}. \tag{5.26}$$

The difference equation (2.1) gives $U_{n-k+1} = AU_{n-k} + (1+B)U_{n-k-1} + D_{n-k+1} - D_{n-k-1}$. Using this in (5.26) and algebraically simplifying produces

$$|B_n| \le U_{n-k+1} - D_{n-k+1} + (1+B)U_{n-k}J_{k-2} + D_n - D_{n-k}J_{k-2} + A\sum_{i=1}^{k-1}(1+B)^{i-1}D_{n-i}, \tag{5.27}$$

where the fact that $J_{k-1} - A = (1+B)J_{k-2}$ has been applied. An algebraic rearrangement of the right hand side of (5.27) now produces

$$|B_n| \le (1-J_{k-2})[U_{n-k+1} - (1+B)U_{n-k}] + J_{k-2}U_{n-k+1} + (1+B)U_{n-k} - D_{n-k+1} + D_n - D_{n-k}J_{k-2} + A\sum_{i=1}^{k-1}(1+B)^{i-1}D_{n-i}. \tag{5.28}$$

Noting that $J_{k-2} \ge 1$ for all k and applying (5.15) to the bracketed term in the right hand side of (5.28) now produces

$$|B_n| \le (1-J_{k-2})[D_{n-k+1} - D_{n-k}] + J_{k-2}U_{n-k+1} + (1+B)U_{n-k} - D_{n-k+1} + D_n - D_{n-k}J_{k-2} + A\sum_{i=1}^{k-1}(1+B)^{i-1}D_{n-i}.$$

Invoking the difference equation in (2.1) again will give

$$|B_n| \le U_{n-k+2} - D_{n-k+2} + (1+B)U_{n-k+1}J_{k-3} + D_n - D_{n-k+1}J_{k-3} + A\sum_{i=1}^{k-2}(1+B)^{i-1}D_{n-i}. \tag{5.29}$$

The discourse between (5.27) – (5.29) is purely algebraic, justified via the difference equation in (2.1). Observe that the bounds for $|B_n|$ in (5.27) and (5.29) are similar in form, except that k is replaced by $k-1$. As such, one can continue iterating the arguments in (5.27) – (5.29) until $k=3$. This will give

$$|B_n| \le U_{n-1} - D_{n-1} + (1+B)U_{n-2}J_0 + D_n - D_{n-2}J_0 + AD_{n-1}. \tag{5.30}$$

Now use $J_0 = A$ in (5.30), employ (2.1) and regroup terms to get

$$|B_n| \le U_n + D_{n-2} + (1-A)[U_{n-1} - (1+B)U_{n-2}] - D_{n-1} - D_{n-2}A + AD_{n-1}. \tag{5.31}$$

Applying (5.20) once more to the bracketed terms in (5.31) and $A \ge 1$ to get

$$\begin{aligned} |B_n| &\le U_n + D_{n-2} + (1-A)(D_{n-1} - D_{n-2}) - D_{n-1} - D_{n-2}A + AD_{n-1} \\ &= U_n. \end{aligned} \tag{5.32}$$

This completes the arguments for Theorem 2.1 in the case where $n \in P$. The discourse for the case where $n \in N$ is similar and is hence omitted. □

Proof of Lemma 5.1. The convolution identity (5.14) is easy to verify directly from (2.1). To prove (5.16), return to (5.11) with the facts that $j \in N$ for $1 \le j \le n-1$ to get $|B_2| = D_2$, $B_n = A\sum_{j=2}^{n-1} |B_j| + D_n$, and $|B_j| = (1+B)|B_{j-1}| - D_{j-1} + D_j$ for $3 \le j \le n-1$.

To prove (5.15), we get an induction started by applying (2.1) with $n=2$ and $n=3$:

$$\begin{aligned} U_3 - (1+B)U_2 &= AD_2 + D_3 - (1+B)D_2 \\ &= (A-B)D_2 + D_3 - D_2 \\ &\ge 0, \end{aligned} \tag{5.33}$$

where the last inequality follows from $A \ge B$, $D_2 \ge 0$ and $D_3 \ge D_2$. Equation (5.15) with $i = 4$ follows from the inequalities $A \ge 1$ and $A \ge B$:

$$\begin{aligned} U_4 - (1+B)U_3 &= [AU_3 + (1+B)U_2 + D_4 - D_2] - (1+B)[AD_2 + D_3] \\ &= (A-1)(A-B)D_2 + (A-B)D_3 + D_4 - D_3 \\ &\ge D_4 - D_3, \end{aligned} \tag{5.34}$$

where the last inequality follows from $A \ge 1$, $A \ge B$, $D_3 \ge 0$ and $D_4 \ge D_3$.

For the general inductive step, take an $n > 4$ and suppose that $U_i - (1+B)U_{i-1} \ge D_i - D_{i-1}$ for $3 \le i \le n-1$. Then (2.1) gives

$$\begin{aligned} U_n - (1+B)U_{n-1} &= [AU_{n-1} + (1+B)U_{n-2} + D_n - D_{n-2}] \\ &\qquad - (1+B)[AU_{n-2} + (1+B)U_{n-3} + D_{n-1} - D_{n-3}] \\ &= A[U_{n-1} - (1+B)U_{n-2}] + (1+B)[U_{n-2} - (1+B)U_{n-3}] \\ &\qquad + D_n - D_{n-2} - (1+B)D_{n-1} + (1+B)D_{n-3}. \end{aligned} \tag{5.35}$$

Applying the inductive hypothesis to the bracketed terms in (5.35) and collecting terms gives the inequality

$$\begin{aligned} U_n - (1+B)U_{n-1} &\ge A(D_{n-1} - D_{n-2}) + (1+B)(D_{n-2} - D_{n-3}) + D_n - D_{n-2} \\ &\qquad - (1+B)D_{n-1} + (1+B)D_{n-3} \\ &= D_n - D_{n-1} + (A-B)[D_{n-1} - D_{n-2}]. \end{aligned} \tag{5.36}$$

The assumed monotonicity of D_k in k and $A \ge B$ give

$$U_n - (1+B)U_{n-1} \ge D_n - D_{n-1} \tag{5.37}$$

and the proof is complete. □

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