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A NOTE ON CERTAIN INEQUALITIES FOR p-VALENT FUNCTIONS

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ABSTRACT. We use a parabolic region to prove certain inequalities for uniformly p-valent functions in the open unit disk D.

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1. INTRODUCTION

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in \mathbb{N} = 1, 2, 3, \dots),$$

which are analytic and multivalent in the open unit disk $D = \{z : z \in \mathbb{C};\ |z| < 1\}$.

A function $f(z) \in A(p)$ is said to be in $SP_p(\alpha)$, the class of uniformly p-valent starlike functions (or, uniformly starlike when $p = 1$) of order α if it satisfies the condition

(1.1) $$\Re e \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} \geq \left| \frac{zf'(z)}{f(z)} - p \right|.$$

Replacing f in (1.1) by $zf'(z)$, we obtain the condition

(1.2) $$\Re e \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \left| \frac{zf''(z)}{f'(z)} - (p-1) \right|$$

required for the function f to be in the subclass UCV_p of uniformly p-valent convex functions (or, uniformly convex when $p = 1$) of order α. Uniformly p-valent starlike and p-valent convex functions were first introduced [4] when $p = 1, \alpha = 0$ and [2] when $p \geq 1, p \in \mathbb{N}$ and then studied by various authors.

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We set

$$\Omega_\alpha = \left\{u + iv, u - \alpha > \sqrt{(u-p)^2 + v^2}\right\}$$

with $q(z) = \frac{zf'(z)}{f(z)}$ or $q(z) = 1 + \frac{zf''(z)}{f'(z)}$ and consider the functions which map D onto the parabolic domain Ω_α such that $q(z) \in \Omega_\alpha$.

By the properties of the domain Ω_α, we have

$$\Re e(q(z)) > \Re e(Q_\alpha(z)) > \frac{p+\alpha}{2}, \tag{1.3}$$

where

$$Q_\alpha(z) = p + \frac{2(p-\alpha)}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

Futhermore, a function $f(z) \in A(p)$ is said to be uniformly p-valent close-to-convex (or, uniformly close-to-convex when $p = 1$) of order α in D if it also satisfies the inequality

$$\Re e\left\{\frac{zf'(z)}{g(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{g(z)} - p\right|$$

for some $g(z) \in SP_p(\alpha)$.

We note that a function $h(z)$ is p-valent convex in D if and only if $zh'(z)$ is p-valent starlike in D (see, for details, [1], [3], and [6]).

In order to obtain our main results, we need the following lemma:

Lemma 1.1 (Jack's Lemma [5])**.** *Let the function $w(z)$ be (non-constant) analytic in D with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = cw(z_0),$$

c is real and $c \geq 1$.

2. Certain Results for the Multivalent Functions

Making use of Lemma 1.1, we first give the following theorem:

Theorem 2.1. *Let $f(z) \in A(p)$. If $f(z)$ satisfies the following inequality:*

$$\Re e\left(\frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p}\right) < 1 + \frac{2}{3p}, \tag{2.1}$$

then $f(z)$ is uniformly p-valent starlike in D.

Proof. We define $w(z)$ by

$$\frac{zf'(z)}{f(z)} - p = \frac{p}{2}w(z), \qquad (p \in \mathbb{N}, z \in D). \tag{2.2}$$

Then $w(z)$ is analytic in D and $w(0) = 0$.Furthermore, by logarithmically differentiating (2.2), we find that

$$1 + \frac{zf''(z)}{f'(z)} - p = \frac{p}{2}w(z) + \frac{zw'(z)}{2 + w(z)}, \qquad (p \in \mathbb{N}, z \in D)$$

which, in view of (2.1), readily yields

$$\frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} = 1 + \frac{zw'(z)}{\frac{p}{2}w(z)(2 + w(z))}, \qquad (p \in \mathbb{N}, z \in D). \tag{2.3}$$

Suppose now that there exists a point $z_0 \in D$ such that

$$\max |w(z)| : |z| \leq |z_0| = |w(z_0)| = 1; \qquad (w(z_0) \neq 1);$$

and, let $w(z_0) = e^{i\theta}(\theta \neq -\pi)$. Then, applying the Lemma 1.1, we have

$$z_0 w'(z_0) = c w(z_0), \qquad c \geq 1. \tag{2.4}$$

From (2.3) – (2.4), we obtain

$$\begin{aligned} \Re e \left(\frac{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - p}{\frac{z_0 f'(z_0)}{f(z_0)} - p} \right) &= \Re e \left(1 + \frac{z_0 w'(z_0)}{\frac{p}{2} w(z_0)(2 + w(z_0))} \right) \\ &= \Re e \left(1 + \frac{2c}{p} \frac{1}{(2 + w(z_0))} \right) \\ &= 1 + \frac{2c}{p} \Re e \left(\frac{1}{(2 + w(z_0))} \right) \\ &= 1 + \frac{2c}{p} \Re e \left(\frac{1}{(2 + e^{i\theta})} \right) \quad (\theta \neq -\pi) \\ &= 1 + \frac{2c}{3p} \geq 1 + \frac{2}{3p} \end{aligned}$$

which contradicts the hypothesis (2.1). Thus, we conclude that $|w(z)| < 1$ for all $z \in D$; and equation (2.2) yields the inequality

$$\left| \frac{z f'(z)}{f(z)} - p \right| < \frac{p}{2}, \qquad (p \in \mathbb{N}, z \in D)$$

which implies that $\frac{z f'(z)}{f(z)}$ lie in a circle which is centered at p and whose radius is $\frac{p}{2}$ which means that $\frac{z f'(z)}{f(z)} \in \Omega$, and so

$$\Re e \left\{ \frac{z f'(z)}{f(z)} \right\} \geq \left| \frac{z f'(z)}{f(z)} - p \right| \tag{2.5}$$

i.e. $f(z)$ is uniformly p-valent starlike in D. □

Using (2.5), we introduce a sufficient coefficient bound for uniformly p-valent starlike functions in the following theorem:

Theorem 2.2. *Let* $f(z) \in A(p)$. *If*

$$\sum_{k=2}^{\infty} (2k + p - \alpha) |a_{k+p}| < p - \alpha.$$

then $f(z) \in SP_p(\alpha)$.

Proof. Let

$$\sum_{k=2}^{\infty} (2k + p - \alpha) |a_{k+p}| < p - \alpha.$$

It is sufficient to show that

$$\left| \frac{z f'(z)}{f(z)} - (p + \alpha) \right| < \frac{p + \alpha}{2}.$$

We find that

$$\left|\frac{zf'(z)}{f(z)} - (p+\alpha)\right| = \left|\frac{-\alpha + \sum_{k=2}^{\infty}(k-\alpha)a_{k+p}z^{k-1}}{1+\sum_{k=2}^{\infty} a_{k+p}z^{k-1}}\right| < \frac{\alpha + \sum_{k=2}^{\infty}(k-\alpha)\left|a_{k+p}\right|}{1-\sum_{k=2}^{\infty}\left|a_{k+p}\right|}, \tag{2.6}$$

$$2\alpha + \sum_{k=2}^{\infty}(2k+p-\alpha)\left|a_{k+p}\right| < p+\alpha. \tag{2.7}$$

This shows that the values of the function

$$\Phi(z) = \frac{zf'(z)}{f(z)} \tag{2.8}$$

lie in a circle which is centered at $(p+\alpha)$ and whose radius is $\frac{p+\alpha}{2}$,which means that $\frac{zf'(z)}{f(z)} \in \Omega_{\alpha}$. Hence $f(z) \in SP_p(\alpha)$. ☐

The following diagram shows $\Omega_{\frac{1}{2}}$ when $p = 3$ and the circle is centered at $\frac{7}{2}$ with radius $\frac{7}{4}$:

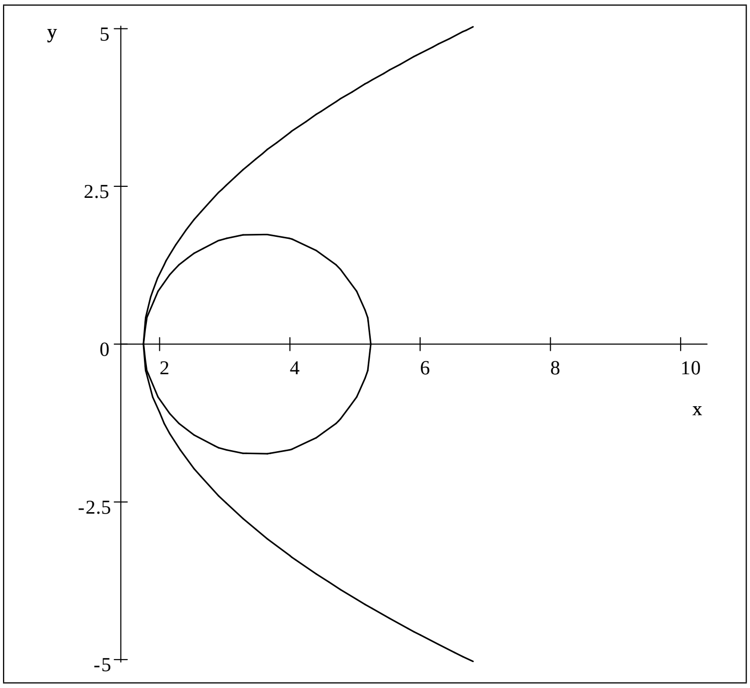

Figure 1.

Next, we determine the sufficient coefficient bound for uniformly p-valent convex functions.

Theorem 2.3. *Let $f(z) \in A(p)$. If $f(z)$ satisfies the following inequality*

$$\Re e\left(\frac{1+\frac{zf'''(z)}{f''(z)}-p}{1+\frac{zf''(z)}{f'(z)}-p}\right) < 1+\frac{2}{3p-2}, \tag{2.9}$$

then $f(z)$ is uniformly $p-$valent convex in D.

Proof. If we define $w(z)$ by

$$1+\frac{zf''(z)}{f'(z)}-p = \frac{p}{2}w(z), \qquad (p \in \mathbb{N}, z \in D), \tag{2.10}$$

then $w(z)$ satisfies the conditions of Jack's Lemma. Making use of the same technique as in the proof of Theorem 2.2, we can easily get the desired proof of Theorem 2.4. ☐

Theorem 2.4. *Let $f(z) \in A(p)$. If*

$$\sum_{k=2}^{\infty}(k+p)(2k+p-\alpha)\left|a_{k+p}\right| < p(p-\alpha), \tag{2.11}$$

then $f(z) \in UCV_p(\alpha)$.

Proof. It is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}-(p+\alpha)\right| < \frac{p+\alpha}{2}.$$

Making use of the same technique as in the proof of Theorem 2.3, we can prove inequality (2.8). □

The following theorems give the sufficient conditions for uniformly p-valent close-to-convex functions.

Theorem 2.5. *Let $f(z) \in A(p)$. If $f(z)$ satisfies the following inequality*

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right) < p-\frac{2}{3}, \tag{2.12}$$

then $f(z)$ is uniformly p-valent close-to-convex in D.

Proof. If we define $w(z)$ by

$$\frac{f'(z)}{z^{p-1}}-p=\frac{p}{2}w(z), \qquad (p\in\mathbb{N}, z\in D), \tag{2.13}$$

then clearly, $w(z)$ is analytic in D and $w(0)=0$. Furthermore, by logarithmically differentiating (2.10), we find that

$$\frac{zf''(z)}{f'(z)}=(p-1)+\frac{zw'(z)}{2+w(z)}. \tag{2.14}$$

Therefore, by using the conditions of Jack's Lemma and (2.11), we have

$$\begin{aligned}\Re e\left(\frac{z_0f''(z_0)}{f'(z_0)}\right) &= (p-1)+c\Re e\left(\frac{w(z_0)}{2+w(z_0)}\right)\\ &= p-1+\frac{c}{3} > p-\frac{2}{3}\end{aligned}$$

which contradicts the hypotheses (2.9). Thus, we conclude that $|w(z)|<1$ for all $z\in D$; and equation (2.10) yields the inequality

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| < \frac{p}{2}, \qquad (p\in\mathbb{N}, z\in D)$$

which implies that $\frac{f'(z)}{z^{p-1}} \in \Omega$, which means

$$\Re e\left\{\frac{f'(z)}{z^{p-1}}\right\} \geq \left|\frac{f'(z)}{z^{p-1}}-p\right|$$

and, hence $f(z)$ is uniformly p-valent close-to-convex in D. □

Theorem 2.6. *Let $f(z) \in A(p)$. If*

$$\sum_{k=2}^{\infty}(k+p)\left|a_{k+p}\right| < \frac{p-\alpha}{2},$$

then $f(z) \in UCC_p(\alpha)$.

By taking $p = 1$ in Theorems 2.2 and 2.6 respectively, we have

Corollary 2.7. *Let $f(z) \in A(1)$. If $f(z)$ satisfies the following inequality:*

$$\Re e \left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)} - 1} \right) < \frac{5}{3},$$

then $f(z)$ is uniformly starlike in D.

Corollary 2.8. *Let $f(z) \in A(1)$. If $f(z)$ satisfy the following inequality*

$$\Re e \left(\frac{zf''(z)}{f'(z)} \right) < \frac{1}{3},$$

then $f(z)$ is uniformly close-to-convex in D.

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