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ON HARMONIC FUNCTIONS BY THE HADAMARD PRODUCT

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Abstract

A function $f = u + iv$ defined in the domain $D \subset \mathbb{C}$ is harmonic in D if u, v are real harmonic. Such functions can be represented as $f = h + \bar{g}$ where h, g are analytic in D. In this paper the class of harmonic functions constructed by the Hadamard product in the unit disk, and properties of some of its subclasses are examined.

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Contents

1. Introduction

Let U denote the open unit disk in $\mathbb{C}$ and let $f = u + iv$ be a complex valued harmonic function on U. Since u and v are real parts of analytic functions, f admits a representation $f = h + \overline{g}$ for two functions h and g, analytic on U.

The Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The necessary and sufficient conditions for f to be local univalent and sense-preserving is $J_f(z) > 0$, $z \in U$ [1].

Many mathematicians studied the class of harmonic univalent and sense-preserving functions on U and its subclasses [2, 5].

Here we discuss two classes obtained by the Hadamard product.

2. The Class $\widetilde{P}_H^0(\alpha)$

Let P_H denote the class of all functions $f = h + \bar{g}$ so that $\operatorname{Re} f > 0$ and $f(0) = 1$ where h and g are analytic on U.

If the function $f_z + \overline{f_{\bar{z}}} = h' + \overline{g'}$ belongs to P_H for the analytic and normalized functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n, \tag{2.1}$$

then the class of functions $f = h + \bar{g}$ is denoted by $\widetilde{P}_H^0$ [5].

The function

$$t_\alpha(z) = z + \frac{1}{1+\alpha} z^2 + \cdots + \frac{1}{1+(n-1)\alpha} z^n + \cdots \tag{2.2}$$

is analytic on U when α is a complex number different from $-1, -\frac{1}{2}, -\frac{1}{3}, \dots$.

For $f \in \widetilde{P}_H^0$, we denote, by $\widetilde{P}_H^0(\alpha)$, the class of functions defined by

$$F = f * (t_\alpha + \overline{t_\alpha}). \tag{2.3}$$

Here $f * (t_\alpha + \overline{t_\alpha})$ is the Hadamard product of the functions f and $t_\alpha + \overline{t_\alpha}$. Therefore

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{1+(n-1)\alpha} z^n + \sum_{n=2}^{\infty} \overline{\frac{b_n}{1+(n-1)\alpha} z^n} \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n}, \qquad z \in U \end{aligned} \tag{2.4}$$

is in $\widetilde{P}_H^0(\alpha)$.

Conversely, if F is in the form (2.4), with a_n, b_n being the coefficients of $f \in \widetilde{P}_H^0$, then $F \in \widetilde{P}_H^0(\alpha)$.

Furthermore, if $\alpha = 0$, then as $F = f$, we have $\widetilde{P}_H^0(0) = \widetilde{P}_H^0$. Moreover $\widetilde{P}_H^0(\infty) = \{I : I(z) \equiv z,\ z \in U\}$ and since $I \in \widetilde{P}_H^0$, $\widetilde{P}_H^0 \cap \widetilde{P}_H^0(\alpha) \neq \phi$.

Theorem 2.1. *If $F \in \widetilde{P}_H^0(\alpha)$ then there exists $f \in \widetilde{P}_H^0$ so that*

$$\alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + (1-\alpha)F(z) = f(z). \tag{2.5}$$

Conversely, for any function $f \in \widetilde{P}_H^0$, there exists $F \in \widetilde{P}_H^0(\alpha)$ satisfying (2.5).

Proof. Let $F \in \widetilde{P}_H^0(\alpha)$. If $f \in \widetilde{P}_H^0$, then since

$$\alpha z t_\alpha^{'}(z) + (1-\alpha)t_\alpha(z) = t_0(z),$$

as $F = f * (t_\alpha + \overline{t_\alpha})$ we obtain that

$$f(z) = \alpha[f(z) * (zt_\alpha^{'}(z) + \overline{zt_\alpha^{'}(z)})] + (1-\alpha)[f(z) * (t_\alpha(z) + \overline{t_\alpha}(z))].$$

Therefore,

$$f(z) = \alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + (1-\alpha)F(z).$$

Conversely, for $f \in \widetilde{P}_H^0$, from (2.1), (2.2) and (2.5),

$$z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} = z + \sum_{n=2}^{\infty} [1+(n-1)\alpha]A_n z^n + \sum_{n=2}^{\infty} \overline{[1+(n-1)\alpha]B_n z^n}.$$

From these one obtains

$$A_n = \frac{a_n}{1+(n-1)\alpha} \quad \text{and} \quad B_n = \frac{b_n}{1+(n-1)\alpha}. \tag{2.6}$$

Therefore,

$$\begin{aligned} F(z) &= z + \sum_{n=2}^{\infty} \frac{a_n}{1+(n-1)\alpha} z^n + \sum_{n=2}^{\infty} \overline{\frac{b_n}{1+(n-1)\alpha} z^n} \\ &= f(z) * [t_\alpha(z) + \overline{t_\alpha}(z)]. \end{aligned}$$

□

Corollary 2.2. *A function* $F = H + \overline{G}$ *of the form (2.4) belongs to* $\widetilde{P}_H^0(\alpha)$, *if and only if*

$$\mathrm{Re}\{z(\alpha H''(z) + \overline{\alpha} G''(z)) + H'(z) + G'(z)\} > 0, \quad z \in U. \tag{2.7}$$

Proof. If $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$, then from Theorem 2.1

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}] = h(z) + \overline{g(z)} \in \widetilde{P}_H^0$$

and $h' + \overline{g'} \in P_H$. Hence

$$\begin{aligned} 0 &< \mathrm{Re}\{h'(z) + g'(z)\} \\ &= \mathrm{Re}\{\alpha z H''(z) + \alpha H'(z) + (1-\alpha)H'(z) \\ &\qquad + \overline{\alpha} z G''(z) + \overline{\alpha} G'(z) + (1-\overline{\alpha})G'(z)\} \\ &= \mathrm{Re}\{z(\alpha H''(z) + \overline{\alpha} G''(z)) + H'(z) + G'(z)\}. \end{aligned}$$

Conversely, if the function $F = H + \overline{G}$ of the form (2.4) satisfies (2.7), then by Theorem 2.1, $h' + \overline{g'} \in P_H$ and the function

$$f(z) = h(z) + \overline{g(z)} = \alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)(H(z) + G(z))$$

is from the class $\widetilde{P}_H^0$. Hence by Theorem 2.1, $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$. □

Proposition 2.3. *$\widetilde{P}_H^0(\alpha)$ is convex and compact.*

Proof. Let $F_1 = H_1 + \overline{G}_1,\ F_2 = H_2 + \overline{G}_2 \in \widetilde{P}_H^0(\alpha)$ and let $\lambda \in [0,1]$. Then

$$\begin{aligned}
\operatorname{Re}\{z[\alpha(\lambda H_1''(z) + (1-\lambda)H_2''(z)\bar{\alpha}(\lambda G_1''(z) + (1-\lambda)G_2''(z))] \\
+ \lambda[H_1'(z) + G_1'(z)] + (1-\lambda)[H_2'(z) + G_2'(z)]\} \\
= \lambda \operatorname{Re}\{z[\alpha H_1''(z) + \bar{\alpha}G_1''(z)] + H_1'(z) + G_1'(z)\} \\
+ (1-\lambda)\operatorname{Re}\{z[\alpha H_2''(z) + \bar{\alpha}G_2''(z)] + H_2'(z) + G_2'(z)\} \\
> 0.
\end{aligned}$$

Hence, from Corollary 2.2, $\lambda\ F_1 + (1-\lambda)F_2 \in \widetilde{P}_H^0(\alpha)$. Therefore, $\widetilde{P}_H^0(\alpha)$ is convex.

On the other hand, let $F_n = H_n + \overline{G}_n \in \widetilde{P}_H^0(\alpha)$ and let $F_n \to F = H + \overline{G}$. By Corollary 2.2,

$$\alpha[zH_n'(z) + \overline{zG_n'(z)}] + (1-\alpha)[H_n(z) + \overline{G_n(z)}] \in \widetilde{P}_H^0.$$

Since $\widetilde{P}_H^0$ is compact, [5],

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}] \in \widetilde{P}_H^0.$$

Hence, by Theorem 2.1, $F = H + \bar{G} \in \widetilde{P}_H^0(\alpha)$. Therefore, $\widetilde{P}_H^0(\alpha)$ is compact. □

Proposition 2.4. *If* $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$ *and* $|z| = r < 1$ *then*

$$-r + 2\ln(1+r) \le \operatorname{Re}\{\alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}]\}$$
$$\le -r - 2\ln(1-r).$$

Equality is obtained for the function (2.3) where

$$f(z) = 2z + \ln(1-z) - 3\overline{z} - 3\ln(1-\overline{z}), \quad z \in U.$$

Proof. From Theorem 2.1, if $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$, then there exists $f = h + \bar{g} \in \widetilde{P}_H^0$ so that

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1-\alpha)[H(z) + \overline{G(z)}] = f(z).$$

Since by [5, Proposition 2.2]

$$-r + 2\ln(1+r) \le \operatorname{Re} f(z) \le -r - 2\ln(1-r),$$

the proof is complete. □

Proposition 2.5. *If* $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$ *and* $\operatorname{Re}\alpha > 0$, *then there exists an* $f \in \widetilde{P}_H^0$ *so that*

$$F(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-2} f(z\zeta)\, d\zeta, \quad z \in U. \tag{2.8}$$

Proof. Since

$$t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{\frac{1}{\alpha}-1} \frac{z}{1-z\zeta}\, d\zeta, \quad |\zeta| \le 1, \quad \operatorname{Re}\alpha > 0,$$

and for $f = h + \overline{g} \in \widetilde{P}_H^0$

$$h(z) * \frac{z}{1 - z\zeta} = \frac{h(z\zeta)}{\zeta}, \quad g(z) * \frac{z}{1 - z\zeta} = \frac{g(z\zeta)}{\zeta},$$

we have

$$H(z) = h(z) * t_\alpha(z) = \frac{1}{\alpha} \int_0^1 \zeta^{\frac{1}{\alpha}-2} h(z\zeta)\, d\zeta$$

and

$$G(z) = g(z) * t_\alpha(z) = \frac{1}{\alpha} \int_0^1 \zeta^{\frac{1}{\alpha}-2} g(z\zeta)\, d\zeta.$$

Hence F is type (2.8). □

Theorem 2.6. *If* $\operatorname{Re} \alpha > 0$, *then* $\widetilde{P}_H^0(\alpha) \subset \widetilde{P}_H^0$. *Further, for any* $0 < \operatorname{Re} \alpha_1 \leq \operatorname{Re} \alpha_2$, $\widetilde{P}_H^0(\alpha_2) \subset \widetilde{P}_H^0(\alpha_1)$.

Proof. Let $F \in \widetilde{P}_H^0(\alpha)$ and $\operatorname{Re}\ \alpha > 0$. Then there exists $f \in \widetilde{P}_H^0$ so that

$$F = H + \overline{G} = f * (t_\alpha + \overline{t_\alpha}) = (h * t_\alpha) + (\overline{g * t_\alpha}).$$

Hence, $0 < \operatorname{Re}\{h' + \overline{g'}\} = \operatorname{Re}\{h' + g'\}$ and since $\operatorname{Re} \alpha > 0$, $\operatorname{Re}\{H' + G'\} > 0$, and $H(0) = 0$, $H'(0) = 1$, $G(0) = G'(0) = 0$ and hence $F = H + \overline{G} \in \widetilde{P}_H^0$.

For $0 < \operatorname{Re} \alpha_1 \leq \operatorname{Re} \alpha_2$, if $F \in \widetilde{P}_H^0(\alpha_2)$, from Corollary 2.2

$$\begin{aligned} 0 &< \operatorname{Re}\{z(\alpha_2 H''(z) + \overline{\alpha_2} G''(z)) + H'(z) + G'(z)\} \\ &\leq \operatorname{Re}\{z(\alpha_1 H''(z) + \overline{\alpha_1} G''(z)) + H'(z) + G'(z)\} \end{aligned}$$

we get $F \in \widetilde{P}_H^0(\alpha_1)$. □

Remark 2.1. For some values of α, $\widetilde{P}_H^0(\alpha) \subset \widetilde{P}_H^0$ is not true. It is known [5, Corollary 2.5] that the sharp inequalities

$$|a_n| \leq \frac{2n-1}{n} \quad \text{and} \quad |b_n| \leq \frac{2n-3}{n} \tag{2.9}$$

are true. Hence, for example, the function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2n-1}{n} z^n + \sum_{n=2}^{\infty} \frac{2n-3}{n} \overline{z^n}$$

belongs to $\widetilde{P}_H^0$. In this case

$$F(z) = z + \sum_{n=2}^{\infty} \frac{2n-1}{n[1+(n-1)\alpha]} z^n + \sum_{n=2}^{\infty} \overline{\frac{2n-3}{n[1+(n-1)\alpha]} z^n}$$

belongs to the class $\widetilde{P}_H^0(\alpha)$ for $\alpha \in \mathbb{C}$, $\alpha \neq -1/n$, $n \in N$. However, for $\operatorname{Re}\alpha \in \left(-\frac{|\alpha|^2}{3}, 0\right)$, $\alpha \neq -1, -\frac{1}{2}, \ldots$ as the coefficient conditions of $\widetilde{P}_H^0$ given in (2.9) are not satisfied, $F \notin \widetilde{P}_H^0$. Hence for each $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \in \left(-\frac{|\alpha|^2}{3}, 0\right)$, $\alpha \neq -1, -\frac{1}{2}, \ldots$, $\widetilde{P}_H^0(\alpha) - \widetilde{P}_H^0 \neq \phi$.

Theorem 2.7. *Let* $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$. *Then*

(i) $||A_n| - |B_n|| \leq \dfrac{2}{n|1+(n-1)\alpha|}, \quad n \geq 1$

(ii) If F is sense-preserving, then

$$|A_n| \leq \frac{2n-1}{n} \frac{1}{|1+(n-1)\alpha|}, \quad n = 1, 2, \dots$$

and

$$|B_n| \leq \frac{2n-3}{n} \frac{1}{|1+(n-1)\alpha|}, \quad n = 2, 3, \dots$$

Equality occurs for the functions of type (2.3) where

$$f(z) = \frac{2z}{1-z} + \ln(1-z) - \frac{3\bar{z} - \bar{z}^2}{1-\bar{z}} - 3\ln(1-\bar{z}), \quad z \in U.$$

Proof. By (2.6),

$$||A_n| - |B_n|| = \frac{1}{|1+(n-1)\alpha|} ||a_n| - |b_n|| .$$

Also by [5, Theorem 2.3], we have

$$||a_n| - |b_n|| \leq \frac{2}{n}$$

the required results are obtained.

On the other hand, from (2.6) and from the coefficient relations in $\widetilde{P}_H^0$ given in (2.9), we obtain the coefficient inequalities for $\widetilde{P}_H^0(\alpha)$. □

3. The Class $P_H(\beta, \alpha)$

Let $f = h + \overline{g}$ for analytic functions

$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

on U. The class $P_H(\beta)$ of all functions with $\operatorname{Re} f(z) > \beta$, $0 \le \beta < 1$ and $f(0) = 1$ is studied in [5].

Let us consider the function

$$k_\alpha(z) = 1 + \frac{1}{1+\alpha} z + \cdots + \frac{1}{1+n\alpha} z^n + \cdots, \quad \alpha \in \mathbb{C},\ \alpha \neq -1, -\frac{1}{2}, \ldots \tag{3.1}$$

which is analytic on U.

For $f \in P_H(\beta)$, let us denote the class of functions

$$F = f * (k_\alpha + \overline{k_\alpha}) = (h * k_\alpha) + (\overline{g * k_\alpha}) = H + \overline{G}, \tag{3.2}$$

by $P_H(\beta, \alpha)$. If $\alpha = 0$, then since $F = f$, $P_H(\beta, 0) = P_H(\beta)$.

Therefore,

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{a_n}{1+n\alpha} z^n + \sum_{n=1}^{\infty} \overline{\frac{b_n}{1+n\alpha} z^n} \\ &= 1 + \sum_{n=1}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n}, \qquad z \in U \end{aligned} \tag{3.3}$$

Theorem 3.1. *If $F \in P_H(\beta,\alpha)$ then there exists an $f \in P_H(\beta)$, so that*

$$\alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + F(z) = f(z). \tag{3.4}$$

Conversely, for $f \in P_H(\beta)$, there is a solution of (3.4) belonging to $P_H(\beta,\alpha)$.

Proof. Since $k_0(z) = \alpha z k'_\alpha(z) + k_\alpha(z)$, for $f \in P_H(\beta)$, using the fact that, $f = f * (k_0 + \overline{k_0})$,

$$f(z) = \alpha[f(z) * (zk'_\alpha(z) + \overline{zk'_\alpha(z)})] + [f(z) * (k_\alpha(z) + \overline{k_\alpha(z)})]$$

is obtained. Hence, for $F \in P_H(\beta,\alpha)$

$$f(z) = \alpha[zF_z(z) + \overline{z}F_{\overline{z}}(z)] + F(z).$$

Conversely, let $f = h + \overline{g} \in P_H(\beta)$ be given by (3.4). Hence, we can write

$$h(z) = \alpha z H'(z) + H(z), \quad g(z) = \alpha z G'(z) + G(z). \tag{3.5}$$

From the system (3.5) the analytic functions H and G are in the form

$$H(z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{1+n\alpha} z^n = h(z) * k_\alpha(z),$$

$$G(z) = \sum_{n=1}^{\infty} \frac{b_n}{1+n\alpha} z^n = g(z) * k_\alpha(z).$$

Hence the function $F = H + \overline{G}$ belongs to the class $P_H(\beta,\alpha)$. □

Corollary 3.2. *The necessary and sufficient conditions for a function F of form (3.3) to belong to $P_H(\beta,\alpha)$ are*

$$\mathrm{Re}\{z(\alpha H'(z)+\overline{\alpha}G'(z))+H(z)+G(z)\}>\beta,\quad z\in U. \tag{3.6}$$

Proof. If $F\in P_H(\beta,\alpha)$ then by Theorem 3.1,

$$\begin{aligned}\beta &< \mathrm{Re}\{f(z)\}\\ &= \mathrm{Re}\{\alpha[zF_z(z)+\overline{z}F_{\overline{z}}(z)]+F(z)\}\\ &= \mathrm{Re}\{z(\alpha H'(z)+\overline{\alpha}G'(z))+H(z)+G(z)\},\ z\in U.\end{aligned}$$

Conversely, if a function $F=H+\overline{G}$ of form (3.3) satisfies (3.6), then

$$z\alpha H'(z)+H(z)+\alpha\overline{zG'(z)}+\overline{G(z)}\in P_H(\beta).$$

Hence, from Theorem 3.1, we have $F=H+\bar{G}\in P_H(\beta,\alpha)$. □

Proposition 3.3. *If $F\in P_H(\beta,\alpha)$, $\mathrm{Re}\,\alpha>0$ then there exists an $f\in P_H(\beta)$ so that*

$$F(z)=\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}f(zt)dt,\quad z\in U. \tag{3.7}$$

The converse is also true.

Proof. Since

$$k_\alpha(z)=\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}\frac{1}{1-zt}dt,\quad \mathrm{Re}\ \alpha>0,$$

and for $f = h + \overline{g} \in P_H(\beta)$,

$$h(z) * \frac{1}{1-zt} = h(zt) \quad \text{and} \quad g(z) * \frac{1}{1-zt} = g(zt),$$

we obtain

$$H(z) = h(z) * k_\alpha(z) = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1} h(zt)dt$$

and

$$G(z) = g(z) * k_\alpha(z) = \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1} g(zt)dt.$$

Therefore, $F = H + \overline{G}$ is of type (3.7). □

Theorem 3.4. *Let $F \in P_H(\beta, \alpha)$. Then*

(i) $||A_n| - |B_n|| \leq \frac{2(1-\beta)}{|1+n\alpha|}, \quad n \geq 1$

(ii) If F is sense- preserving, then for $n = 1, 2, \ldots$

$$|A_n| \leq \frac{(1-\beta)(n+1)}{|1+n\alpha|} \quad \textit{and} \quad |B_n| \leq \frac{(1-\beta)(n-1)}{|1+n\alpha|}.$$

Equality is valid for the functions of type (3.2) where

$$f(z) = \operatorname{Re}\left\{\frac{1+(1-2\beta)z}{1-z}\right\} + i \operatorname{Im}\left\{\frac{1+z}{1-z}\right\}. \tag{3.8}$$

Proof. Let $F \in P_H(\beta, \alpha)$. Then from (3.3), as the coefficient relation for $P_H(\beta)$ is

$$||a_n| - |b_n|| \leq 2(1-\beta)$$

[5, Proposition 3.4], the required inequalities are obtained.

On the other hand, from (3.3), as the coefficient relations for $P_H(\beta)$ are

$$|a_n| \leq (1-\beta)(n+1) \quad \text{and} \quad |b_n| \leq (1-\beta)(n-1)$$

the required inequalities are obtained. □

Proposition 3.5. *If $F = H + \overline{G} \in P_H(\beta, \alpha)$, then for $X = \{\eta : |\eta| = 1\}$ and $z \in U$,*

$$H(z) + G(z) = 2(1-\beta) \int_{|\eta|=1} k_\alpha(\eta z)\, d\mu(\eta).$$

Here μ is the probability measure defined on the Borel sets on X.

Proof. From [5, Corollary 3.3] there exists a probability measure μ defined on the Borel sets on X so that

$$h(z) + g(z) = \int_{|\eta|=1} \frac{1 + (1-2\beta)\, z\eta}{1 - z\eta}\, d\mu(\eta).$$

Taking the Hadamard product of both sides by $k_\alpha(z)$, we get

$$\begin{aligned} &H(z) + G(z) \\ &= \int_{|\eta\ |=1} \left\{ \left(k_\alpha(z) * \frac{1}{1 - z\eta} \right) + (1-2\beta)\eta \left(k_\alpha(z) * \frac{z}{1 - z\eta} \right) \right\} d\mu(\eta) \end{aligned}$$

$$= \int_{|\eta|=1} \left\{ k_\alpha(\eta z) + (1-2\beta)\eta \frac{k_\alpha(\eta z)}{\eta} \right\} d\mu(\eta).$$

□

Theorem 3.6. *If* $\operatorname{Re}\alpha \geq 0$*, then* $P_H(\beta, \alpha) \subset P_H(\beta)$*. Further if* $0 \leq \operatorname{Re}\alpha_1 \leq \operatorname{Re}\alpha_2$*, then* $P_H(\beta, \alpha_2) \subset P_H(\beta, \alpha_1)$*.*

Proof. Let $F \in P_H(\beta, \alpha)$ and $\operatorname{Re}\alpha \geq 0$. Then as $\operatorname{Re}\{h' + g'\} > \beta$, we have $\operatorname{Re}\{H' + G'\} > \beta$ and $F(0) = 1$. Hence $F \in P_H(\beta)$. Further as $0 \leq \operatorname{Re}\alpha_1 \leq \operatorname{Re}\alpha_2$, for $F \in P_H(\beta, \alpha_2)$

$$\begin{aligned} \beta &< \operatorname{Re}\{z(\alpha_2 H'(z) + \overline{\alpha}_2 G'(z)) + H(z) + G(z)\} \\ &< \operatorname{Re}\{z(\alpha_1 H'(z) + \overline{\alpha}_1 G'(z)) + H(z) + G(z)\}. \end{aligned}$$

Therefore, by Corollary 3.2, $F \in P_H(\beta, \alpha_1)$. □

For $f \in P_H$, the class $B_H(\alpha)$ consisting of the functions $F = f * (k_\alpha + \overline{k_\alpha})$ is studied in [2]. The relation between the classes $P_H(\beta, \alpha)$ and $B_H(\alpha)$ is given as follows.

Proposition 3.7. *For* $\operatorname{Re}\alpha \geq 0$*,* $P_H(\beta, \alpha) \subset B_H(\alpha)$*.*

Proof. If $F \in P_H(\beta, \alpha)$ then there exists an $f \in P_H(\beta)$ so that $F = f * (k_\alpha + \overline{k_\alpha})$. Since $\operatorname{Re} f(z) > \beta$, $f(0) = 1$ and $0 \leq \beta < 1$, $\operatorname{Re} f(z) > 0$. Hence, $f \in P_H$. By the definition of $B_H(\alpha)$, $F \in B_H(\alpha)$. □

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