

Journal of Inequalities in Pure and Applied Mathematics

ON HARMONIC FUNCTIONS BY THE HADAMARD PRODUCT

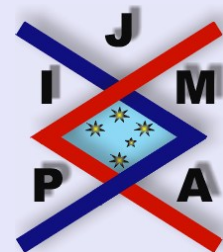
M. ÖZTÜRK, S. YALÇIN AND M. YAMANKARADENİZ

Uludag University,
Faculty of Science,
Department of Mathematics
16059 Bursa/Turkey.
EMail: ometin@uludag.edu.tr

©2000 School of Communications and Informatics, Victoria University of Technology

ISSN (electronic): 1443-5756

040-01



volume 3, issue 1, article 9,
2002.

*Received 11 May, 2001;
accepted 11 October, 2001.*

Communicated by: S.S. Dragomir

Abstract

Contents

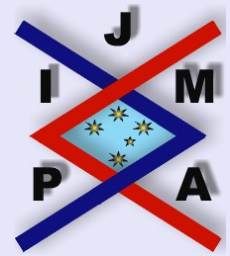


Home Page

Go Back

Close

Quit



Abstract

A function $f = u + iv$ defined in the domain $D \subset \mathbb{C}$ is harmonic in D if u, v are real harmonic. Such functions can be represented as $f = h + \bar{g}$ where h, g are analytic in D . In this paper the class of harmonic functions constructed by the Hadamard product in the unit disk, and properties of some of its subclasses are examined.

2000 Mathematics Subject Classification: 30C45, 31A05.

Key words: Harmonic functions, Hadamard product and extremal problems.

Contents

1	Introduction	3
2	The Class $\tilde{P}_H^0(\alpha)$	4
3	The Class $P_H(\beta, \alpha)$	12
	References	

On Harmonic Functions Constructed by the Hadamard Product

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 2 of 18

1. Introduction

Let U denote the open unit disk in \mathbb{C} and let $f = u + iv$ be a complex valued harmonic function on U . Since u and v are real parts of analytic functions, f admits a representation $f = h + \bar{g}$ for two functions h and g , analytic on U .

The Jacobian of f is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. The necessary and sufficient conditions for f to be local univalent and sense-preserving is $J_f(z) > 0, z \in U$ [1].

Many mathematicians studied the class of harmonic univalent and sense-preserving functions on U and its subclasses [2, 5].

Here we discuss two classes obtained by the Hadamard product.



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 3 of 18

2. The Class $\tilde{P}_H^0(\alpha)$

Let P_H denote the class of all functions $f = h + \bar{g}$ so that $\operatorname{Re} f > 0$ and $f(0) = 1$ where h and g are analytic on U .

If the function $f_z + \bar{f}_{\bar{z}} = h' + \bar{g}'$ belongs to P_H for the analytic and normalized functions

$$(2.1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n,$$

then the class of functions $f = h + \bar{g}$ is denoted by \tilde{P}_H^0 [5].

The function

$$(2.2) \quad t_\alpha(z) = z + \frac{1}{1+\alpha} z^2 + \cdots + \frac{1}{1+(n-1)\alpha} z^n + \cdots$$

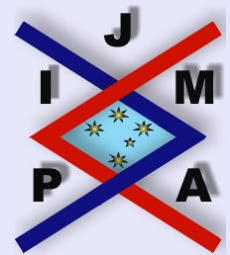
is analytic on U when α is a complex number different from $-1, -\frac{1}{2}, -\frac{1}{3}, \dots$

For $f \in \tilde{P}_H^0$, we denote, by $\tilde{P}_H^0(\alpha)$, the class of functions defined by

$$(2.3) \quad F = f * (t_\alpha + \bar{t}_\alpha).$$

Here $f * (t_\alpha + \bar{t}_\alpha)$ is the Hadamard product of the functions f and $t_\alpha + \bar{t}_\alpha$. Therefore

$$(2.4) \quad \begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{1+(n-1)\alpha} z^n + \sum_{n=2}^{\infty} \frac{\overline{b_n}}{1+(n-1)\alpha} z^n \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n}, \quad z \in U \end{aligned}$$



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 4 of 18

is in $\tilde{P}_H^0(\alpha)$.

Conversely, if F is in the form (2.4), with a_n, b_n being the coefficients of $f \in \tilde{P}_H^0$, then $F \in \tilde{P}_H^0(\alpha)$.

Furthermore, if $\alpha = 0$, then as $F = f$, we have $\tilde{P}_H^0(0) = \tilde{P}_H^0$. Moreover $\tilde{P}_H^0(\infty) = \{I : I(z) \equiv z, z \in U\}$ and since $I \in \tilde{P}_H^0, \tilde{P}_H^0 \cap \tilde{P}_H^0(\alpha) \neq \phi$.

Theorem 2.1. *If $F \in \tilde{P}_H^0(\alpha)$ then there exists $f \in \tilde{P}_H^0$ so that*

$$(2.5) \quad \alpha[zF_z(z) + \bar{z}F_{\bar{z}}(z)] + (1 - \alpha)F(z) = f(z).$$

Conversely, for any function $f \in \tilde{P}_H^0$, there exists $F \in \tilde{P}_H^0(\alpha)$ satisfying (2.5).

Proof. Let $F \in \tilde{P}_H^0(\alpha)$. If $f \in \tilde{P}_H^0$, then since

$$\alpha z t'_\alpha(z) + (1 - \alpha)t_\alpha(z) = t_0(z),$$

as $F = f * (t_\alpha + \bar{t}_\alpha)$ we obtain that

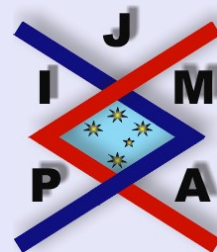
$$f(z) = \alpha[f(z) * (z t'_\alpha(z) + \overline{z t'_\alpha(z)})] + (1 - \alpha)[f(z) * (t_\alpha(z) + \bar{t}_\alpha(z))].$$

Therefore,

$$f(z) = \alpha[zF_z(z) + \bar{z}F_{\bar{z}}(z)] + (1 - \alpha)F(z).$$

Conversely, for $f \in \tilde{P}_H^0$, from (2.1), (2.2) and (2.5),

$$z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n z^n} = z + \sum_{n=2}^{\infty} [1 + (n-1)\alpha] A_n z^n + \sum_{n=2}^{\infty} \overline{[1 + (n-1)\alpha] B_n z^n}.$$



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 5 of 18

From these one obtains

$$(2.6) \quad A_n = \frac{a_n}{1 + (n-1)\alpha} \quad \text{and} \quad B_n = \frac{b_n}{1 + (n-1)\alpha}.$$

Therefore,

$$\begin{aligned} F(z) &= z + \sum_{n=2}^{\infty} \frac{a_n}{1 + (n-1)\alpha} z^n + \sum_{n=2}^{\infty} \frac{\overline{b_n}}{1 + (n-1)\alpha} z^n \\ &= f(z) * [t_\alpha(z) + \overline{t_\alpha}(z)]. \end{aligned}$$

□

Corollary 2.2. A function $F = H + \overline{G}$ of the form (2.4) belongs to $\tilde{P}_H^0(\alpha)$, if and only if

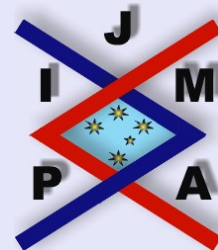
$$(2.7) \quad \operatorname{Re}\{z(\alpha H''(z) + \overline{\alpha} G''(z)) + H'(z) + G'(z)\} > 0, \quad z \in U.$$

Proof. If $F = H + \overline{G} \in \tilde{P}_H^0(\alpha)$, then from Theorem 2.1

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1 - \alpha)[H(z) + \overline{G(z)}] = h(z) + \overline{g(z)} \in \tilde{P}_H^0$$

and $h' + \overline{g'} \in P_H$. Hence

$$\begin{aligned} 0 &< \operatorname{Re}\{h'(z) + g'(z)\} \\ &= \operatorname{Re}\{\alpha z H''(z) + \alpha H'(z) + (1 - \alpha)H'(z) \\ &\quad + \overline{\alpha z G''(z)} + \overline{\alpha G'(z)} + (1 - \overline{\alpha})G'(z)\} \\ &= \operatorname{Re}\{z(\alpha H''(z) + \overline{\alpha} G''(z)) + H'(z) + G'(z)\}. \end{aligned}$$



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 6 of 18

Conversely, if the function $F = H + \overline{G}$ of the form (2.4) satisfies (2.7), then by Theorem 2.1, $h' + \overline{g'} \in P_H$ and the function

$$f(z) = h(z) + \overline{g(z)} = \alpha[zH'(z) + \overline{zG'(z)}] + (1 - \alpha)(H(z) + G(z))$$

is from the class \tilde{P}_H^0 . Hence by Theorem 2.1, $F = H + \overline{G} \in \tilde{P}_H^0(\alpha)$. \square

Proposition 2.3. $\tilde{P}_H^0(\alpha)$ is convex and compact.

Proof. Let $F_1 = H_1 + \overline{G}_1$, $F_2 = H_2 + \overline{G}_2 \in \tilde{P}_H^0(\alpha)$ and let $\lambda \in [0, 1]$. Then

$$\begin{aligned} & \operatorname{Re}\{z[\alpha(\lambda H_1''(z) + (1 - \lambda)H_2''(z))\bar{\alpha}(\lambda G_1''(z) + (1 - \lambda)G_2''(z))] \\ & \quad + \lambda[H_1'(z) + G_1'(z)] + (1 - \lambda)[H_2'(z) + G_2'(z)]\} \\ &= \lambda \operatorname{Re}\{z[\alpha H_1''(z) + \bar{\alpha}G_1''(z)] + H_1'(z) + G_1'(z)\} \\ & \quad + (1 - \lambda) \operatorname{Re}\{z[\alpha H_2''(z) + \bar{\alpha}G_2''(z)] + H_2'(z) + G_2'(z)\} \\ & > 0. \end{aligned}$$

Hence, from Corollary 2.2, $\lambda F_1 + (1 - \lambda)F_2 \in \tilde{P}_H^0(\alpha)$. Therefore, $\tilde{P}_H^0(\alpha)$ is convex.

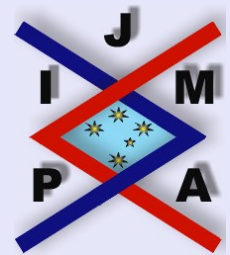
On the other hand, let $F_n = H_n + \overline{G}_n \in \tilde{P}_H^0(\alpha)$ and let $F_n \rightarrow F = H + \overline{G}$. By Corollary 2.2,

$$\alpha[zH_n'(z) + \overline{zG_n'(z)}] + (1 - \alpha)[H_n(z) + \overline{G_n(z)}] \in \tilde{P}_H^0.$$

Since \tilde{P}_H^0 is compact, [5],

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1 - \alpha)[H(z) + \overline{G(z)}] \in \tilde{P}_H^0.$$

Hence, by Theorem 2.1, $F = H + \overline{G} \in \tilde{P}_H^0(\alpha)$. Therefore, $\tilde{P}_H^0(\alpha)$ is compact. \square



On Harmonic Functions
Constructed by the Hadamard
Product

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents

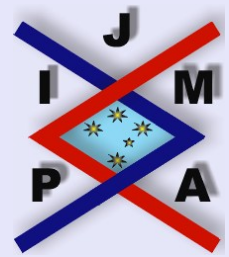


Go Back

Close

Quit

Page 7 of 18



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 8 of 18

Proposition 2.4. If $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$ and $|z| = r < 1$ then

$$\begin{aligned} -r + 2 \ln(1 + r) &\leq \operatorname{Re}\{\alpha[zH'(z) + \overline{zG'(z)}] + (1 - \alpha)[H(z) + \overline{G(z)}]\} \\ &\leq -r - 2 \ln(1 - r). \end{aligned}$$

Equality is obtained for the function (2.3) where

$$f(z) = 2z + \ln(1 - z) - 3\bar{z} - 3 \ln(1 - \bar{z}), \quad z \in U.$$

Proof. From Theorem 2.1, if $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$, then there exists $f = h + \bar{g} \in \widetilde{P}_H^0$ so that

$$\alpha[zH'(z) + \overline{zG'(z)}] + (1 - \alpha)[H(z) + \overline{G(z)}] = f(z).$$

Since by [5, Proposition 2.2]

$$-r + 2 \ln(1 + r) \leq \operatorname{Re} f(z) \leq -r - 2 \ln(1 - r),$$

the proof is complete. \square

Proposition 2.5. If $F = H + \overline{G} \in \widetilde{P}_H^0(\alpha)$ and $\operatorname{Re} \alpha > 0$, then there exists an $f \in \widetilde{P}_H^0$ so that

$$(2.8) \quad F(z) = \frac{1}{\alpha} \int_0^1 \zeta^{\frac{1}{\alpha}-2} f(z\zeta) d\zeta, \quad z \in U.$$

Proof. Since

$$t_\alpha(z) = \frac{1}{\alpha} \int_0^1 \zeta^{\frac{1}{\alpha}-1} \frac{z}{1 - z\zeta} d\zeta, \quad |\zeta| \leq 1, \quad \operatorname{Re} \alpha > 0,$$

and for $f = h + \bar{g} \in \tilde{P}_H^0$

$$h(z) * \frac{z}{1-z\zeta} = \frac{h(z\zeta)}{\zeta}, \quad g(z) * \frac{z}{1-z\zeta} = \frac{g(z\zeta)}{\zeta},$$

we have

$$H(z) = h(z) * t_\alpha(z) = \frac{1}{\alpha} \int_0^1 \zeta^{\frac{1}{\alpha}-2} h(z\zeta) d\zeta$$

and

$$G(z) = g(z) * t_\alpha(z) = \frac{1}{\alpha} \int_0^1 \zeta^{\frac{1}{\alpha}-2} g(z\zeta) d\zeta.$$

Hence F is type (2.8). □

Theorem 2.6. *If $\operatorname{Re} \alpha > 0$, then $\tilde{P}_H^0(\alpha) \subset \tilde{P}_H^0$. Further, for any $0 < \operatorname{Re} \alpha_1 \leq \operatorname{Re} \alpha_2$, $\tilde{P}_H^0(\alpha_2) \subset \tilde{P}_H^0(\alpha_1)$.*

Proof. Let $F \in \tilde{P}_H^0(\alpha)$ and $\operatorname{Re} \alpha > 0$. Then there exists $f \in \tilde{P}_H^0$ so that

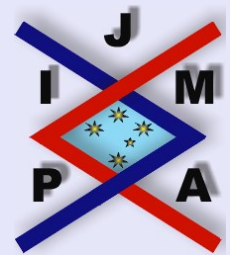
$$F = H + \bar{G} = f * (t_\alpha + \bar{t}_\alpha) = (h * t_\alpha) + (\overline{g * t_\alpha}).$$

Hence, $0 < \operatorname{Re}\{h' + \bar{g}'\} = \operatorname{Re}\{h' + g'\}$ and since $\operatorname{Re} \alpha > 0$, $\operatorname{Re}\{H' + G'\} > 0$, and $H(0) = 0$, $H'(0) = 1$, $G(0) = G'(0) = 0$ and hence $F = H + \bar{G} \in \tilde{P}_H^0$.

For $0 < \operatorname{Re} \alpha_1 \leq \operatorname{Re} \alpha_2$, if $F \in \tilde{P}_H^0(\alpha_2)$, from Corollary 2.2

$$\begin{aligned} 0 &< \operatorname{Re}\{z(\alpha_2 H''(z) + \bar{\alpha}_2 G''(z)) + H'(z) + G'(z)\} \\ &\leq \operatorname{Re}\{z(\alpha_1 H''(z) + \bar{\alpha}_1 G''(z)) + H'(z) + G'(z)\} \end{aligned}$$

we get $F \in \tilde{P}_H^0(\alpha_1)$. □



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 9 of 18

Remark 2.1. For some values of α , $\tilde{P}_H^0(\alpha) \subset \tilde{P}_H^0$ is not true. It is known [5, Corollary 2.5] that the sharp inequalities

$$(2.9) \quad |a_n| \leq \frac{2n-1}{n} \quad \text{and} \quad |b_n| \leq \frac{2n-3}{n}$$

are true. Hence, for example, the function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2n-1}{n} z^n + \sum_{n=2}^{\infty} \frac{2n-3}{n} z^n$$

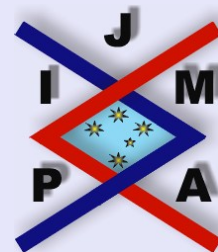
belongs to \tilde{P}_H^0 . In this case

$$F(z) = z + \sum_{n=2}^{\infty} \frac{2n-1}{n[1+(n-1)\alpha]} z^n + \sum_{n=2}^{\infty} \frac{2n-3}{n[1+(n-1)\alpha]} z^n$$

belongs to the class $\tilde{P}_H^0(\alpha)$ for $\alpha \in \mathbb{C}$, $\alpha \neq -1/n$, $n \in \mathbb{N}$. However, for $\text{Re } \alpha \in \left(-\frac{|\alpha|^2}{3}, 0\right)$, $\alpha \neq -1, -\frac{1}{2}, \dots$ as the coefficient conditions of \tilde{P}_H^0 given in (2.9) are not satisfied, $F \notin \tilde{P}_H^0$. Hence for each $\alpha \in \mathbb{C}$ with $\text{Re } \alpha \in \left(-\frac{|\alpha|^2}{3}, 0\right)$, $\alpha \neq -1, -\frac{1}{2}, \dots$, $\tilde{P}_H^0(\alpha) - \tilde{P}_H^0 \neq \emptyset$.

Theorem 2.7. Let $F = H + \overline{G} \in \tilde{P}_H^0(\alpha)$. Then

$$(i) \quad ||A_n| - |B_n|| \leq \frac{2}{n|1+(n-1)\alpha|}, \quad n \geq 1$$



On Harmonic Functions
Constructed by the Hadamard
Product

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 10 of 18

(ii) If F is sense-preserving, then

$$|A_n| \leq \frac{2n-1}{n} \frac{1}{|1+(n-1)\alpha|}, \quad n = 1, 2, \dots$$

and

$$|B_n| \leq \frac{2n-3}{n} \frac{1}{|1+(n-1)\alpha|}, \quad n = 2, 3, \dots$$

Equality occurs for the functions of type (2.3) where

$$f(z) = \frac{2z}{1-z} + \ln(1-z) - \frac{3\bar{z} - \bar{z}^2}{1-\bar{z}} - 3\ln(1-\bar{z}), \quad z \in U.$$

Proof. By (2.6),

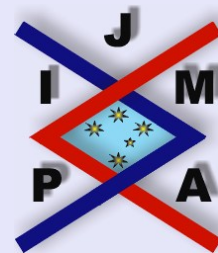
$$\| |A_n| - |B_n| \| = \frac{1}{|1+(n-1)\alpha|} \| |a_n| - |b_n| \|.$$

Also by [5, Theorem 2.3], we have

$$\| |a_n| - |b_n| \| \leq \frac{2}{n}$$

the required results are obtained.

On the other hand, from (2.6) and from the coefficient relations in \tilde{P}_H^0 given in (2.9), we obtain the coefficient inequalities for $\tilde{P}_H^0(\alpha)$. \square



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 11 of 18

3. The Class $P_H(\beta, \alpha)$

Let $f = h + \bar{g}$ for analytic functions

$$h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

on U . The class $P_H(\beta)$ of all functions with $\operatorname{Re} f(z) > \beta$, $0 \leq \beta < 1$ and $f(0) = 1$ is studied in [5].

Let us consider the function

$$(3.1) \quad k_\alpha(z) = 1 + \frac{1}{1+\alpha}z + \dots + \frac{1}{1+n\alpha}z^n + \dots, \quad \alpha \in \mathbb{C}, \quad \alpha \neq -1, -\frac{1}{2}, \dots$$

which is analytic on U .

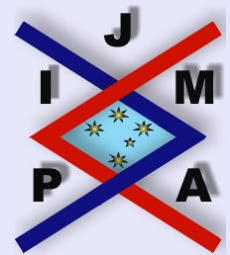
For $f \in P_H(\beta)$, let us denote the class of functions

$$(3.2) \quad F = f * (k_\alpha + \overline{k_\alpha}) = (h * k_\alpha) + (\overline{g * k_\alpha}) = H + \overline{G},$$

by $P_H(\beta, \alpha)$. If $\alpha = 0$, then since $F = f$, $P_H(\beta, 0) = P_H(\beta)$.

Therefore,

$$(3.3) \quad \begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{a_n}{1+n\alpha} z^n + \sum_{n=1}^{\infty} \overline{\frac{b_n}{1+n\alpha}} z^n \\ &= 1 + \sum_{n=1}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} z^n, \quad z \in U \end{aligned}$$



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents

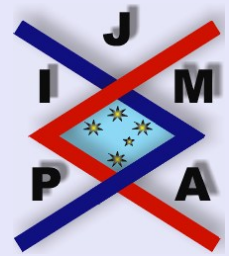


Go Back

Close

Quit

Page 12 of 18



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 13 of 18

Theorem 3.1. If $F \in P_H(\beta, \alpha)$ then there exists an $f \in P_H(\beta)$, so that

$$(3.4) \quad \alpha[zF_z(z) + \bar{z}F_{\bar{z}}(z)] + F(z) = f(z).$$

Conversely, for $f \in P_H(\beta)$, there is a solution of (3.4) belonging to $P_H(\beta, \alpha)$.

Proof. Since $k_0(z) = \alpha z k'_\alpha(z) + k_\alpha(z)$, for $f \in P_H(\beta)$, using the fact that, $f = f * (k_0 + \bar{k}_0)$,

$$f(z) = \alpha[f(z) * (zk'_\alpha(z) + \overline{zk'_\alpha(z)})] + [f(z) * (k_\alpha(z) + \overline{k_\alpha(z)})]$$

is obtained. Hence, for $F \in P_H(\beta, \alpha)$

$$f(z) = \alpha[zF_z(z) + \bar{z}F_{\bar{z}}(z)] + F(z).$$

Conversely, let $f = h + \bar{g} \in P_H(\beta)$ be given by (3.4). Hence, we can write

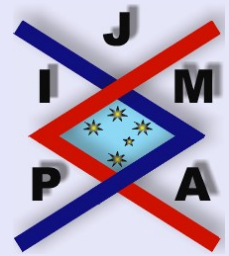
$$(3.5) \quad h(z) = \alpha z H'(z) + H(z), \quad g(z) = \alpha z G'(z) + G(z).$$

From the system (3.5) the analytic functions H and G are in the form

$$H(z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{1+n\alpha} z^n = h(z) * k_\alpha(z),$$

$$G(z) = \sum_{n=1}^{\infty} \frac{b_n}{1+n\alpha} z^n = g(z) * k_\alpha(z).$$

Hence the function $F = H + \bar{G}$ belongs to the class $P_H(\beta, \alpha)$. □



Corollary 3.2. *The necessary and sufficient conditions for a function F of form (3.3) to belong to $P_H(\beta, \alpha)$ are*

$$(3.6) \quad \operatorname{Re}\{z(\alpha H'(z) + \bar{\alpha}G'(z)) + H(z) + G(z)\} > \beta, \quad z \in U.$$

Proof. If $F \in P_H(\beta, \alpha)$ then by Theorem 3.1,

$$\begin{aligned} \beta &< \operatorname{Re}\{f(z)\} \\ &= \operatorname{Re}\{\alpha[zF_z(z) + \bar{z}F_{\bar{z}}(z)] + F(z)\} \\ &= \operatorname{Re}\{z(\alpha H'(z) + \bar{\alpha}G'(z)) + H(z) + G(z)\}, \quad z \in U. \end{aligned}$$

Conversely, if a function $F = H + \bar{G}$ of form (3.3) satisfies (3.6), then

$$z\alpha H'(z) + H(z) + \alpha z\overline{G'(z)} + \overline{G(z)} \in P_H(\beta).$$

Hence, from Theorem 3.1, we have $F = H + \bar{G} \in P_H(\beta, \alpha)$. □

Proposition 3.3. *If $F \in P_H(\beta, \alpha)$, $\operatorname{Re} \alpha > 0$ then there exists an $f \in P_H(\beta)$ so that*

$$(3.7) \quad F(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} f(zt) dt, \quad z \in U.$$

The converse is also true.

Proof. Since

$$k_\alpha(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} \frac{1}{1-zt} dt, \quad \operatorname{Re} \alpha > 0,$$

**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 14 of 18

and for $f = h + \bar{g} \in P_H(\beta)$,

$$h(z) * \frac{1}{1-zt} = h(zt) \quad \text{and} \quad g(z) * \frac{1}{1-zt} = g(zt),$$

we obtain

$$H(z) = h(z) * k_\alpha(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} h(zt) dt$$

and

$$G(z) = g(z) * k_\alpha(z) = \frac{1}{\alpha} \int_0^1 t^{\frac{1}{\alpha}-1} g(zt) dt.$$

Therefore, $F = H + \bar{G}$ is of type (3.7). □

Theorem 3.4. *Let $F \in P_H(\beta, \alpha)$. Then*

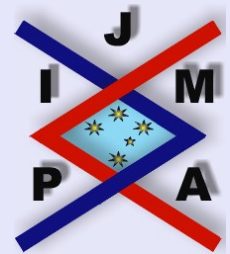
$$(i) \quad ||A_n| - |B_n|| \leq \frac{2(1-\beta)}{|1+n\alpha|}, \quad n \geq 1$$

(ii) *If F is sense-preserving, then for $n = 1, 2, \dots$*

$$|A_n| \leq \frac{(1-\beta)(n+1)}{|1+n\alpha|} \quad \text{and} \quad |B_n| \leq \frac{(1-\beta)(n-1)}{|1+n\alpha|}.$$

Equality is valid for the functions of type (3.2) where

$$(3.8) \quad f(z) = \operatorname{Re} \left\{ \frac{1 + (1-2\beta)z}{1-z} \right\} + i \operatorname{Im} \left\{ \frac{1+z}{1-z} \right\}.$$



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents

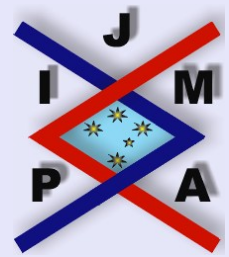


Go Back

Close

Quit

Page 15 of 18



Proof. Let $F \in P_H(\beta, \alpha)$. Then from (3.3), as the coefficient relation for $P_H(\beta)$ is

$$||a_n| - |b_n|| \leq 2(1 - \beta)$$

[5, Proposition 3.4], the required inequalities are obtained.

On the other hand, from (3.3), as the coefficient relations for $P_H(\beta)$ are

$$|a_n| \leq (1 - \beta)(n + 1) \quad \text{and} \quad |b_n| \leq (1 - \beta)(n - 1)$$

the required inequalities are obtained. \square

Proposition 3.5. *If $F = H + \overline{G} \in P_H(\beta, \alpha)$, then for $X = \{\eta : |\eta| = 1\}$ and $z \in U$,*

$$H(z) + G(z) = 2(1 - \beta) \int_{|\eta|=1} k_\alpha(\eta z) d\mu(\eta).$$

Here μ is the probability measure defined on the Borel sets on X .

Proof. From [5, Corollary 3.3] there exists a probability measure μ defined on the Borel sets on X so that

$$h(z) + g(z) = \int_{|\eta|=1} \frac{1 + (1 - 2\beta)z\eta}{1 - z\eta} d\mu(\eta).$$

Taking the Hadamard product of both sides by $k_\alpha(z)$, we get

$$\begin{aligned} & H(z) + G(z) \\ &= \int_{|\eta|=1} \left\{ \left(k_\alpha(z) * \frac{1}{1 - z\eta} \right) + (1 - 2\beta)\eta \left(k_\alpha(z) * \frac{z}{1 - z\eta} \right) \right\} d\mu(\eta) \end{aligned}$$

**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 16 of 18

$$= \int_{|\eta|=1} \left\{ k_\alpha(\eta z) + (1 - 2\beta)\eta \frac{k_\alpha(\eta z)}{\eta} \right\} d\mu(\eta).$$

□

Theorem 3.6. *If $\operatorname{Re} \alpha \geq 0$, then $P_H(\beta, \alpha) \subset P_H(\beta)$. Further if $0 \leq \operatorname{Re} \alpha_1 \leq \operatorname{Re} \alpha_2$, then $P_H(\beta, \alpha_2) \subset P_H(\beta, \alpha_1)$.*

Proof. Let $F \in P_H(\beta, \alpha)$ and $\operatorname{Re} \alpha \geq 0$. Then as $\operatorname{Re}\{h' + g'\} > \beta$, we have $\operatorname{Re}\{H' + G'\} > \beta$ and $F(0) = 1$. Hence $F \in P_H(\beta)$. Further as $0 \leq \operatorname{Re} \alpha_1 \leq \operatorname{Re} \alpha_2$, for $F \in P_H(\beta, \alpha_2)$

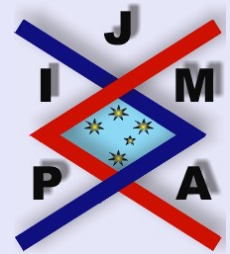
$$\begin{aligned} \beta &< \operatorname{Re}\{z(\alpha_2 H'(z) + \bar{\alpha}_2 G'(z)) + H(z) + G(z)\} \\ &< \operatorname{Re}\{z(\alpha_1 H'(z) + \bar{\alpha}_1 G'(z)) + H(z) + G(z)\}. \end{aligned}$$

Therefore, by Corollary 3.2, $F \in P_H(\beta, \alpha_1)$. □

For $f \in P_H$, the class $B_H(\alpha)$ consisting of the functions $F = f * (k_\alpha + \overline{k_\alpha})$ is studied in [2]. The relation between the classes $P_H(\beta, \alpha)$ and $B_H(\alpha)$ is given as follows.

Proposition 3.7. *For $\operatorname{Re} \alpha \geq 0$, $P_H(\beta, \alpha) \subset B_H(\alpha)$.*

Proof. If $F \in P_H(\beta, \alpha)$ then there exists an $f \in P_H(\beta)$ so that $F = f * (k_\alpha + \overline{k_\alpha})$. Since $\operatorname{Re} f(z) > \beta$, $f(0) = 1$ and $0 \leq \beta < 1$, $\operatorname{Re} f(z) > 0$. Hence, $f \in P_H$. By the definition of $B_H(\alpha)$, $F \in B_H(\alpha)$. □



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

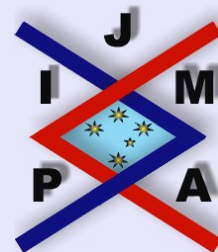
Close

Quit

Page 17 of 18

References

- [1] J. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Series A.I, Math.*, **9** (1984), 3–25.
- [2] Z.J. JAKUBOWSKI, W. MAJCHRZAK AND K. SKALSKA, Harmonic mappings with a positive real part, *Materialy XIV Konferencijiz Teorii Zagadnien Exstremalnych*, Lodz, 17–24, 1993.
- [3] H. LEWY, On the non vanishing of the Jacobian in certain one to one mappings, *Bull. Amer. Math. Soc.*, **42** (1936), 689–692.
- [4] H. SILVERMAN, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.*, **220** (1998), 283–289.
- [5] S. YALÇIN, M. ÖZTÜRK AND M. YAMANKARADENIZ, On some subclasses of harmonic functions, in *Mathematics and Its Applications*, Kluwer Acad. Publ.; *Functional Equations and Inequalities*, **518** (2000), 325–331.



**On Harmonic Functions
Constructed by the Hadamard
Product**

Metin Öztürk, Sibel Yalçın and
Mümin Yamankaradeniz

Title Page

Contents



Go Back

Close

Quit

Page 18 of 18