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INTERPOLATION FUNCTIONS OF SEVERAL MATRIX VARIABLES

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ABSTRACT. An interpolation theorem of Donoghue is extended to interpolation of tensor products. The result is related to Korányi's work on monotone matrix functions of several variables.

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1. STATEMENT AND PROOF OF THE MAIN RESULT

Recall the definition of an interpolation function (of one variable). Let $A \in M_n(\mathbb{C}) := \mathcal{L}(\ell_2^n)$ be a positive definite matrix. A real function h defined on $\sigma(A)$ is said to belong to the class C_A of *interpolation functions with respect to* A if

$$T \in M_n(\mathbb{C}), \quad T^*T \leq 1, \quad T^*AT \leq A \tag{1.1}$$

imply

$$T^*h(A)T \leq h(A). \tag{1.2}$$

(Here $A \leq B$ means that $B - A$ is positive semidefinite). By Donoghue's theorem (cf. [4, Theorem 1], see also [1, Theorem 7.1]), the functions in C_A are precisely those representable in the form

$$h(\lambda) = \int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda} d\rho(t), \quad \lambda \in \sigma(A), \tag{1.3}$$

for some positive Radon measure ρ on the compactified half-line $[0, \infty]$. Thus, by Löwner's theorem (see [6] or [3]), C_A is precisely the set of restrictions to $\sigma(A)$ of the positive *matrix monotone* functions on $\mathbb{R}_+$, in the sense that $A, B \in M_n(\mathbb{C})$ positive definite and $A \leq B$ imply $h(A) \leq h(B)$. Before we proceed, it is important to note that

$$h \in C_A \quad \text{implies} \quad h^{\frac{1}{2}} \in C_A \tag{1.4}$$

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because the function $\lambda \mapsto \lambda^{\frac{1}{2}}$ is matrix monotone and the class of matrix monotone functions is a semi-group under composition.

Given two positive definite matrices $A_i \in M_{n_i}(\mathbb{C})$, define the class C_{A_1,A_2} of *interpolation functions with respect to* A_1, A_2 as the set of functions h defined on $\sigma(A_1) \times \sigma(A_2)$ having the following property:

$$T_i \in M_{n_i}(\mathbb{C}) \quad T_i^* T_i \leq 1 \quad T_i^* A_i T_i \leq A_i, \quad i = 1, 2 \tag{1.5}$$

imply

$$(T_1 \otimes T_2)^* h(A_1, A_2)(T_1 \otimes T_2) \leq h(A_1, A_2). \tag{1.6}$$

(Here (cf. [8])

$$h(A_1, A_2) = \sum_{(\lambda_1,\lambda_2)\in\sigma(A_1)\times\sigma(A_2)} h(\lambda_1, \lambda_2) E_{\lambda_1} \otimes F_{\lambda_2},$$

where E, F are the spectral resolutions of A_1, A_2).

Note that if $h = h_1 \otimes h_2$ is an elementary tensor where $h_i \in C_{A_i}$, then $h \in C_{A_1,A_2}$, because then (1.5) yields

$$\begin{aligned}(T_1 \otimes T_2)^* h(A_1, A_2)(T_1 \otimes T_2) &= (T_1^* h_1(A_1) T_1) \otimes (T_2^* h_2(A_2) T_2) \\ &\leq h_1(A_1) \otimes h_2(A_2) = h(A_1, A_2),\end{aligned}$$

i.e. (1.6) holds. Since by (1.3) each function

$$\lambda \mapsto \frac{(1+t)\lambda}{1+t\lambda}$$

is in C_A for any A, and since the class C_{A_1,A_2} is a convex cone, closed under pointwise convergence, it follows that functions of the type

$$h(\lambda_1, \lambda_2) = \int_{[0,\infty]^2} \frac{(1+t_1)\lambda_1}{1+t_1\lambda_1} \frac{(1+t_2)\lambda_2}{1+t_2\lambda_2} d\rho(t_1, t_2), \tag{1.7}$$

where ρ is a positive Radon measure on $[0, \infty]^2$ are in C_{A_1,A_2} for all A_1, A_2. We have thus proved the easy part of our main theorem:

Theorem 1.1. *Let h be a real function defined on $\sigma(A_1) \times \sigma(A_2)$. Then $h \in C_{A_1,A_2}$ iff h is representable in the form (1.7) for some positive Radon measure ρ.*

It remains to prove "⇒". Let us make some preliminary observations:

(i) ([2, Lemma 2.2]) The class C_{A_1,A_2} is *unitarily invariant* in the sense that if A_1 and A_2 are unitarily equivalent to A_1' and A_2' respectively, then $h \in C_{A_1,A_2}$ implies $h \in C_{A_1',A_2'}$. (Indeed,

$$h(U_1^* A_1 U_1, U_2^* A_2 U_2) = (U_1 \otimes U_2)^* h(A_1, A_2)(U_1 \otimes U_2)$$

for all unitaries U_1, U_2).

(ii) ([2, Lemma 2.1]) The class C_{A_1,A_2} *respects compressions to invariant subspaces* in the sense that if $f \in C_{A_1,A_2}$ and A_1', A_2' are compressions of A_1, A_2 respectively to invariant subspaces, then $h \in C_{A_1',A_2'}$. (Indeed,

$$(E \otimes F) h(A_1, A_2)(E \otimes F) = (E \otimes F) h(EA_1E, FA_2F)(E \otimes F)$$

whenever E, F are orthogonal projections commuting with A_1, A_2 respectively).

(iii) If λ_2^* is any (fixed) eigenvalue of A_2 and the function $h_{\lambda_2^*} : \sigma(A_1) \to \mathbb{R}$ is defined by $h_{\lambda_2^*}(\lambda_1) = h(\lambda_1, \lambda_2^*)$, then

$$\begin{aligned} h(A_1, \lambda_2^* F_{\lambda_2^*}) &= \sum_{\lambda_1 \in \sigma(A_1)} h(\lambda_1, \lambda_2^*)(E_{\lambda_1} \otimes F_{\lambda_2^*}) \\ &= \left(\sum_{\lambda_1 \in \sigma(A_1)} h_{\lambda_2^*}(\lambda_1) E_{\lambda_1} \right) \otimes F_{\lambda_2^*} \\ &= h_{\lambda_2^*}(A_1) \otimes F_{\lambda_2^*}. \end{aligned}$$

(iv) By symmetry, of course (with fixed λ_1^* in $\sigma(A_1)$ and $h_{\lambda_1^*}(\lambda_2) = h(\lambda_1^*, \lambda_2)$),

$$h(\lambda_1^* E_{\lambda_1^*}, A_2) = E_{\lambda_1^*} \otimes h_{\lambda_1^*}(A_2).$$

Lemma 1.2. *Let $h \in C_{A_1,A_2}$ and let λ_1^*, λ_2^* be fixed eigenvalues of A_1 and A_2 respectively. Then $h_{\lambda_1^*}^{\frac{1}{2}} \in C_{A_2}$ and $h_{\lambda_2^*}^{\frac{1}{2}} \in C_{A_1}$.*

Proof. By symmetry of the problem, it suffices to prove the statement about $h_{\lambda_2^*}^{\frac{1}{2}}$. If $h \in C_{A_1,A_2}$, then by (iii),

$$h(A_1, \lambda_2^* F_{\lambda_2^*}) = h_{\lambda_2^*}(A_1) \otimes F_{\lambda_2^*}.$$

Let f_2^* be a fixed non-zero vector in the range of $F_{\lambda_2^*}$ and put $c = (F_{\lambda_2^*} f_2^*, f_2^*) > 0$. Put $T_2 = F_{\lambda_2^*}$ and let T_1 be any matrix fulfilling $T_1^* T_1 \leq 1$ and $T_1^* A_1 T_1 \leq A_1$; then plainly T_1, T_2 satisfy condition (1.5). Thus, since $h \in C_{A_1, \lambda_2^* F_{\lambda_2^*}}$, we get from (1.6)

$$\begin{aligned} ((T_1 \otimes T_2)^* h(A_1, \lambda_2^* F_{\lambda_2^*})(T_1 \otimes T_2)(f_1 \otimes f_2^*), f_1 \otimes f_2^*) - (h(A_1, \lambda_2^* F_{\lambda_2^*})(f_1 \otimes f_2^*), f_1 \otimes f_2^*) \\ = c((T_1^* h_{\lambda_2^*}(A_1) T_1 f_1, f_1) - (h_{\lambda_2^*}(A_1) f_1, f_1)) \leq 0, \quad f_1 \in M_{n_1}(\mathbb{C}). \end{aligned}$$

This yields $T_1^* h_{\lambda_2^*}(A_1) T_1 \leq h_{\lambda_2^*}(A_1)$, $T_1 \in M_{n_1}(\mathbb{C})$, i.e. $h_{\lambda_2^*} \in C_{A_1}$. In view of (1.4), $h_{\lambda_2^*}^{\frac{1}{2}} \in C_{A_1}$. □

Let h be a fixed function in the class C_{A_1,A_2}. Replacing the matrices A_1, A_2 by $c_1 A_1$, $c_2 A_2$ for suitable constants $c_1, c_2 > 0$, we can assume without loss of generality that

$$(1,1) \in \sigma(A_1) \times \sigma(A_2). \tag{1.8}$$

Define C to be the C^*-algebra of continuous functions $[0, \infty] \to \mathbb{C}$ with the supremum norm, and denote (for fixed $\lambda \in \mathbb{R}_+$) by e_λ the function

$$e_\lambda(t) = \frac{(1+t)\lambda}{1+t\lambda} \in C, \quad t \in [0, \infty].$$

Let two finite-dimensional subspaces V_1, V_2 be defined by

$$V_i = \operatorname{span}\{e_{\lambda_i} : \lambda_i \in \sigma(A_i)\} \subset C, \quad i = 1, 2.$$

Then (1.8) yields that the unit $1 = e_1(t) \in C$ belongs to $V_1 \cap V_2$. For fixed $\lambda_i^* \in \sigma(A_i)$, define two linear functionals

$$\phi_{\lambda_1^*} : V_2 \to \mathbb{C}, \quad \phi_{\lambda_2^*} : V_1 \to \mathbb{C}$$

by

$$\phi_{\lambda_1^*} \left(\sum_{\lambda_2 \in \sigma(A_2)} a_{\lambda_2} e_{\lambda_2} \right) = \sum_{\lambda_2 \in \sigma(A_2)} a_{\lambda_2} h_{\lambda_1^*}(\lambda_2)^{\frac{1}{2}},$$

and

$$\phi_{\lambda_2^*}\left(\sum_{\lambda_1\in\sigma(A_1)} a_{\lambda_1}e_{\lambda_1}\right) = \sum_{\lambda_1\in\sigma(A_1)} a_{\lambda_1}h_{\lambda_2^*}(\lambda_1)^{\frac{1}{2}}$$

respectively. We then have the following lemma:

Lemma 1.3. *The functional $\phi_{\lambda_1^*}$ is positive on V_2 in the sense that if $u\in V_2$ satisfies $u(t)\geq 0$ for all $t>0$, then $\phi_{\lambda_1^*}(u)\geq 0$. Similarly, $\phi_{\lambda_2^*}$ is a positive functional on V_1.*

Proof of Lemma 1.3. This follows from Lemma 1.2 and Lemma 7.1 of [1]. □

Proof of Theorem 1.1. Consider now the bilinear form

$$\phi : V_1\times V_2\to\mathbb{C}$$

defined by

$$\phi\left(\sum_{\lambda_1\in\sigma(A_1)} a_{\lambda_1}e_{\lambda_1}, \sum_{\lambda_2\in\sigma(A_2)} a_{\lambda_2}e_{\lambda_2}\right) = \sum_{(\lambda_1^*,\lambda_2^*)\in\sigma(A_1)\times\sigma(A_2)} \phi_{\lambda_1^*}\left(\sum_{\lambda_2\in\sigma(A_2)} a_{\lambda_2}e_{\lambda_2}\right)\phi_{\lambda_2^*}\left(\sum_{\lambda_1\in\sigma(A_1)} a_{\lambda_1}e_{\lambda_1}\right). \tag{1.9}$$

By Lemma 1.3, ϕ is *positive* on $V_1\times V_2$ in the sense that $u_i\in V_i$, $u_i\geq 0$ implies $\phi(u_1,u_2)\geq 0$. Hence (since the V_i's contain the function 1),

$$\|\phi\| = \sup\{|\phi(u_1,u_2)| : u_i\in V_i, \|u_i\|_\infty\leq 1, i=1,2\} = \phi(1,1). \tag{1.10}$$

Now ϕ lifts to a linear functional

$$\tilde{\phi} : V_1\otimes V_2\to\mathbb{C},$$

which is positive on $V_1\otimes V_2$, because

$$\|\tilde{\phi}\| = \|\phi\| = \phi(1,1) = \tilde{\phi}(1).$$

The Hahn–Banach theorem yields an extension $\Phi : C\otimes C = C([0,\infty]^2)\to\mathbb{C}$ of $\tilde{\phi}$ of the same norm. Thus the positivity of $\tilde{\phi}$ yields

$$\|\Phi\| = \|\tilde{\phi}\| = \tilde{\phi}(1) = \Phi(1),$$

i.e. Φ is a positive functional on $C([0,\infty]^2)$. Hence, the Riesz representation theorem provides us with a positive Radon measure ρ on $[0,\infty]^2$ such that

$$\Phi(u) = \int_{[0,\infty]^2} u(t_1,t_2)d\rho(t_1,t_2), \quad u\in C([0,\infty]^2). \tag{1.11}$$

A simple rewriting yields that (1.9) equals

$$\sum_{(\lambda_1^*,\lambda_2^*)\in\sigma(A_1)\times\sigma(A_2)}\left(a_{\lambda_1^*}a_{\lambda_2^*}h(\lambda_1^*,\lambda_2^*) + \sum_{(\lambda_1,\lambda_2)\neq(\lambda_1^*,\lambda_2^*)} a_{\lambda_1}a_{\lambda_2}h(\lambda_1^*,\lambda_2)^{\frac{1}{2}}h(\lambda_1,\lambda_2^*)^{\frac{1}{2}}\right).$$

Inserting the latter expression into (1.11) yields

$$\begin{aligned} h(\lambda_1^*,\lambda_2^*) &= \phi(\lambda_1^*,\lambda_2^*) \\ &= \Phi(e_{\lambda_1^*}\otimes e_{\lambda_2^*}) \\ &= \int_{[0,\infty]^2}\frac{(1+t_1)\lambda_1^*}{1+t_1\lambda_1^*}\frac{(1+t_2)\lambda_2^*}{1+t_2\lambda_2^*}d\rho(t_1,t_2). \end{aligned}$$

Since λ_1^*, λ_2^* are arbitrary, the theorem is proved. □

Remark 1.4. It is easy to modify the above proof to obtain a representation theorem for interpolation functions of more than two matrix variables (where the latter set of functions is interpreted in the obvious way).

2. KORÁNYI'S THEOREM

Consider the class of functions which are *monotone* according to the definition of Korányi [8] [1], $A_1 \leq A_1'$ and $A_2 \leq A_2'$ imply

$$h(A_1', A_2') - h(A_1', A_2) - h(A_1, A_2') - h(A_1, A_2) \geq 0. \tag{2.1}$$

The functions

$$h_t(\lambda) = \frac{(1+t)\lambda}{1+t\lambda}$$

are monotone of one variable $(0 \leq t \leq \infty)$, whence with $h_{t_1 t_2} = h_{t_1} \otimes h_{t_2}$ (cf. [8, p. 544]),

$$\begin{aligned} &h_{t_1 t_2}(A_1', A_2') - h_{t_1 t_2}(A_1', A_2) - h_{t_1 t_2}(A_1, A_2') - h_{t_1 t_2}(A_1, A_2) \\ &\qquad = (h_{t_1}(A_1') - h_{t_1}(A_1)) \otimes (h_{t_2}(A_2') - h_{t_2}(A_2)) \geq 0, \end{aligned}$$

i.e. $h_{t_1 t_2}$ is monotone. Since the class of monotone functions of two variables is closed under pointwise convergence, the latter inequality can be integrated, which yields that all functions of the form (1.7) are monotone. Hence we have proved the easy half of the following theorem of A. Korányi, cf. [8, Theorem 4], cf. also [9].

Theorem 2.1. *Let h be a positive function on $\mathbb{R}_+^2$. Assume that (a) the first partial derivatives and the mixed second partial derivatives of h exist and are continuous. Then h is monotone iff h is representable in the form (1.7) for some positive Radon measure ρ on $[0,\infty]^2$.*

Remark 2.2. According to Korányi the differentiability condition (a) was imposed "in order to avoid lengthy computations which are of no interest for the main course of our investigation" ([8, bottom of p. 541]).

Let us denote a function h defined on $\mathbb{R}_+^2$ an *interpolation function* if $h \in C_{A_1, A_2}$ for any positive matrices A_1, A_2. Theorem 1.1 and Theorem 2.1 then yield the following corollary, which nicely generalizes the one-variable case.

Corollary 2.3. *The set of interpolation functions coincides with the set of monotone functions satisfying (a).*

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