POSITIVE SOLUTIONS FOR SECOND-ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE ON A HALF-LINE

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Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents

44 >>>

4

Page 1 of 20

Go Back

Full Screen

Close

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Abstract:

This paper deals with the second order boundary value problem with integral boundary conditions on a half-line:

$$(p(t)x'(t))' + g(t)f(t,x(t)) = 0$$
, a.e. in $(0,\infty)$, $x(0) = \int_0^\infty x(s)g(s)ds$, $\lim_{t \to \infty} p(t)x'(t) = p(0)x'(0)$.

A new result on the existence of positive solutions is obtained. The interesting points are: firstly, the boundary value problem involved in the integral boundary condition on unbounded domains; secondly, we employ a new tool - the recent Leggett-Williams norm-type theorem for coincidences and obtain positive solutions. Also, an example is constructed to illustrate that our result here is valid.



Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



44





>>

Page 2 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Contents

1 Introduction	4
----------------	---

- 2 Related Lemmas 6
- 3 Main Result



Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge

vol. 10, iss. 1, art. 9, 2009

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

1. Introduction

In this paper, we study the existence of positive solutions to the following boundary value problem at resonance:

$$(1.1) (p(t)x'(t))' + g(t)f(t,x(t)) = 0, a.e.in (0,\infty),$$

(1.2)
$$x(0) = \int_0^\infty x(s)g(s)ds, \quad \lim_{t \to \infty} p(t)x'(t) = p(0)x'(0),$$

where $g \in L^1[0,\infty)$ with g(t) > 0 on $[0,\infty)$ and $\int_0^\infty g(s)ds = 1, \ p \in C[0,\infty) \cap C^1(0,\infty), \frac{1}{p} \in L^1[0,\infty), \int_0^\infty \frac{1}{p(t)}dt \leq 1 \text{ and } p(t) > 0 \text{ on } [0,\infty).$

Second-order boundary value problems (in short: BVPs) on infinite intervals, arising from the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium [10], have received much attention, to identify a few, we refer the readers to [9] – [11] and references therein. For example, in [9], Lian and Ge studied the following second-order BVPs on a half-line

(1.3)
$$x''(t) = f(t, x(t), x'(t)), \qquad 0 < t < \infty,$$

(1.4)
$$x(0) = x(\eta), \quad \lim_{t \to \infty} x'(t) = 0$$

and

$$(1.5) x''(t) = f(t, x(t), x'(t)) + e(t), 0 < t < \infty,$$

(1.6)
$$x(0) = x(\eta), \quad \lim_{t \to \infty} x'(t) = 0,$$

By using Mawhin's continuity theorem, they obtained the existence results.

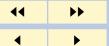


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 4 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

N. Kosmanov in [11] considered the second-order nonlinear differential equation at resonance

$$(1.7) (p(t)u'(t))' = f(t, u(t), u'(t)), a.e. in (0, \infty)$$

with two sets of boundary conditions:

(1.8)
$$u'(0) = 0, \quad \sum_{i=1}^{n} \kappa_i u_i(T_i) = \lim_{t \to \infty} u(t)$$

and

(1.9)
$$u(0) = 0, \quad \sum_{i=1}^{n} \kappa_i u_i(T_i) = \lim_{t \to \infty} u(t).$$

The author established existence theorems by the coincidence degree theorem of Mawhin under the condition that $\sum_{i=1}^{n} \kappa_i = 1$.

Although the existing literature on solutions of BVPs is quite wide, to the best of our knowledge, only a few papers deal with the existence of positive solutions to BVPs at resonance. In particular, there has been no work done for the boundary value problems with integral boundary conditions on a half-line, such as the BVP (1.1) – (1.2). Moreover, our main approach is different from the existing ones and our main ingredient is the Leggett-Williams norm-type theorem for coincidences obtained by O'Regan and Zima [4], which is a new tool used to study the existence of positive solutions for nonlocal BVPs at resonance. An example is constructed to illustrate that our result here is valid and almost sharp.



Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge

vol. 10, iss. 1, art. 9, 2009

Title Page

Contents





Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

2. Related Lemmas

For the convenience of the reader, we review some standard facts on Fredholm operators and cones in Banach spaces. Let X,Y be real Banach spaces. Consider a linear mapping $L: \operatorname{dom} L \subset X \to Y$ and a nonlinear operator $N: X \to Y$. Assume that $1^{\circ} L$ is a Fredholm operator of index zero, i.e., $\operatorname{Im} L$ is closed and $\operatorname{dim} \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

The assumption 1° implies that there exist continuous projections $P: X \to X$ and $Q: Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Moreover, since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J: \operatorname{Im} Q \to \operatorname{Ker} L$. Denote by L_p the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L$. Clearly, L_p is an isomorphism from $\operatorname{Ker} P \cap \operatorname{dom} L$ to $\operatorname{Im} L$, we denote its inverse by $K_p: \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$. It is known (see [3]) that the coincidence equation L = Nx is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Let C be a cone in X such that

- (i) $\mu x \in C$ for all $x \in C$ and $\mu \ge 0$,
- (ii) $x, -x \in C$ implies $x = \theta$.

It is well known that C induces a partial order in X by

$$x \leq y$$
 if and only if $y - x \in C$.

The following property is valid for every cone in a Banach space X.

Lemma 2.1 ([7]). Let C be a cone in X. Then for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that

$$||x+u|| \ge \sigma(u)||x||$$
 for all $x \in C$.

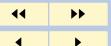


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 6 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Let $\gamma: X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I - Q)N$$
 and $\Psi_{\gamma} := \Psi \circ \gamma$.

In order to prove the existence result, we present here a definition.

Definition 2.2. $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is called a g-Carathéodory function if

- (A1) for each $u \in \mathbb{R}$, the mapping $t \mapsto f(t, u)$ is Lebesgue measurable on $[0, \infty)$,
- (A2) for a.e. $t \in [0, \infty)$, the mapping $u \mapsto f(t, u)$ is continuous on \mathbb{R} ,
- (A3) for each l>0 and $g\in L^1[0,\infty)$, there exists $\alpha_l:[0,\infty)\to [0,\infty)$ satisfying $\int_0^\infty g(s)\alpha_l(s)ds<\infty$ such that

$$|u| \le l$$
 implies $|f(t, u)| \le \alpha_l(t)$ for a.e. $t \in [0, \infty)$.

We make use of the following result due to O'Regan and Zima.

Theorem 2.3 ([4]). Let C be a cone in X and let Ω_1 , Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that 1° and the following conditions hold.

- 2° N is L-compact, that is, $QN: X \to Y$ is continuous and bounded and $K_p(I-Q)N: X \to X$ is compact on every bounded subset of X,
- $3^{\circ} Lx \neq \lambda Nx \text{ for all } x \in C \cap \partial \Omega_2 \cap \operatorname{Im} L \text{ and } \lambda \in (0,1),$
- $4^{\circ} \ \gamma$ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,
- $5^{\circ} \operatorname{deg}_{B}\{[I-(P+JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_{2}, 0\} \neq 0, \text{ where } \operatorname{deg}_{B} \text{ denotes the } Brouwer \, degree,$



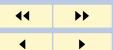
Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge

vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 7 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

6° there exists $u_0 \in C \setminus \{0\}$ such that $||x|| \leq \sigma(u_0)||\Psi x||$ for $x \in C(u_0) \cap \partial \Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ such that $||x + u_0|| \geq \sigma(u_0)||x||$ for every $x \in C$,

$$7^{\circ} (P + JQN)\gamma(\partial\Omega_2) \subset C$$
,

8° $\Psi_{\gamma}(\overline{\Omega}_2 \setminus \Omega_1) \subset C$. Then the equation Lx = Nx has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For simplicity of notation, we set

(2.1)
$$\omega := \int_0^\infty \left(\int_0^s \frac{1}{p(\tau)} d\tau \right) g(s) ds$$

and

$$G(t,s) = \begin{cases} \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \left[\int_s^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr d\tau \right. \\ - \int_0^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr \int_0^\tau g(r) dr d\tau \right] \\ + 1 + \int_0^t \frac{1}{p(\tau)} \int_0^\tau g(r) dr d\tau - \int_s^t \frac{1}{p(\tau)} d\tau, \quad 0 \le s < t < \infty, \\ \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \left[\int_s^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr d\tau \right. \\ - \int_0^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr \int_0^\tau g(r) dr d\tau \right] \\ + 1 + \int_0^t \frac{1}{p(\tau)} \int_0^\tau g(r) dr d\tau, \qquad 0 \le t \le s < \infty. \end{cases}$$

Note that $G(t,s) \ge 0$ for $t,s \in [0,1]$, and set

(2.2)
$$0 < \kappa \le \min \left\{ 1, \frac{1}{\sup\limits_{t,s \in [0,\infty)} G(t,s)} \right\}.$$



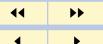
Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge

vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 8 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

3. Main Result

We work in the Banach spaces

(3.1)
$$X = \left\{ x \in C[0, \infty) : \lim_{t \to \infty} x(t) \text{ exists} \right\}$$

and

(3.2)
$$Y = \left\{ y : [0, \infty) \to \mathbb{R} : \int_0^\infty g(t)|y(t)|dt < \infty \right\}$$

with the norms $||x||_X = \sup_{t \in [0,\infty)} |x(t)|$ and $||y||_Y = \int_0^\infty g(t)|y(t)|dt$, respectively.

Define the linear operator $L: \mathrm{dom}\, L \subset X \to Y$ and the nonlinear operator $N: X \to Y$ with

$$(3.3) \quad \operatorname{dom} L = \left\{ x \in X : \lim_{t \to \infty} p(t) x'(t) \quad \text{exists,} \quad x, px' \in AC[0, \infty) \right.$$

$$\text{and} \quad gx, (px')' \in L^1[0, \infty), x(0) = \int_0^\infty x(s) g(s) ds$$

$$\text{and} \quad \lim_{t \to \infty} p(t) x'(t) = p(0) x'(0) \right\}$$

by
$$Lx(t)=-\frac{1}{g(t)}(p(t)x'(t))'$$
 and $Nx(t)=f(t,x(t)), t\in [0,\infty)$, respectively. Then
$$\operatorname{Ker} L=\{x\in \operatorname{dom} L: x(t)\equiv c \quad \text{on } [0,\infty)\}$$

and

$$\operatorname{Im} L = \left\{ y \in Y : \int_0^\infty g(s)y(s)ds = 0 \right\}.$$

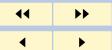


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 9 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Next, define the projections $P:X\to X$ by $(Px)(t)=\int_0^\infty g(s)x(s)ds$ and $Q:Y\to Y$ by

$$(Qy)(t) = \int_0^\infty g(s)y(s)ds.$$

Clearly, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. So $\dim \operatorname{Ker} L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$. Notice that $\operatorname{Im} L$ is closed, L is a Fredholm operator of index zero.

Note that the inverse $K_p: \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of L_p is given by

$$(K_p y)(t) = \int_0^\infty k(t, s)g(s)y(s)ds,$$

where

$$(3.4) \quad k(t,s) := \begin{cases} \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau - \int_s^t \frac{1}{p(\tau)} d\tau, & 0 \le s < t < \infty, \\ \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau, & 0 \le t \le s < \infty. \end{cases}$$

It is easy to see that $|k(t,s)| \leq 2 \int_0^\infty \frac{1}{p(s)} ds$.

In order to apply Theorem 2.3, we have to prove that N is L-compact, that is, QN is continuous and bounded and $K_p(I-Q)N$ is compact on every bounded subset of X. Since the Arzelà-Ascoli theorem fails in the noncompact interval case, we will use the following criterion.

Theorem 3.1 ([10]). Let $M \subset \left\{x \in C[0,\infty) : \lim_{t \to \infty} x(t) \text{ exists}\right\}$. Then M is relatively compact if the following conditions hold:

- (B1) all functions from M are uniformly bounded,
- (B2) all functions from M are equicontinuous on any compact interval of $[0, \infty)$,

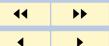


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 10 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

(B3) all functions from M are equiconvergent at infinity, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $|f(t) - f(\infty)| < \varepsilon$ for all t > T and $f \in M$.

Lemma 3.2. If $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a g-Carathéodory function, then N is L-compact.

Proof. Suppose that $\Omega \subset X$ is a bounded set. Then there exists l>0 such that $||x||_X \leq l$ for $x \in \overline{\Omega}$. Since f is a g-Carathéodory function, there exists $\alpha_l \in L^1[0,\infty)$ satisfying $\alpha_l(t)>0$, $t\in (0,\infty)$ and $\int_0^\infty g(s)\alpha_l(s)ds < \infty$ such that for a.e. $t\in [0,\infty)$, $|f(t,x(t))|\leq \alpha_l(t)$ for $x\in \overline{\Omega}$. Then for $x\in \overline{\Omega}$,

$$||QNx||_Y = \int_0^\infty g(t) \left| \int_0^\infty g(s)f(s,x(s))ds \right| dt \le \int_0^\infty g(s)\alpha_l(s)ds < \infty,$$

which implies that QN is bounded on $\overline{\Omega}$.

Next, we show that $K_p(I-Q)N$ is compact, i.e., $K_p(I-Q)N$ maps bounded sets into relatively compact ones. Furthermore, denote $K_{P,Q}=K_P(I-Q)N$ (see [9], [11]). For $x\in\overline{\Omega}$, one gets

$$|(K_{P,Q}x)(t)| \leq \int_0^\infty \left| k(t,s)g(s) \left[f(s,x(s)) - \int_0^\infty g(\tau)f(\tau,x(\tau))d\tau \right] \right| ds$$

$$\leq 2 \int_0^\infty \frac{1}{p(\tau)} d\tau \left[\int_0^\infty g(s)|f(s,x(s))|ds$$

$$+ \int_0^\infty g(s) \int_0^\infty g(\tau)|f(\tau,x(\tau))|d\tau ds \right]$$

$$\leq 4 \int_0^\infty \frac{1}{p(\tau)} d\tau \int_0^\infty g(s)\alpha_l(s)ds < \infty,$$



Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge

vol. 10, iss. 1, art. 9, 2009

Title Page

Contents





Page 11 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

that is, $K_{P,Q}(\overline{\Omega})$ is uniformly bounded. Meanwhile, for any $t_1, t_2 \in [0, T]$ with T a positive constant,

$$\begin{split} &|(K_{P,Q}x)(t_1)-(K_{P,Q}x)(t_2)|\\ &=\left|\frac{1}{\omega}\int_0^\infty\int_s^\infty\int_s^\tau\frac{1}{p(r)}drg(\tau)d\tau g(s)\right[f(s,x(s))\\ &-\int_0^\infty g(\tau)f(\tau,x(\tau))d\tau\right]ds\int_{t_2}^{t_1}\frac{1}{p(\tau)}d\tau\\ &-\left\{\int_0^{t_1}\int_s^{t_1}\frac{1}{p(\tau)}d\tau g(s)\left[f(s,x(s))-\int_0^\infty g(\tau)f(\tau,x(\tau))d\tau\right]ds\right.\\ &-\int_0^{t_2}\int_s^{t_2}\frac{1}{p(\tau)}d\tau g(s)\left[f(s,x(s))-\int_0^\infty g(\tau)f(\tau,x(\tau))d\tau\right]ds\right\}\Big|\\ &\leq\frac{1}{\omega}\left[\int_0^\infty g(s)\int_0^s\frac{1}{p(\tau)}\int_0^\tau g(r)|f(r,x(r))|drd\tau ds+\int_0^\infty g(\tau)|f(\tau,x(\tau))|d\tau\\ &\cdot\int_0^\infty g(s)\int_0^s\frac{1}{p(\tau)}\int_0^\tau g(r)drd\tau ds\right]\cdot\left|\int_{t_2}^{t_1}\frac{1}{p(\tau)}d\tau\right|\\ &+\left|\int_{t_2}^{t_1}\frac{1}{p(s)}\left[\int_0^s g(\tau)|f(\tau,x(\tau))|d\tau+\int_0^s g(\tau)\int_0^\infty g(r)|f(r,x(\tau))|drd\tau\right]ds\right|\\ &\leq\left[\int_0^\infty g(r)|f(r,x(r))|dr+\int_0^\infty g(\tau)|f(\tau,x(\tau))|d\tau\cdot\int_0^\infty g(r)dr\right]\left|\int_{t_2}^{t_1}\frac{1}{p(\tau)}d\tau\right|\\ &+2\int_0^\infty g(s)\alpha_l(s)ds\cdot\left|\int_s^{t_1}\frac{1}{p(\tau)}d\tau\right|-0,\qquad \text{uniformly as }|t_1-t_2|\to 0, \end{split}$$

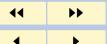


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 12 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

which means that $K_{P,Q}(\overline{\Omega})$ is equicontinuous. In addition, we claim that $K_{P,Q}(\overline{\Omega})$ is equiconvergent at infinity. In fact,

$$\begin{split} &|(K_{P,Q}x)(\infty) - (K_{P,Q}x)(t)| \\ &\leq \frac{1}{\omega} \int_0^\infty \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau g(s) \bigg[|f(s,x(s))| + \int_0^\infty g(\tau) |f(\tau,x(\tau))| d\tau \bigg] ds \cdot \int_t^\infty \frac{1}{p(\tau)} d\tau \\ &+ \int_t^\infty \frac{1}{p(s)} ds \left[\int_0^s g(\tau) |f(\tau,x(\tau)| d\tau + \int_0^s g(\tau) \int_0^\infty g(r) |f(r,x(r))| dr d\tau \right] ds \\ &\leq 4 \int_0^\infty g(s) \alpha_l(s) ds \cdot \int_t^\infty \frac{1}{p(\tau)} d\tau \to 0, \quad \text{uniformly as } t \to \infty. \end{split}$$

Hence, Theorem 3.1 implies that $K_p(I-Q)N(\overline{\Omega})$ is relatively compact. Furthermore, since f satisfies g-Carathéodory conditions, the continuity of QN and $K_p(I-Q)N$ on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem. This completes the proof.

Now, we state our main result on the existence of positive solutions for the BVP (1.1) - (1.2).

Theorem 3.3. Assume that

- (H1) $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a g-Carathéodory function,
- (H2) there exist positive constants $b_1, b_2, b_3, c_1, c_2, B$ with

(3.5)
$$B > \frac{c_2}{c_1} + 2\left(\frac{b_2c_2}{b_1c_1} + \frac{b_3}{b_1}\right) \int_0^\infty \frac{1}{p(s)} ds$$



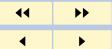
Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge

vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 13 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

such that

$$-\kappa x \le f(t, x),$$

 $f(t, x) \le -c_1 x + c_2,$
 $f(t, x) \le -b_1 |f(t, x)| + b_2 x + b_3$

for $t \in [0, \infty)$, $x \in [0, B]$,

(H3) there exist $b \in (0, B)$, $t_0 \in [0, \infty)$, $\rho \in (0, 1]$ and $\delta \in (0, 1)$. For each $t \in [0, \infty)$, $\frac{f(t, x)}{x^{\rho}}$ is non-increasing on $x \in (0, b]$ with

(3.6)
$$\int_0^\infty G(t_0, s)g(s)\frac{f(s, b)}{b}ds \ge \frac{1 - \delta}{\delta^{\rho}}.$$

Then the BVP (1.1) – (1.2) has at least one positive solution on $[0, \infty)$.

Proof. Consider the cone

$$C = \{x \in X : x(t) \ge 0 \text{ on } [0, \infty)\}.$$

Let

$$\Omega_1 = \{ x \in X : \delta ||x||_X < |x(t)| < b \text{ on } [0, \infty) \}$$

and

$$\Omega_2 = \{ x \in X : ||x||_X < B \}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\overline{\Omega}_1 = \{x \in X : \delta ||x||_X \le |x(t)| \le b \quad \text{on} \quad [0, \infty)\} \subset \Omega_2$$

(see [4]). Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let J = I and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that 4° holds.

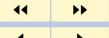


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 14 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

In order to prove 3°, suppose that there exist $x_0 \in \partial\Omega_2 \cap C \cap \text{dom } L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 Nx_0$, then $(p(t)x_0'(t))' + \lambda_0 g(t)f(t,x_0(t)) = 0$ for all $t \in [0,\infty)$. In view of (H2), we have

$$-\frac{1}{\lambda_0 g(t)} (p(t)x_0'(t))' = f(t, x_0(t)) \le -b_1 \frac{1}{\lambda_0 g(t)} |(p(t)x_0'(t))'| + b_2 x_0(t) + b_3.$$

Hence,

$$(3.7) -(p(t)x_0'(t))' \le -b_1|(p(t)x_0'(t))'| + \lambda_0 b_2 g(t)x_0(t) + \lambda_0 b_3 g(t).$$

Integrating both sides of (3.7) from 0 to ∞ , one gets

$$0 = -\int_0^\infty (p(t)x_0'(t))'dt$$

$$\leq -b_1 \int_0^\infty |(p(t)x_0'(t))'|dt + \lambda_0 b_2 \int_0^\infty g(t)x_0(t)dt + \lambda_0 b_3 \int_0^\infty g(t)dt,$$

which gives

(3.8)
$$\int_0^\infty |(p(t)x_0'(t))'|dt < \frac{b_2}{b_1} \int_0^\infty g(t)x_0(t)dt + \frac{b_3}{b_1}.$$

Similarly, from (H2), we also obtain

$$(3.9) \qquad \int_0^\infty g(t)x_0(t)dt \le \frac{c_2}{c_1}.$$

On the other hand,

(3.10)
$$x_0(t) = \int_0^\infty g(t)x_0(t)dt + \int_0^\infty k(t,s)(p(s)x_0'(s))'ds$$

$$\leq \int_0^\infty g(t)x_0(t)dt + \int_0^\infty |k(t,s)| \cdot |(p(s)x_0'(s))'|ds.$$



Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Contents

Contents

Page 15 of 20

Full Screen

Go Back

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Then, (3.8)-(3.9) yield

$$B = ||x_0||_X \le \frac{c_2}{c_1} + 2\left(\frac{b_2c_2}{b_1c_1} + \frac{b_3}{b_1}\right) \int_0^\infty \frac{1}{p(s)} ds,$$

which contradicts (3.5).

To prove 5°, consider $x \in \operatorname{Ker} L \cap \overline{\Omega}_2$. Then $x(t) \equiv c$ on $[0, \infty)$. Let

$$H(c,\lambda) = c - \lambda|c| - \lambda \int_0^\infty g(s)f(s,|c|)ds$$

for $c \in [-B, B]$ and $\lambda \in [0, 1]$. It is easy to show that $0 = H(c, \lambda)$ implies $c \ge 0$. Suppose $0 = H(B, \lambda)$ for some $\lambda \in (0, 1]$. Then, (3.5) leads to

$$0 \le B(1-\lambda) = \lambda \int_0^\infty g(s)f(s,B)ds \le \lambda(-c_1B + c_2) < 0,$$

which is a contradiction. In addition, if $\lambda=0$, then B=0, which is impossible. Thus, $H(x,\lambda)\neq 0$ for $x\in \operatorname{Ker} L\cap\partial\Omega_2$ and $\lambda\in [0,1]$. As a result,

$$\deg_B\{H(\cdot,1),\operatorname{Ker} L\cap\Omega_2,0\}=\deg_B\{H(\cdot,0),\operatorname{Ker} L\cap\Omega_2,0\}.$$

However,

$$\deg_B\{H(\cdot,0),\operatorname{Ker} L\cap\Omega_2,0\}=\deg_B\{I,\operatorname{Ker} L\cap\Omega_2,0\}=1.$$

Then,

$$\deg_B\{[I-(P+JQN)\gamma]_{\operatorname{Ker} L}, \operatorname{Ker} L\cap\Omega_2, 0\} = \deg_B\{H(\cdot,1), \operatorname{Ker} L\cap\Omega_2, 0\} \neq 0.$$

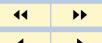


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 16 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Next, we prove 8° . Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0, \infty)$,

$$(\Psi_{\gamma}x)(t) = \int_0^\infty g(s)|x(s)|ds + \int_0^\infty g(s)f(s,|x(s)|)ds$$

$$+ \int_0^\infty k(t,s)g(s) \left[f(s,|x(s)|) - \int_0^\infty g(\tau)f(\tau,|x(\tau)|)d\tau \right] ds$$

$$= \int_0^\infty g(s)|x(s)|ds + \int_0^\infty G(t,s)g(s)f(s,|x(s)|)ds$$

$$\geq \int_0^\infty (1 - \kappa G(t,s))g(s)|x(s)|ds \geq 0.$$

Hence, $\Psi_{\gamma}(\overline{\Omega}_2 \setminus \Omega_1) \subset C$, i.e. 8° holds.

Since for $x \in \partial \Omega_2$,

$$(P+JQN)\gamma x = \int_0^\infty g(s)|x(s)|ds + \int_0^\infty g(s)f(s,|x(s)|)ds$$
$$\geq \int_0^\infty (1-\kappa)g(s)|x(s)|ds \geq 0,$$

then, $(P + JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$, and 7° holds.

It remains to verify 6°. Let $u_0(t) \equiv 1$ on $[0, \infty)$. Then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0, \infty)\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial \Omega_1$. Then x(t) > 0 on $[0, \infty)$, $0 < ||x||_X \le b$ and $x(t) \ge \delta ||x||_X$ on $[0, \infty)$. For every $x \in C(u_0) \cap \partial \Omega_1$, by (H3)

$$(\Psi x)(t_0) = \int_0^\infty g(s)x(s)ds + \int_0^\infty G(t_0, s)g(s)f(s, x(s))ds$$

$$\geq \delta ||x||_X + \int_0^\infty G(t_0, s)g(s)\frac{f(s, x(s))}{x^{\rho}(s)}x^{\rho}(s)ds$$

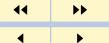


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 17 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

$$\geq \delta ||x||_{X} + \delta^{\rho} ||x||_{X}^{\rho} \int_{0}^{\infty} G(t_{0}, s) g(s) \frac{f(s, b)}{b^{\rho}} ds$$

$$= \delta ||x||_{X} + \delta^{\rho} ||x||_{X} \frac{b^{1-\rho}}{||x||_{X}^{1-\rho}} \int_{0}^{\infty} G(t_{0}, s) g(s) \frac{f(s, b)}{b} ds$$

$$\geq \delta ||x||_{X} + \delta^{\rho} ||x||_{X} \int_{0}^{\infty} G(t_{0}, s) g(s) \frac{f(s, b)}{b} ds \geq ||x||_{X}.$$

Thus, $||x||_X \leq \sigma(u_0)||\Psi x||_X$ for all $x \in C(u_0) \cap \partial \Omega_1$.

In addition, 1° holds and Lemma 3.2 yields 2°. Then, by Theorem 2.3, the BVP (1.1)-(1.2) has at least one positive solution x^* on $[0,\infty)$ with $b \leq ||x^*||_X \leq B$. This completes the proof of Theorem 3.3.

Remark 1. Note that with the projection P(x) = x(0), Conditions 7° and 8° of Theorem 2.3 are no longer satisfied.

To illustrate how our main result can be used in practice, we present here an example.

Example 3.1. Consider the following BVP

(3.12)
$$\begin{cases} 2(e^t x'(t))' + e^{-t} f(t, x(t)) = 0, & \text{a.e. in } (0, \infty), \\ x'(0) = \lim_{t \to \infty} e^t x'(t), & x(0) = \int_0^\infty e^{-s} x(s) ds. \end{cases}$$

Corresponding to the BVP (1.1) – (1.2), $p(t)=2e^t$, $g(t)=e^{-t}$ and $f(t,x)=(t-\frac{1}{2})e^{-2t}x+e^{-t}x^2$. We can get $\omega=\frac{1}{4}$ and

(3.13)
$$G(t,s)$$

$$=\begin{cases} \frac{13}{12} + \frac{1}{6}(e^{-t} - 3e^{-s}) + \frac{1}{4}(e^{-2t} + 2e^{-2s}) - \frac{1}{2}e^{-(t+2s)}, & 0 \le s \le t < \infty, \\ \frac{13}{12} - \frac{1}{3}e^{-t} + \frac{1}{4}(e^{-2t} + 2e^{-2s}) - \frac{1}{2}e^{-(t+2s)}, & 0 \le t \le s < \infty. \end{cases}$$

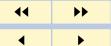


Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 18 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

Obviously, $G(t,s) \geq 0$ for $t,s \in [0,+\infty)$. Choose $\kappa = \frac{1}{2}$, B = 5, $c_1 = \frac{2}{5}$, $c_2 = \frac{1}{2}e^{-\frac{3}{2}}$, $b_1 = \frac{1}{2}$, $b_2 = \frac{3}{2}$ and $b_3 = \frac{3}{2}e^{-\frac{3}{2}}$ such that (H2) holds, and take $b = \frac{5}{4}$, $t_0 = 0$, $\rho = 1$ and $\delta = \frac{4}{9}$ such that (H3) is satisfied. Then thanks to Theorem 3.3, the BVP (3.12) has a positive solution on $[0,\infty)$.



Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756

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Positive Solutions for BVP at Resonance on a Half-Line

Aijun Yang and Weigao Ge vol. 10, iss. 1, art. 9, 2009

Title Page

Contents



Page 20 of 20

Go Back

Full Screen

Close

journal of inequalities in pure and applied mathematics

issn: 1443-5756