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REVERSE CONVOLUTION INEQUALITIES AND APPLICATIONS TO INVERSE HEAT SOURCE PROBLEMS

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ABSTRACT. We introduce reverse convolution inequalities obtained recently and at the same time, we give new type reverse convolution inequalities and their important applications to inverse source problems. We consider the inverse problem of determining $f(t)$, $0 < t < T$, in the heat source of the heat equation $\partial_t u(x,t) = \Delta u(x,t) + f(t)\varphi(x)$, $x \in \mathbb{R}^n$, $t > 0$ from the observation $u(x_0, t)$, $0 < t < T$, at a remote point x_0 away from the support of φ. Under an a priori assumption that f changes the signs at most N-times, we give a conditional stability of Hölder type, as an example of applications.

Key words and phrases: Convolution, Heat source, Weighted convolution inequalities, Young's inequality, Hölder's inequality, Reverse Hölder's inequality, Green's function, Stability in inverse problems, Volterra's equation, Conditional stability of Hölder type, Analytic semigroup, Interpolation inequality, Sobolev inequality.

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1. INTRODUCTION

For the Fourier convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi,$$

the Young's inequality

$$\|f * g\|_r \le \|f\|_p \|g\|_q, \quad f \in L_p(\mathbb{R}), g \in L_q(\mathbb{R}), \tag{1.1}$$
$$r^{-1} = p^{-1} + q^{-1} - 1 \quad (p, q, r > 0),$$

is fundamental. Note, however, that for the typical case of $f, g \in L_2(\mathbb{R})$, the inequality (1.1) does not hold. In a series of papers [12] – [16] (see also [5]) we obtained the following weighted $L_p\,(p > 1)$ norm inequality for convolution

Proposition 1.1. *(*[15]*). For two nonvanishing functions $\rho_j \in L_1(\mathbb{R})\,(j = 1, 2)$, the $L_p\,(p > 1)$ weighted convolution inequality*

$$\left\| \left((F_1\rho_1) * (F_2\rho_2) \right) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \le \|F_1\|_{L_p(\mathbb{R}, |\rho_1|)} \|F_2\|_{L_p(\mathbb{R}, |\rho_2|)} \tag{1.2}$$

holds for $F_j \in L_p(\mathbb{R}, |\rho_j|)\,(j = 1, 2)$. Equality holds here if and only if

$$F_j(x) = C_j e^{\alpha x}, \tag{1.3}$$

where α is a constant such that $e^{\alpha x} \in L_p(\mathbb{R}, |\rho_j|)\,(j = 1, 2)$. Here

$$\|F\|_{L_p(\mathbb{R}, |\rho|)} = \left\{ \int_{-\infty}^{\infty} |F(x)|^p |\rho(x)|\, dx \right\}^{\frac{1}{p}}.$$

Unlike the Young's inequality, inequality (1.2) holds also in case $p = 2$.

Note that the proof of Proposition 1.1 is direct and fairly elementary. Indeed, we use only Hölder's inequality and Fubini's theorem for exchanging the orders of integrals for the proof. So, for various type convolutions, we can also obtain similar type convolution inequalities, see [17] for various convolutions.

In many cases of interest, the convolution is given in the form

$$\rho_2(x) \equiv 1, \quad F_2(x) = G(x), \tag{1.4}$$

where $G(x - \xi)$ is some Green's function. Then inequality (1.2) takes the form

$$\| (F\rho) * G\|_p \le \|\rho\|_p^{1-\frac{1}{p}} \|G\|_p \|F\|_{L_p(\mathbb{R}, |\rho|)}, \tag{1.5}$$

where ρ, F, and G are such that the right hand side of (1.5) is finite.

Inequality (1.5) enables us to estimate the output function

$$\int_{-\infty}^{\infty} F(\xi)\rho(\xi) G(x - \xi)\, d\xi \tag{1.6}$$

in terms of the input function F in the related differential equation. We are also interested in the reverse type inequality for (1.5), namely, we wish to estimate the input function F by means of the output (1.6). This kind of estimates is important in inverse problems. One estimate is obtained by using the following famous reverse Hölder inequality

Proposition 1.2. *(*[18]*, see also* [10, p. 125–126]*). For two positive functions f and g satisfying*

$$0 < m \le \frac{f}{g} \le M < \infty \tag{1.7}$$

on the set X, and for $p,q>1,\ p^{-1}+q^{-1}=1$,

$$\left(\int_X f d\mu\right)^{\frac{1}{p}}\left(\int_X g d\mu\right)^{\frac{1}{q}} \le A_{p,q}\left(\frac{m}{M}\right)\int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{1.8}$$

if the right hand side integral converges. Here

$$A_{p,q}(t)=p^{-\frac{1}{p}}q^{-\frac{1}{q}}\frac{t^{-\frac{1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}}\left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

Then, by using Proposition 1.2 we obtain, as in the proof of Proposition 1.1, the following

Proposition 1.3. *(*[16]*). Let F_1 and F_2 be positive functions satisfying*

$$0<m_1^{\frac{1}{p}}\le F_1(x)\le M_1^{\frac{1}{p}}<\infty,\quad 0<m_2^{\frac{1}{p}}\le F_2(x)\le M_2^{\frac{1}{p}}<\infty,\quad p>1,\quad x\in\mathbb{R}. \tag{1.9}$$

Then for any positive continuous functions ρ_1 and ρ_2, we have the reverse L_p–weighted convolution inequality

$$\left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-1}\|F_1\|_{L_p(R,\rho_1)}\|F_2\|_{L_p(R,\rho_2)}\le\left\|\left((F_1\rho_1)*(F_2\rho_2)\right)(\rho_1*\rho_2)^{\frac{1}{p}-1}\right\|_p. \tag{1.10}$$

Inequality (1.10) should be understood in the sense that if the right hand side is finite, then so is the left hand side, and in this case the inequality holds.

In formula (1.10) replacing ρ_2 by 1, and $F_2(x-\xi)$ by $G(x-\xi)$, and integrating with respect to x from c to d we arrive at the following inequality

$$\begin{aligned}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p}\left(\int_{-\infty}^{\infty}\rho(\xi)\,d\xi\right)^{p-1}\int_{-\infty}^{\infty}F^p(\xi)\,\rho(\xi)\,d\xi\int_{c-\xi}^{d-\xi}G^p(x)dx\\ \le\int_c^d\left(\int_{-\infty}^{\infty}F(\xi)\,\rho(\xi)\,G(x-\xi)\,d\xi\right)^p dx,\end{aligned} \tag{1.11}$$

if positive continuous functions ρ, F, and G satisfy

$$0<m^{\frac{1}{p}}\le F(\xi)G(x-\xi)\le M^{\frac{1}{p}},\quad x\in[c,d],\quad \xi\in\mathbb{R}. \tag{1.12}$$

Inequality (1.11) is especially important when $G(x-\xi)$ is a Green's function. We gave various concrete examples in [16] from the viewpoint of stability in inverse problems.

2. REMARKS FOR REVERSE HÖLDER INEQUALITIES

In connection with Proposition 1.2 which gives Proposition 1.3, Izumino and Tominaga [8] consider the upper bound of

$$\left(\sum a_k^p\right)^{\frac{1}{p}}\left(\sum b_k^q\right)^{\frac{1}{q}}-\lambda\sum a_kb_k$$

for $\lambda>0$, for $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$ and for positive numbers $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$, in detail. In their different approach, they showed that the constant $A_{p,q}(t)$ in Proposition 1.2 is best possible in a sense. Note that the proof of Proposition 1.2 is quite involved. In connection with Proposition 1.2 we note that the following version whose proof is surprisingly simple

Theorem 2.1. *In Proposition 1.2, replacing f and g by f^p and g^q, respectively, we obtain the reverse Hölder type inequality*

$$\left(\int_X f^p d\mu\right)^{\frac{1}{p}}\left(\int_X g^q d\mu\right)^{\frac{1}{q}}\le\left(\frac{m}{M}\right)^{-\frac{1}{pq}}\int_X fg d\mu. \tag{2.1}$$

Proof. Since $\frac{f^p}{g^q} \le M$, $g \ge M^{-\frac{1}{q}} f^{\frac{p}{q}}$, therefore

$$fg \ge M^{-\frac{1}{q}} f^{1+\frac{p}{q}} = M^{-\frac{1}{q}} f^p$$

and so,

$$\left\{ \int f^p d\mu \right\}^{\frac{1}{p}} \le M^{\frac{1}{pq}} \left\{ \int fg d\mu \right\}^{\frac{1}{p}}. \tag{2.2}$$

On the other hand, since $m \le \frac{f^p}{g^q}$, $f \ge m^{\frac{1}{p}} g^{\frac{q}{p}}$, hence

$$\int fg d\mu \ge \int m^{\frac{1}{p}} g^{1+\frac{q}{p}} d\mu = m^{\frac{1}{p}} \int g^q d\mu,$$

and so,

$$\left\{ \int fg d\mu \right\}^{\frac{1}{q}} \ge m^{\frac{1}{pq}} \left\{ \int g^q d\mu \right\}^{\frac{1}{q}}.$$

Combining with (2.2), we have the desired inequality

$$\begin{aligned} \left\{ \int f^p d\mu \right\}^{\frac{1}{p}} \left\{ \int g^q d\mu \right\}^{\frac{1}{q}} &\le M^{\frac{1}{pq}} \left\{ \int fg d\mu \right\}^{\frac{1}{p}} m^{\frac{-1}{pq}} \left\{ \int fg d\mu \right\}^{\frac{1}{q}} \\ &= \left(\frac{m}{M} \right)^{\frac{-1}{pq}} \int fg d\mu. \end{aligned}$$

□

3. NEW REVERSE CONVOLUTION INEQUALITIES

In reverse convolution inequality (1.10), similar type inequalities for $m_1 = m_2 = 0$ are also important as we see from our example in Section 4. For these, we obtain a new reverse convolution inequality.

Theorem 3.1. *Let* $p \ge 1, \delta > 0, 0 \le \alpha < T$, *and* $f, g \in L_\infty(0,T)$ *satisfy*

$$0 \le f, g \le M < \infty, \qquad 0 < t < T. \tag{3.1}$$

Then

$$\|f\|_{L_p(\alpha,T)} \|g\|_{L_p(0,\delta)} \le M^{\frac{2p-2}{p}} \left(\int_\alpha^{T+\delta} \left(\int_\alpha^t f(s) g(t-s) ds \right) dt \right)^{\frac{1}{p}}. \tag{3.2}$$

In particular, for

$$(f * g)(t) = \int_0^t f(t-s) g(s) ds, \qquad 0 < t < T$$

and for $\alpha = 0$, *we have*

$$\|f\|_{L_p(0,T)} \|g\|_{L_p(0,\delta)} \le M^{\frac{2p-2}{p}} \|f * g\|_{L_1(0,T+\delta)}^{\frac{1}{p}}.$$

Proof. Since $0 \le f, g \le M$ for $0 \le t \le T$, we have

$$\begin{aligned} \int_\alpha^t f(s)^p g(t-s)^p ds &= \int_\alpha^t f(s)^{p-1} g(t-s)^{p-1} f(s) g(t-s) ds \\ &\le M^{2p-2} \int_\alpha^t f(s) g(t-s) ds. \end{aligned} \tag{3.3}$$

Hence

$$\int_\alpha^{T+\delta}\left(\int_\alpha^t f(s)^p g(t-s)^p ds\right)dt \le M^{2p-2}\int_\alpha^{T+\delta}\left(\int_\alpha^t f(s)g(t-s)ds\right)dt.$$

On the other hand, we have

$$\begin{aligned}\int_\alpha^{T+\delta}\left(\int_\alpha^t f(s)^p g(t-s)^p ds\right)dt &= \int_\alpha^{T+\delta}\left(\int_s^{T+\delta} g(t-s)^p dt\right) f(s)^p ds\\ &= \int_\alpha^{T+\delta}\left(\int_0^{T+\delta-s} g(\eta)^p d\eta\right) f(s)^p ds\\ &\ge \int_\alpha^{T}\left(\int_0^{T+\delta-s} g(\eta)^p d\eta\right) f(s)^p ds\\ &\ge \int_\alpha^{T}\left(\int_0^{\delta} g(\eta)^p d\eta\right) f(s)^p ds\\ &= \|f\|^p_{L_p(\alpha,T)}\|g\|^p_{L_p(0,\delta)}.\end{aligned}$$

Thus the proof of Theorem 3.1 is complete. □

4. APPLICATIONS TO INVERSE SOURCE HEAT PROBLEMS AND RESULTS

We consider the heat equation with a heat source:

$$\partial_t u(x,t) = \Delta u(x,t) + f(t)\varphi(x), \quad x\in\mathbb{R}^n,\ t>0 \tag{4.1}$$

$$u(x,0) = 0, \qquad x\in\mathbb{R}^n. \tag{4.2}$$

We assume that φ is a given function and satisfies

$$\begin{cases}\varphi\ge 0, \quad \not\equiv 0 & \text{in } \mathbb{R}^n,\\ \varphi \text{ has compact support}, \ \varphi\in C^\infty(\mathbb{R}^n), & \text{if } n\ge 4 \text{ and}\\ \varphi\in L_2(\mathbb{R}^n), & \text{if } n\le 3.\end{cases} \tag{4.3}$$

Our problem is to derive a conditional stability in the determination of $f(t)$, $0<t<T$, from the observation

$$u(x_0,t), \qquad 0<t<T, \tag{4.4}$$

where $x_0\notin \operatorname{supp}\varphi$.

We are interested only in the case of $x_0\notin\operatorname{supp}\varphi$, because in the case where x_0 is in the interior of $\operatorname{supp}\varphi$, the problem can be reduced to a Volterra integral equation of the second kind by differentiation in t formula (4.8) stated below. Moreover $x_0\notin\operatorname{supp}\varphi$ means that our observation (4.4) is done far from the set where the actual process is occuring, and the design of the observation point is easy.

Let

$$K(x,t) = \frac{1}{(2\sqrt{\pi t})^n}e^{-\frac{|x|^2}{4t}}, \quad x\in\mathbb{R}^n,\ t>0. \tag{4.5}$$

Then the solution u to (4.1) and (4.2) is represented by

$$u(x,t) = \int_0^t\int_{\mathbb{R}^n} K(x-y,t-s)f(s)\varphi(y)dyds, \quad x\in\mathbb{R}^n,\ t>0 \tag{4.6}$$

(e.g., Friedman [6]). Therefore, setting

$$\mu_{x_0}(t) = \int_{\mathbb{R}^n} K(x_0 - y, t)\varphi(y)dy, \qquad t > 0, \tag{4.7}$$

we have

$$u(x_0, t) \equiv h_{x_0}(t) = \int_0^t \mu_{x_0}(t-s)f(s)ds, \quad 0 < t < T, \tag{4.8}$$

which is a Volterra integral equation of the first kind with respect to f. Since

$$\lim_{t\downarrow 0} \frac{d^k \mu_{x_0}}{dt^k}(t) = (\Delta^k \varphi)(x_0) = 0, \quad k \in \mathbb{N} \cup \{0\}$$

by $x_0 \not\in \operatorname{supp} \varphi$ (e.g., [6]), the equation (4.8) cannot be reduced to a Volterra equation of the second kind by differentiating in t. Hence, even though, for any $m \in \mathbb{N}$, we take the C^m-norms for data h, the equation (4.8) is ill-posed, and we cannot expect a better stability such as of Hölder type under suitable a priori boundedness.

In Cannon and Esteva [3], an estimate of logarithmic type is proved: let $n = 1$ and $\varphi = \varphi(x)$ be the characteristic function of an interval $(a, b) \subset \mathbb{R}$. Set

$$\mathcal{V}_M = \left\{ f \in C^2[0,\infty); f(0) = 0, \quad \left\| \frac{df}{dt} \right\|_{C[0,\infty)}, \left\| \frac{d^2 f}{dt^2} \right\|_{C[0,\infty)} \le M \right\}. \tag{4.9}$$

Let $x_0 \not\in (a, b)$. Then, for $T > 0$, there exists a constant $C = C(M, a, b, x_0) > 0$ such that

$$|f(t)| \le \frac{C}{|\log \|u(x_0,\cdot)\|_{L_2(0,\infty)}|^2}, \quad 0 \le t \le T, \tag{4.10}$$

for all $f \in \mathcal{V}_M$. The stability rate is logarithmic and worse than any rate of Hölder type: $\|u(x_0,\cdot)\|^{\alpha}_{L_2(0,\infty)}$ for any $\alpha > 0$. For (4.10), the condition $f \in \mathcal{V}_M$ prescribes a priori information and (4.10) is called conditional stability within the admissible set $\mathcal{V}_M$. The rate of conditional stability heavily depends on the choice of admissible sets and an observation point x_0. As for other inverse problems for the heat equation, we can refer to Cannon [2], Cannon and Esteva [4], Isakov [7] and the references therein.

We arbitrarily fix $M > 0$ and $N \in \mathbb{N}$. Let

$$\mathcal{U} = \{f \in C[0,T]; \|f\|_{C[0,T]} \le M, f \text{ changes the signs at most } N\text{-times}\}. \tag{4.11}$$

We take $\mathcal{U}$ as an admissible set of unknowns f. Then, within $\mathcal{U}$, we can show an improved conditional stability of Hölder type:

Theorem 4.1. *Let φ satisfy (4.3), and $x_0 \not\in \operatorname{supp} \varphi$. We set*

$$p > \begin{cases} \dfrac{4}{4-n}, & n \le 3, \\ 1, & n \ge 4. \end{cases} \tag{4.12}$$

Then, for an arbitrarily given $\delta > 0$, there exists a constant $C = C(x_0, \varphi, T, p, \delta, \mathcal{U}) > 0$ such that

$$\|f\|_{L_p(0,T)} \le C \|u(x_0,\cdot)\|^{\frac{1}{pN}}_{L_1(0,T+\delta)} \tag{4.13}$$

for any $f \in \mathcal{U}$.

We will see that $\lim_{\delta\to 0} C = \infty$ and, in order to estimate f over the time interval $(0, T)$, we have to observe $u(x_0, \cdot)$ over a longer time interval $(0, T + \delta)$.

Remark 4.2. In the case of $n \geq 4$, we can relax the regularity of φ to $H^\alpha(\mathbb{R}^n)$ with some $\alpha > 0$, but we will not go into the details. In the case of $n \leq 3$, if we assume that $\varphi \in C^\infty(\mathbb{R}^n)$ in (4.3), then in Theorem 4.1 we can take any $p > 1$.

Remark 4.3. As a subset of $\mathcal{U}$, we can take, for example,

$$\mathcal{P}_N = \{f; f \text{ is a polynomial whose order is at most } N \text{ and } \|f\|_{C[0,T]} \leq M\}.$$

The condition $f \in \mathcal{U}$ is quite restrictive at the expense of the practically reasonable estimate of Hölder type (4.4).

Remark 4.4. The a priori boundedness $\|f\|_{C[0,T]} \leq M$ is necessary for the stability.

Example 4.1. Let $n = 1$, $p > 2$ and

$$\varphi(x) = \begin{cases} 0 & |x| < r,\ |x| > \mathbb{R}, \\ \dfrac{\sqrt{\pi}}{R - r}, & r < |x| < \mathbb{R}. \end{cases} \tag{4.14}$$

We set $x_0 = 0$. Then, by (4.7), we have

$$\mu_0(t) = \frac{1}{2\sqrt{\pi t}} \int_{r<|y|<\mathbb{R}} e^{-\frac{y^2}{4t}} \varphi(y)dy, \tag{4.15}$$

so that

$$\frac{1}{\sqrt{t}} e^{-\frac{R^2}{4t}} \leq \mu_0(t) \leq \frac{1}{\sqrt{t}} e^{-\frac{r^2}{4t}}, \qquad t > 0. \tag{4.16}$$

We choose f_n as

$$f_n(t) = \frac{1}{\sqrt{t}} e^{-\frac{1}{nt}}, \qquad t > 0,\ n \in \mathbb{N}. \tag{4.17}$$

Then f_n does not change the signs in $(0, T)$ and $\lim_{n\to\infty} \max_{0\leq t\leq T} |f_n(t)| = \infty$. The corresponding solution $u_n(x, t)$ of (4.1) – (4.2) with f_n is estimated as follows:

$$|u_n(0, t)| = \left| \int_0^t \mu_0(t - s) f_n(s) ds \right| \leq \int_0^t \frac{1}{\sqrt{t - s}} e^{-\frac{r^2}{4(t-s)}} \frac{1}{\sqrt{s}} e^{-\frac{1}{ns}} ds,$$

and so

$$\begin{aligned} \int_0^T |u_n(0, t)| dt &\leq \int_0^T \left(\int_0^t \frac{1}{\sqrt{t - s}} e^{-\frac{r^2}{4(t-s)}} \frac{1}{\sqrt{s}} e^{-\frac{1}{ns}} ds \right) dt \\ &= \int_0^T \left(\int_s^T \frac{1}{\sqrt{t - s}} e^{-\frac{r^2}{4(t-s)}} dt \right) \frac{1}{\sqrt{s}} e^{-\frac{1}{ns}} ds \\ &\leq \int_0^T \left(\int_0^T \frac{1}{\sqrt{\eta}} e^{-\frac{r^2}{4\eta}} d\eta \right) \frac{1}{\sqrt{s}} ds \\ &= 2\sqrt{T} \int_0^T \frac{1}{\sqrt{\eta}} e^{-\frac{r^2}{4\eta}} d\eta. \end{aligned}$$

Next

$$\int_0^T f_n(t)^p dt = n^{\frac{p}{2}-1} \int_0^{nT} e^{-\frac{p}{\eta}} \eta^{-\frac{p}{2}} d\eta.$$

Therefore, for any $\gamma \in (0,1)$, we have

$$\frac{\left(\int_0^T f_n(t)^p dt\right)^{\frac{1}{p}}}{\left(\int_0^T u_n(0,t)dt\right)^{\gamma}} \geq \frac{n^{\frac{1}{2}-\frac{1}{p}}\left(\int_0^{nT} e^{-\frac{p}{\eta}}\eta^{-\frac{p}{2}}d\eta\right)^{\frac{1}{p}}}{\left(2\sqrt{T}\int_0^T \frac{1}{\sqrt{\eta}}e^{-\frac{r^2}{4\eta}}d\eta\right)^{\gamma}} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty$$

by $p > 2$. Hence the stability of the type (4.4) is impossible for $p > 2$.

Remark 4.5. For our stability, the finiteness of changes of signs is essential. In fact, we take

$$f_n(t) = \cos nt, \qquad 0 \leq t \leq T,\ n \in \mathbb{N}. \tag{4.18}$$

Then f_n oscillates very frequently and we cannot take any finite partition of $(0,T)$ where the condition on signs in (4.1) holds true. We note that we can take $M = 1$, that is, $\|f_n\|_{C[0,T]} \leq 1$ for $n \in \mathbb{N}$. We denote the solution to (4.1) – (4.2) for $f = f_n$ by $u_n(x,t)$. Then

$$\begin{aligned} u_n(x_0,t) &= \int_0^t \mu_{x_0}(t-s)f_n(s)ds \\ &= \int_0^t \mu_{x_0}(s)f_n(t-s)ds \\ &= \cos nt \int_0^t \mu_{x_0}(s)\cos nsds - \sin nt \int_0^t \mu_{x_0}(s)\sin nsds. \end{aligned}$$

By $\mu_{x_0} \in L_1(0,T)$, the Riemann-Lebesgue lemma yields $\lim_{n\to\infty} u_n(x_0,t) = 0$ for all $t \in [0, T+\delta]$. Moreover we readily see that

$$|u_n(x_0,t)| \leq \int_0^{T+\delta} \mu_{x_0}(s)ds < \infty, \qquad n \in \mathbb{N},\ 0 \leq t \leq T+\delta.$$

Therefore, by the Lebesgue convergence theorem, we can conclude that

$$\lim_{n\to\infty} \|u_n(x_0,\cdot)\|_{L_1(0,T+\delta)} = 0.$$

For $n = 1$, we can choose $p = 2$ in Theorem 4.1. We have

$$\int_0^T f_n(t)^2 dt = \frac{T}{2} + \frac{\sin 2nT}{4n},$$

so that $\lim_{n\to\infty} \|f_n\|_{L_2(0,T)} \neq 0$. Thus any stability cannot hold for f_n, $n \in \mathbb{N}$.

5. PROOF OF THEOREM 4.1

Suppose that f changes the signs at $0 < t_1 < t_2 < \ldots < t_I = T$, $I \leq N$. Without loss of generality, we may assume that $f \geq 0$ on $(0, t_1)$. Since $f \in \mathcal{U}$, we see that f satisfies (3.1). Meanwhile since $\mu_{x_0}(t)$ is positive and bounded, for some constant $B > 0$, $B\mu_{x_0}(t)$ satisfies (3.1). We apply Theorem 3.1 on $(0,t_1)$, setting $\alpha = 0$ and $g(t) = B\mu_{x_0}(t)$. Setting $C_1 = B^{1-\frac{1}{p}}\|\mu_{x_0}\|_{L_p(0,\delta)}$ ($C_1 > 0$), we obtain

$$\|f\|_{L_p(0,t_1)} \leq C_1^{-1} M^{\frac{2p-2}{p}} \|u(x_0,\cdot)\|_{L_1(0,t_1+\delta)}^{\frac{1}{p}}. \tag{5.1}$$

Next we will prove

$$|u(x_0,t_1)| \leq C_2 \|u(x_0,\cdot)\|_{L_1(0,t_1+\delta)}^{\frac{1}{p}}, \tag{5.2}$$

where the constant $C_2 > 0$ depends on φ, T, δ, p. Henceforth the constants $C_j > 0$, $j \geq 2$, are independent of the choice of $0 < t_1 < t_2 < \cdots < t_N < T$.

Proof of (5.2). Let $L_2(\mathbb{R}^n)$ be the usual L_2-space with the norm $\|\cdot\|$ and let $-A$ be the operator defined by

(5.3) $$(-Au)(x) = \Delta u(x), \quad x \in \mathbb{R}^n, \qquad \mathcal{D}(A) = H^2(\mathbb{R}^n).$$

Then $-A$ generates an analytic semigroup e^{-tA}, $t > 0$ and, by the definition of $H^{2\ell}(\mathbb{R}^n)$ and the interpolation inequality (e.g., Lions and Magenes [9]), we see that

$$(\mathcal{D}A^\ell) = H^{2\ell}(\mathbb{R}^n), \qquad \|u\|_{H^{2\ell}(\mathbb{R}^n)} \leq C_3(\ell)\|A^\ell u\|, \quad u \in \mathcal{D}(A^\ell).$$

Moreover

(5.4) $$\|A^\ell e^{-tA}\| \leq C_3(\ell)t^{-\ell}$$

and

(5.5) $$u(t) = u(\cdot, t) = \int_0^t e^{-(t-s)A} f(s)\varphi(\cdot)ds, \qquad t > 0$$

(e.g., Pazy [11]). By the Sobolev inequality: $H^{2\ell}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ if $4\ell > n$ (e.g., Adams [1]), we have $\mathcal{D}(A^\ell) \subset L^\infty(\mathbb{R}^n)$ and

(5.6) $$\|u\|_{L_\infty(\mathbb{R}^n)} \leq C_4(\ell)\|A^\ell u\|, \qquad u \in \mathcal{D}(A^\ell).$$

Case: $n \leq 3$.
We can take

(5.7) $$\ell = \frac{n}{4} + \varepsilon_0 < 1 \quad \text{with a sufficiently small } \varepsilon_0 > 0.$$

Let $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Since $p > \frac{4}{4-n}$, we have $q < \frac{4}{n}$, therefore we can choose $\varepsilon_0 > 0$ sufficiently small such that

(5.8) $$q\ell < 1.$$

Hence, by (5.4), (5.5) and the Hölder inequality, we obtain

(5.9) $$\begin{aligned} \|A^\ell u(t_1)\| &\leq \int_0^{t_1} |f(s)|\|A^\ell e^{-(t_1-s)A}\varphi\|ds \\ &\leq C_5 \int_0^{t_1} (t_1 - s)^{-\ell}|f(s)|ds \\ &\leq C_5 \left(\int_0^{t_1} (t_1 - s)^{-q\ell} ds\right)^{\frac{1}{q}} \left(\int_0^{t_1} |f(s)|^p ds\right)^{\frac{1}{p}}. \end{aligned}$$

Consequently, by (5.8), we have

$$\|A^\ell u(t_1)\| \leq C_5 \left(\frac{t_1^{1-q\ell}}{1 - q\ell}\right)^{\frac{1}{q}} \|f\|_{L_p(0,t_1)} \leq C_5 \left(\frac{T}{1 - q\ell}\right)^{\frac{1}{q}} \|f\|_{L_p(0,t_1)}.$$

Therefore (5.1) and (5.6) yield (5.2).

Case: $n \geq 4$.
By $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have $\varphi \in \mathcal{D}(A^\ell)$ for $\ell \in \mathbb{N}$. Therefore the estimate of $\|A^\ell u(t)\|$ is simpler than in (5.9):

$$\begin{aligned}\|A^\ell u(t_1)\| &= \left\| \int_0^{t_1} f(s) e^{-(t_1-s)A} A^\ell \varphi ds \right\| \\ &\leq \int_0^{t_1} |f(s)| \|e^{-(t_1-s)A}\| \|A^\ell \varphi\| ds \\ &\leq C_6' \|f\|_{L_1(0,t_1)} \leq C_6 \|f\|_{L_p(0,t_1)}.\end{aligned}$$

Thus (5.1) and (5.6) complete the proof of (5.2). □

Next we will estimate $\|f\|_{L_p(t_1,t_2)}$. By (4.8), we have

$$-u(x_0,t) = -u(x_0,t_1) + \int_{t_1}^{t} \mu_{x_0}(t-s)(-f(s))ds, \quad t_1 \leq t \leq t_2. \tag{5.10}$$

Taking $\alpha = t_1$, $T = t_2$ in Theorem 3.1, we obtain

$$\begin{aligned}\|f\|_{L_p(t_1,t_2)} &\leq M^{\frac{2p-2}{p}} C_1^{-1} \left(\int_{t_1}^{t_2+\delta} |-u(x_0,t) + u(x_0,t_1)| dt \right)^{\frac{1}{p}} \\ &\leq M^{\frac{2p-2}{p}} C_1^{-1} (\|u(x_0,\cdot)\|_{L_1(t_1,t_2+\delta)} + T|u(x_0,t_1)|)^{\frac{1}{p}}.\end{aligned} \tag{5.11}$$

Therefore we apply (5.2), and

$$\begin{aligned}\|f\|_{L_p(t_1,t_2)} &\leq C_1^{-1} M^{\frac{2p-2}{p}} \|u(x_0,\cdot)\|_{L_1(t_1,t_2+\delta)}^{\frac{1}{p}} + C_1^{-1} M^{\frac{2p-2}{p}} T^{\frac{1}{p}} C_2^{\frac{1}{p}} \|u(x_0,\cdot)\|_{L_1(0,t_1+\delta)}^{\frac{1}{p^2}} \\ &\leq C_1^{-1} M^{\frac{2p-2}{p}} \left\{ \|u(x_0,\cdot)\|_{L_1(t_1,t_2+\delta)}^{\frac{1}{p}-\frac{1}{p^2}} \|u(x_0,\cdot)\|_{L_1(t_1,t_2+\delta)}^{\frac{1}{p^2}} \right. \\ &\qquad \left. + T^{\frac{1}{p}} C_2^{\frac{1}{p}} \|u(x_0,\cdot)\|_{L_1(0,t_1+\delta)}^{\frac{1}{p^2}} \right\} \\ &\leq C_1^{-1} M^{\frac{2p-2}{p}} ((TM')^{\frac{p-1}{p^2}} + T^{\frac{1}{p}} C_2^{\frac{1}{p}}) \|u(x_0,\cdot)\|_{L_1(0,t_2+\delta)}^{\frac{1}{p^2}}.\end{aligned} \tag{5.12}$$

Here, since $u(x_0,t)$ is bounded, we take a positive M' such that $|u(x_0,t)| \leq M'$. By (5.1) and (5.12), we can estimate $\|f\|_{L_p(0,t_2)}$. Continuing this argument until $t_I = T$, we can complete the proof of Theorem 4.1.

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