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REVERSE INEQUALITIES ON CHAOTICALLY GEOMETRIC MEAN VIA SPECHT RATIO, II

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ABSTRACT. In 1967, as a converse of the arithmetic-geometric mean inequality, Mond and Shisha gave an estimate of the difference between the arithmtic mean and the geometric one, which we call it the Mond-Shisha difference. As an application of the Mond-Pečarić method, we show some order relations between the power means of positive operators on a Hilbert space. Among others, we show that the upper bound of the difference between the arithmetic mean and the chaotically geometric one of positive operators coincides with the Mond-Shisha difference.

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1. INTRODUCTION

In 1960, as a converse of the arithmetic-geometric mean inequality, W. Specht [13] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \dots, x_n \in [m, M]$ with $0 < m < M$,

$$\sqrt[n]{x_1 x_2 \cdots x_n} \le \frac{x_1 + x_2 + \cdots + x_n}{n} \le M_h(1) \sqrt[n]{x_1 x_2 \cdots x_n}, \tag{1.1}$$

where $h = \frac{M}{m} (\ge 1)$ is a generalized condition number in the sense of Turing [15] and the Specht ratio $M_h(1)$ is defined for $h \ge 1$ as

$$M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h > 1) \quad \text{and} \quad M_1(1) = 1.$$

On the other hand, Mond and Shisha [11, 12] gave an estimate of the difference between the arithmetic mean and the geometric one: For $x_1, \dots, x_n \in [m, M]$ with $0 < m < M$,

$$0 \le \frac{x_1 + x_2 + \cdots + x_n}{n} - \sqrt[n]{x_1 x_2 \cdots x_n} \le L(m, M) \log M_h(1), \tag{1.2}$$

where the logarithmic mean $L(m, M)$ is defined for $0 < m < M$ as

$$L(m, M) = \frac{M - m}{\log M - \log m} (M \ne m) \quad \text{and} \quad L(m, m) = m.$$

J.I. Fujii and one of the authors [1, 2] showed an operator version of the Mond-Shisha theorem (1.2): Let A be a positive operator on a Hilbert space H satisfying $m \le A \le M$ for some scalars $0 < m < M$. Then

$$(Ax, x) - \exp(\log A\, x, x) \le L(m, M) \log M_h(1) \tag{1.3}$$

holds for every unit vector x in H. Incidentally, if we put $A = \operatorname{diag}(x_1, x_2, \dots, x_n)$ and $x = \frac{1}{\sqrt{n}}(1, 1, \dots, 1)$ in (1.3), then we have (1.2).

Next, we recall the geometric mean in the sense of Kubo-Ando theory [7]: For two positive operators A and B on a Hilbert space H, the geometric mean and arithmetic mean of A and B are defined as follows:

$$A \sharp_\lambda B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}} \quad \text{and} \quad A \nabla_\lambda B = (1 - \lambda)A + \lambda B$$

for $\lambda \in [0, 1]$. Like the numerical case, the arithmetic-geometric mean inequality holds:

$$A \sharp_\lambda B \le A \nabla_\lambda B \qquad \text{for all } \lambda \in [0, 1]. \tag{1.4}$$

Tominaga [14] showed the following inequality, as a reverse inequality of the noncommutative arithmetic-geometric mean inequality (1.4) which differs from (1.3): Let A and B be positive operators on a Hilbert space H satisfying $m \le A, B \le M$ for some scalars $0 < m < M$. Then

$$0 \le A \nabla_\lambda B - A \sharp_\lambda B \le hL(m, M) \log M_h(1) \qquad \text{for all } \lambda \in [0, 1], \tag{1.5}$$

where $h = \frac{M}{m}$. It is considered as another operator version of the Mond-Shisha theorem (1.2).

On the other hand, M. Fujii and R. Nakamoto discussed the monotonicity of a family of power means in [4]. For fixed $A,\ B > 0$ and $\lambda \in [0, 1]$, we put

$$F(r) = (A^r \nabla_\lambda B^r)^{\frac{1}{r}} \ (r \ne 0), \ = e^{\log A \ \nabla_\lambda \ \log B} \ (r = 0).$$

Then the power mean $F(r)$ is monotone increasing on $\mathbb{R}$ under the chaotic order $X \gg Y$, i.e., $\log X \ge \log Y$ for $X,\ Y > 0$, [4, Lemma 2]. In particular, $A \diamondsuit_\lambda B = e^{\log A \ \nabla_\lambda \ \log B}$ is called the chaotically λ-geometric mean. In general, it does not concide with $A \sharp_\lambda B$.

In this note, as a continuation of [3], we consider some order relations between the arithmetic mean and the chaotically geometric one. Among others, we show that if A and B are positive operators on a Hilbert space H satisfying $m \le A, B \le M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$, then

$$-L(m, M) \log M_h(1) \le A \nabla_\lambda B - A \Diamond_\lambda B \le L(m, M) \log M_h(1) \qquad \text{for all } \lambda \in [0, 1].$$

Concluding this section, we have to mention that almost all results in this note are based on our previous result [8, Corollary 4] coming from the Mond-Pečarić method [9]. Namely this note might be understood as an application of the Mond-Pečarić method.

2. PRELIMINARY ON THE MOND-PEČARIĆ METHOD

Let A be a positive operator on a Hilbert space H satisfying $m \le A \le M$ for some scalars $0 < m < M$, and let $f(t)$ be a real valued continuous convex function on $[m, M]$. Mond and Pečarić [9] proved that

$$0 \le (f(A)x, x) - f((Ax, x)) \le \beta(m, M, f) \tag{2.1}$$

holds for every unit vector $x \in H$, where

$$\beta(m, M, f) = \max \left\{ \frac{f(M) - f(m)}{M - m}(t - m) + f(m) - f(t); t \in [m, M] \right\}. \tag{2.2}$$

Similarly, we have the following complementary result of (2.1) for a concave function. If $f(t)$ is concave, then

$$\bar{\beta}(m, M, f) \le (f(A)x, x) - f((Ax, x)) \le 0 \tag{2.3}$$

holds for every unit vector $x \in H$, where

$$\bar{\beta}(m, M, f) = \min \left\{ \frac{f(M) - f(m)}{M - m}(t - m) + f(m) - f(t); t \in [m, M] \right\}. \tag{2.4}$$

The following result is a generalization of (2.1) and based on the idea due to Furuta's work [5, 6]. We cite it here for convenience:

Theorem A ([8]). *Let A_j $(j = 1, 2, \ldots, k)$ be positive operators on a Hilbert space H satisfying $m \le A_j \le M$ for some scalars $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$. Then*

$$0 \le \sum_{j=1}^{k} (f(A_j)x_j, x_j) - f\left(\sum_{j=1}^{k} (A_j x_j, x_j) \right) \le \beta(m, M, f) \tag{2.5}$$

holds for all $k-$tuples $(x_1, \ldots, x_k)$ in H with $\sum_{j=1}^{k} \|x_j\|^2 = 1$, where $\beta(m, M, f)$ is defined as in (2.2).

For the power function $f(t) = t^p$, we know the following fact, which is a reverse inequality of the Hölder-McCarthy inequality:

Theorem B. *Let A be a positive operator on a Hilbert space H satisfying $m \le A \le M$ for some scalars $0 < m < M$ and put $h = \frac{M}{m}$. For each $p > 1$*

$$0 \le (A^p x, x) - (Ax, x)^p \le C(m, M, p) \tag{2.6}$$

holds for every unit vector $x \in H$, where the constant $C(m, M, p)$ ([8, 16]) is defined as

$$C(m, M, p) = \frac{Mm^p - mM^p}{M - m} + (p - 1)\left(\frac{M^p - m^p}{p(M - m)} \right)^{\frac{p}{p-1}} \qquad \text{for all } p > 1. \tag{2.7}$$

We obtain a complement of Theorem B: Under the assumption of Theorem B, for each $0 < p < 1$

$$-\frac{M^p - m^p}{M - m} C\left(m^p, M^p, \frac{1}{p}\right) \leq (A^p x, x) - (Ax, x)^p \leq 0 \tag{2.8}$$

holds for every unit vector $x \in H$. It easily can be proved by the fact that $\bar{\beta}(m, M, t^p) = -\frac{M^p - m^p}{M - m} C\left(m^p, M^p, \frac{1}{p}\right)$ for $0 < p < 1$.

3. REVERSE INEQUALITY ON OPERATOR CONVEXITY

Continuous functions which are convex as real functions need not be operator convex. In this section, we estimate the bounds of the operator convexity for convex functions.

Lemma 3.1. *Let A and B be positive operators on a Hilbert space H satisfying $m \leq A, B \leq M$ for some scalars $0 < m < M$. If $f(t)$ is a real valued continuous convex function on $[m, M]$, then for each $\lambda \in [0, 1]$*

$$-\beta(m, M, f) \leq f(A) \nabla_\lambda f(B) - f(A \nabla_\lambda B) \leq \beta(m, M, f), \tag{3.1}$$

where $\beta(m, M, f)$ is defined as (2.2).

Proof. For each $0 < \lambda < 1$ and unit vector $x \in H$, put $A_1 = A$, $A_2 = B$, $x_1 = \sqrt{1 - \lambda} x$ and $x_2 = \sqrt{\lambda} x$ in Theorem A. Then we have

$$(1 - \lambda)(f(A)x, x) + \lambda(f(B)x, x) \leq f((1 - \lambda)(Ax, x) + \lambda(Bx, x)) + \beta(m, M, f).$$

Hence it follows that

$$\begin{aligned}(((1 - \lambda) f(A) + \lambda f(B))x, x) &\leq f((((1 - \lambda)A + \lambda B)x, x)) + \beta(m, M, f) \\ &\leq (f((1 - \lambda)A + \lambda B)x, x) + \beta(m, M, f)\end{aligned}$$

where the last inequality holds by the convexity of $f(t)$ [9, Theorem 1] or (2.1). Therefore we have

$$f(A) \nabla_\lambda f(B) \leq f(A \nabla_\lambda B) + \beta(m, M, f).$$

Next, since $f(t)$ is convex, it follows that

$$\begin{aligned}(1 - \lambda)(f(A)x, x) + \lambda(f(B)x, x) &\geq (1 - \lambda) f((Ax, x)) + \lambda f((Bx, x)) \\ &\geq f((1 - \lambda)(Ax, x) + \lambda(Bx, x)).\end{aligned}$$

Since $0 < m \leq (1 - \lambda)A + \lambda B \leq M$, it follows from (2.1) that

$$\begin{aligned}f((1 - \lambda)(Ax, x) + \lambda(Bx, x)) &= f(((A \nabla_\lambda B)x, x)) \\ &\geq (f(A \nabla_\lambda B)x, x) - \beta(m, M, f)\end{aligned}$$

holds for every unit vector $x \in H$. Therefore we have

$$-\beta(m, M, f) + f(A \nabla_\lambda B) \leq f(A) \nabla_\lambda f(B).$$

□

We have the following complementary result of Lemma 3.1 for concave functions.

Lemma 3.2. *Let A and B be positive operators on a Hilbert space H satisfying $m \leq A, B \leq M$ for some scalars $0 < m < M$. If $f(t)$ is a real valued continuous concave function on $[m, M]$, then for each $\lambda \in [0, 1]$*

$$-\bar{\beta}(m, M, f) \geq f(A) \nabla_\lambda f(B) - f(A \nabla_\lambda B) \geq \bar{\beta}(m, M, f), \tag{3.2}$$

where $\bar{\beta}(m, M, f)$ is defined as (2.4).

Next, consider the functions $f(t) = t^r$ on $[0, \infty)$. Then $f(t)$ is operator concave if $0 \le r \le 1$, operator convex if $1 \le r \le 2$, and $f(t)$ is not operator convex but it is convex if $r > 2$. By Lemmas 3.1 and 3.2, we obtain the following reverse inequalities on operator convexity and operator concavity for $f(t) = t^r$.

Corollary 3.3. *Let A and B be positive operators on a Hilbert space H satisfying $m \le A, B \le M$ for some scalars $0 < m < M$ and $\lambda \in [0, 1]$.*

(i) *If $0 < r \le 1$, then*

$$-\frac{M^r - m^r}{M - m} C\left(m^r, M^r, \frac{1}{r}\right) \le A^r \nabla_\lambda B^r - (A \nabla_\lambda B)^r \le 0.$$

(ii) *If $1 \le r \le 2$, then*

$$0 \le A^r \nabla_\lambda B^r - (A \nabla_\lambda B)^r \le C(m, M, r).$$

(iii) *If $r > 2$, then*

$$-C(m, M, r) \le A^r \nabla_\lambda B^r - (A \nabla_\lambda B)^r \le C(m, M, r),$$

where $C(m, M, r)$is defined as (2.7).

Proof. Put $f(t) = t^r$ for $r > 1$ in Lemma 3.1, then we obtain $\beta(m, M, f) = C(m, M, r)$. Also, in the case of $0 < r \le 1$, we have $\bar{\beta}(m, M, f) = -\frac{M^r - m^r}{M - m} C(m^r, m^r, \frac{1}{r})$ in Lemma 3.2. □

4. COMPARISON BETWEEN ARITHMETIC AND CHAOTICALLY GEOMETRIC MEANS

Let A and B be positive operators on a Hilbert space H and $\lambda \in [0, 1]$. The operator function $F(r) = (A^r \nabla_\lambda B^r)^{1/r} (r \in \mathbb{R})$ is monotone increasing on $[1, \infty)$ and not monotone increasing on $(0, 1]$ under the usual order. Recently, Nakamoto and one of the authors [4] investigated some properties of the chaotically geometric mean $A \diamondsuit_\lambda B = e^{\log A \nabla_\lambda \log B}$ and showed that the operator function $F(r)$ is monotone increasing on $\mathbb{R}$ under the chaotic order and $F(r)$ converges to $A \diamondsuit_\lambda B$ as $r \to +0$ in the strong operator topology.

In this section, we shall consider some order relations among the chaotically geometric mean, the arithmetic one and the power mean $F(r)$ by using the results in the previous section. The obtained inequality

$$-L(m, M) \log M_h(1) \le A \nabla_\lambda B - A \diamondsuit_\lambda B \le L(m, M) \log M_h(1)$$

is understood as a variant of a reverse Young inequality

$$0 \le A \nabla_\lambda B - A \sharp_\lambda B \le hL(m, M) \log M_h(1)$$

due to Tominaga [14], where $h = \frac{M}{m}$.

Firstly, by virtue of Corollary 3.3, we see an estimate of the bounds of the difference among the family $\{F(r) : r > 0\}$. Incidentally the constant $C(m, M, r)$ is defined as (2.7).

Theorem 4.1. *Let A and B be positive operators on a Hilbert space H satisfying $m \le A, B \le M$ for some scalars $0 < m < M$ and $\lambda \in [0, 1]$.*

(i) *If $0 < r \le 1 \le s$, then*

$$-C\left(m^r, M^r, \frac{1}{r}\right) \le F(s) - F(r) \le C\left(m^r, M^r, \frac{1}{r}\right) + \frac{M - m}{M^s - m^s} C(m, M, s).$$

(ii) *If $0 < 1 \le r \le s$, then*

$$0 \le F(s) - F(r) \le \frac{M - m}{M^s - m^s} C(m, M, s).$$

(iii) *If* $0 < r \le s \le 1$, *then*

$$|F(s) - F(r)| \le C\left(m^r, M^r, \frac{1}{r}\right) + C\left(m^s, M^s, \frac{1}{s}\right).$$

Proof. Suppose that $0 < r \le 1$ or $1 \le \frac{1}{r}$. By (iii) of Corollary 3.3, it follows that

$$-C\left(m, M, \frac{1}{r}\right) \le A^{\frac{1}{r}} \nabla_\lambda B^{\frac{1}{r}} - (A\nabla_\lambda B)^{\frac{1}{r}} \le C\left(m, M, \frac{1}{r}\right).$$

We apply it to $m^r \le A^r, B^r \le M^r$ instead of $m \le A, B \le M$. That is,

$$-C\left(m^r, M^r, \frac{1}{r}\right) \le A\nabla_\lambda B - (A^r \nabla_\lambda B^r)^{\frac{1}{r}} \le C\left(m^r, M^r, \frac{1}{r}\right). \tag{4.1}$$

If $s \ge 1$, then $\frac{1}{s} \le 1$ and by (i) of Corollary 3.3

$$-\frac{M^{1/s} - m^{1/s}}{M - m} C\left(m^{1/s}, M^{1/s}, s\right) \le A^{\frac{1}{s}} \nabla_\lambda B^{\frac{1}{s}} - (A\nabla_\lambda B)^{\frac{1}{s}} \le 0.$$

Since $m^s \le A^s, B^s \le M^s$, we have also

$$-\frac{M - m}{M^s - m^s} C\left(m, M, s\right) \le A\nabla_\lambda B - (A^s \nabla_\lambda B^s)^{\frac{1}{s}} \le 0. \tag{4.2}$$

By using (4.1) and (4.2), it follows that

$$\begin{aligned}
-C\left(m^r, M^r, \frac{1}{r}\right) &\le A\nabla_\lambda B - (A^r\nabla_\lambda B^r)^{1/r} && \text{by (4.1)} \\
&\le (A^s\nabla_\lambda B^s)^{1/s} - (A^r\nabla_\lambda B^r)^{1/r} && \text{by (4.2)} \\
&\le A\nabla_\lambda B + \frac{M - m}{M^s - m^s} C(m, M, s) \\
&\qquad -A\nabla_\lambda B + C\left(m^r, M^r, \frac{1}{r}\right) && \text{by (4.1) and (4.2)} \\
&= \frac{M - m}{M^s - m^s} C(m, M, s) + C\left(m^r, M^r, \frac{1}{r}\right),
\end{aligned}$$

and hence we have (i) in the case of $0 < r \le 1 \le s$.

In the case of $0 < 1 \le r \le s$, we have $1/s \le 1/r \le 1$ and by (4.2)

$$-\frac{M - m}{M^s - m^s} C(m, M, s) \le A\nabla_\lambda B - (A^s\nabla_\lambda B^s)^{1/s}$$

and

$$A\nabla_\lambda B - (A^r\nabla_\lambda B^r)^{1/r} \le 0.$$

Therefore it follows that

$$\begin{aligned}
0 &\le (A^s\nabla_\lambda B^s)^{1/s} - (A^r\nabla_\lambda B^r)^{1/r} \\
&\le A\nabla_\lambda B + \frac{M - m}{M^s - m^s} C(m, M, s) - A\nabla_\lambda B \\
&\le \frac{M - m}{M^s - m^s} C(m, M, s).
\end{aligned}$$

In the case of $0 < r \le s \le 1$, we have $1 < 1/s \le 1/r$ and by (4.1)

$$-C\left(m^r, M^r, \frac{1}{r}\right) \le A\nabla_\lambda B - (A^r\nabla_\lambda B^r)^{\frac{1}{r}} \le C\left(m^r, M^r, \frac{1}{r}\right)$$

and

$$-C\left(m^s, M^s, \frac{1}{s}\right) \le A\nabla_\lambda B - (A^s\nabla_\lambda B^s)^{\frac{1}{s}} \le C\left(m^s, M^s, \frac{1}{s}\right).$$

Therefore it follows that

$$\begin{aligned}-C\left(m^r, M^r, \frac{1}{r}\right) - C\left(m^s, M^s, \frac{1}{s}\right) &\le (A^s\nabla_\lambda B^s)^{1/s} - (A^r\nabla_\lambda B^r)^{1/r} \\ &\le C\left(m^r, M^r, \frac{1}{r}\right) + C\left(m^s, M^s, \frac{1}{s}\right).\end{aligned}$$

□

Though the operator function $F(r)$ converges to $A\Diamond_\lambda B$ as $r \to 0$ in the strong operator topology, $F(s)$ is not generally monotone increasing on $(0, 1]$ under the usual order. Thus, we have the following estimation of the difference between $F(r)$ and $A\Diamond_\lambda B$.

Theorem 4.2. *Let A and B be positive operators on a Hilbert space H satisfying $m \le A, B \le M$ for some scalars $0 < m < M$ and $\lambda \in [0, 1]$. Put $h = \frac{M}{m}$.*

(i) *If $0 < s < 1$, then*

$$\begin{aligned}-C\left(m^s, M^s, \frac{1}{s}\right) - L(m, M)\log M_h(1) &\le F(s) - A\Diamond_\lambda B \\ &\le C\left(m^s, M^s, \frac{1}{s}\right) + L(m, M)\log M_h(1).\end{aligned}$$

(ii) *If $1 < s$, then*

$$-L(m, M)\log M_h(1) \le F(s) - A\Diamond_\lambda B \le \frac{M-m}{M^s - m^s}C(m, M, s) + L(m, M)\log M_h(1).$$

Proof. To prove this, we need the following facts on $C(m, M, r)$ for $0 < m < M$ and $r > 1$ by Yamazaki [16]:

(a) $0 \le C(m, M, r) \le M(M^{r-1} - m^{r-1})$ for $r > 1$
(b) $C(m, M, r) \to 0$ as $r \to 1$
(c) $C\left(m^r, M^r, \frac{1}{r}\right) \to L(m, M)\log M_h(1)$ as $r \to +0$.

In the case of $0 < r \le s \le 1$, if we put $r \to 0$ in (iii) of Theorem 4.1, then $F(r) \to A\Diamond_\lambda B$ and $C\left(m^r, M^r, \frac{1}{r}\right) \to L(m, M)\log M_h(1)$ as $r \to 0$. Therefore we have (i).

In the case of $0 < r \le 1 \le s$, if we put $r \to 0$ in (i) of Theorem 4.1, then we have (ii). □

As a result, we obtain an operator version of the Mond-Shisha theorem (1.2):

Theorem 4.3. *Let A and B be positive operators on a Hilbert space H satisfying $m \le A, B \le M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$. Then*

$$-L(m, M)\log M_h(1) \le A\nabla_\lambda B - A\Diamond_\lambda B \le L(m, M)\log M_h(1)$$

hold for all $\lambda \in [0, 1]$.

Proof. Since $C(m, M, s) \to 0$ as $s \to 1$, we have the conclusion by (ii) of Theorem 4.2. □

By combining Theorem 4.3 and a reverse Young inequality (1.4), we obtain an estimate of the difference between the geometric mean and the chaotically geometric one:

Corollary 4.4. *Let A and B be positive operators on a Hilbert space H satisfying $m \leq A, B \leq M$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$. Then*

$$-(1+h)L(m,M)\log M_h(1) \leq A\sharp_\lambda B - A\Diamond_\lambda B \leq L(m,M)\log M_h(1)$$

holds for all $\lambda \in [0,1]$.

Proof. Since $A\sharp_\lambda B \leq A\nabla_\lambda B$, it follows from Theorem 4.3 that

$$A\sharp_\lambda B - A\Diamond_\lambda B \leq A\nabla_\lambda B - A\Diamond_\lambda B \leq L(m,M)\log M_h(1).$$

By Theorem 4.3 and a reverse Young inequality (1.4), it follows that

$$\begin{aligned} -L(m,M)\log M_h(1) &\leq A\nabla_\lambda B - A\Diamond_\lambda B \\ &\leq A\sharp_\lambda B + hL(m,M)\log M_h(1) - A\Diamond_\lambda B. \end{aligned}$$

□

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