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REFINEMENTS OF REVERSE TRIANGLE INEQUALITIES IN INNER PRODUCT SPACES

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ABSTRACT. Refining some results of S.S. Dragomir, several new reverses of the triangle inequality in inner product spaces are obtained.

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1. INTRODUCTION

It is interesting to know under which conditions the triangle inequality reverses in a normed space X; in other words, we would like to know if there is a constant c with the property that $c\sum_{k=1}^{n}\|x_k\| \leq \|\sum_{k=1}^{n} x_k\|$ for some finite set $x_1,\dots,x_n \in X$. M. Nakai and T. Tada [7] proved that the normed spaces with this property for any finite set $x_1,\dots,x_n \in X$ are only those of finite dimension.

The first authors to investigate the reverse of the triangle inequality in inner product spaces were J. B. Diaz and F. T. Metcalf [2]. They did so by establishing the following result as an extension of an inequality given by M. Petrovich [8] for complex numbers:

Theorem 1.1 (Diaz-Metcalf Theorem)**.** *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$. Suppose the vectors $x_k \in H$, $k \in \{1,\dots,n\}$ satisfy*

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_k, a\rangle}{\|x_k\|}, \qquad k \in \{1,\dots,n\}.$$

Then

$$r\sum_{k=1}^{n}\|x_k\| \leq \left\|\sum_{k=1}^{n} x_k\right\|,$$

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where equality holds if and only if

$$\sum_{k=1}^{n} x_k = r \sum_{k=1}^{n} \|x_k\| a.$$

Inequalities related to the triangle inequality are of special interest; cf. Chapter XVII of [6] and may be applied to obtain inequalities in complex numbers or to study vector-valued integral inequalities [3], [4].

Using several ideas and the notation of [3], [4] we modify or refine some results of S.S. Dragomir to procure some new reverses of the triangle inequality (see also [1]).

We use repeatedly the Cauchy-Schwarz inequality without mentioning it. The reader is referred to [9], [5] for the terminology of inner product spaces.

2. MAIN RESULTS

The following theorem is an improvement of Theorem 2.1 of [4] in which the real numbers r_1, r_2 are not neccesarily nonnegative. The proof seems to be different as well.

Theorem 2.1. *Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H$, $k \in \{1, \dots, n\}$ satisfy*

$$0 \le r_1^2 \|x_k\| \le \mathrm{Re}\langle x_k, r_1 a\rangle, \qquad 0 \le r_2^2 \|x_k\| \le \mathrm{Im}\langle x_k, r_2 a\rangle \tag{2.1}$$

for some $r_1, r_2 \in [-1, 1]$. Then we have the inequality

$$(r_1^2 + r_2^2)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| \le \left\| \sum_{k=1}^{n} x_k \right\|. \tag{2.2}$$

The equality holds in (2.2) if and only if

$$\sum_{k=1}^{n} x_k = (r_1 + i r_2) \sum_{k=1}^{n} \|x_k\| a. \tag{2.3}$$

Proof. If $r_1^2 + r_2^2 = 0$, the theorem is trivial. Assume that $r_1^2 + r_2^2 \neq 0$. Summing inequalities (2.1) over k from 1 to n, we have

$$\begin{aligned}
(r_1^2 + r_2^2) \sum_{k=1}^{n} \|x_k\| &\le \mathrm{Re} \left\langle \sum_{k=1}^{n} x_k, r_1 a \right\rangle + \mathrm{Im} \left\langle \sum_{k=1}^{n} x_k, r_2 a \right\rangle \\
&= \mathrm{Re} \left\langle \sum_{k=1}^{n} x_k, (r_1 + i r_2) a \right\rangle \\
&\le \left| \left\langle \sum_{k=1}^{n} x_k, (r_1 + i r_2) a \right\rangle \right| \\
&\le \left\| \sum_{k=1}^{n} x_k \right\| \|(r_1 + i r_2) a\| \\
&= (r_1^2 + r_2^2)^{\frac{1}{2}} \left\| \sum_{k=1}^{n} x_k \right\|.
\end{aligned}$$

Hence (2.2) holds.

If (2.3) holds, then

$$\left\|\sum_{k=1}^{n} x_k\right\| = \left\|(r_1 + ir_2)\sum_{k=1}^{n}\|x_k\|a\right\| = (r_1^2 + r_2^2)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\|.$$

Conversely, if the equality holds in (2.2), we have

$$\begin{aligned}(r_1^2 + r_2^2)^{\frac{1}{2}}\left\|\sum_{k=1}^{n} x_k\right\| &= (r_1^2 + r_2^2)\sum_{k=1}^{n}\|x_k\| \\ &\le \operatorname{Re}\left\langle\sum_{k=1}^{n} x_k, (r_1 + ir_2)a\right\rangle \\ &\le \left|\left\langle\sum_{k=1}^{n} x_k, (r_1 + ir_2)a\right\rangle\right| \\ &\le (r_1^2 + r_2^2)^{\frac{1}{2}}\left\|\sum_{k=1}^{n} x_k\right\|.\end{aligned}$$

From this we deduce

$$\left|\left\langle\sum_{k=1}^{n} x_k, (r_1 + ir_2)a\right\rangle\right| = \left\|\sum_{k=1}^{n} x_k\right\| \|(r_1 + ir_2)a\|.$$

Consequently there exists $\eta \ge 0$ such that

$$\sum_{k=1}^{n} x_k = \eta(r_1 + ir_2)a.$$

From this we have

$$(r_1^2 + r_2^2)^{\frac{1}{2}}\eta = \|\eta(r_1 + ir_2)a\| = \left\|\sum_{k=1}^{n} x_k\right\| = (r_1^2 + r_2^2)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\|.$$

Hence

$$\eta = \sum_{k=1}^{n}\|x_k\|.$$

□

The next theorem is a refinement of Corollary 1 of [4] since, in the notation of Theorem 2.1, $\sqrt{2 - p_1^2 - p_2^2} \le \sqrt{\alpha_1^2 + \alpha_2^2}$.

Theorem 2.2. *Let a be a unit vector in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$. Suppose the vectors $x_k \in H - \{0\}$, $k \in \{1, \dots, n\}$, are such that*

(2.4) $$\|x_k - a\| \le p_1, \quad \|x_k - ia\| \le p_2, \quad p_1, p_2 \in \left(0, \sqrt{\alpha^2 + 1}\right),$$

where $\alpha = \min_{1\le k\le n}\|x_k\|$. Let

$$\alpha_1 = \min\left\{\frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} : 1 \le k \le n\right\},$$
$$\alpha_2 = \min\left\{\frac{\|x_k\|^2 - p_2^2 + 1}{2\|x_k\|} : 1 \le k \le n\right\}.$$

Then we have the inequality

$$(\alpha_1^2+\alpha_2^2)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\| \le \left\|\sum_{k=1}^{n} x_k\right\|,$$

where the equality holds if and only if

$$\sum_{k=1}^{n} x_k = (\alpha_1+i\alpha_2)\sum_{k=1}^{n}\|x_k\|a.$$

Proof. From the first inequality in (2.4) we have

$$\langle x_k-a, x_k-a\rangle \le p_1^2,$$

$$\|x_k\|^2+1-p_1^2 \le 2\operatorname{Re}\langle x_k,a\rangle, \qquad k=1,\dots,n,$$

and

$$\frac{\|x_k\|^2-p_1^2+1}{2\|x_k\|}\|x_k\| \le \operatorname{Re}\langle x_k,a\rangle.$$

Consequently,

$$\alpha_1\|x_k\| \le \operatorname{Re}\langle x_k,a\rangle.$$

Similarly from the second inequality we obtain

$$\alpha_2\|x_k\| \le \operatorname{Re}\langle x_k,ia\rangle = \operatorname{Im}\langle x_k,a\rangle.$$

Now apply Theorem 2.1 for $r_1=\alpha_1, r_2=\alpha_2$. □

Corollary 2.3. *Let a be a unit vector in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$. Suppose that the vectors $x_k\in H-\{0\}$, $k\in\{1,\dots,n\}$ such that*

$$\|x_k-a\|\le 1, \qquad \|x_k-ia\|\le 1.$$

Then

$$\frac{\alpha}{\sqrt{2}}\sum_{k=1}^{n}\|x_k\| \le \left\|\sum_{k=1}^{n} x_k\right\|,$$

in which $\alpha=\min_{1\le k\le n}\|x_k\|$. The equality holds if and only if

$$\sum_{k=1}^{n} x_k = \alpha\frac{(1+i)}{2}\sum_{k=1}^{n}\|x_k\|a.$$

Proof. Apply Theorem 2.2 for $\alpha_1=\frac{\alpha}{2}=\alpha_2$. □

Theorem 2.4. *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field. Suppose that the vectors $x_k\in H-\{0\}$, $k\in\{1,\dots,n\}$ satisfy*

$$\|x_k-a\|\le p, \quad p\in\left(0,\sqrt{\alpha^2+1}\right), \quad \alpha=\min_{1\le k\le n}\|x_k\|.$$

Then we have the inequality

$$\alpha_1\sum_{k=1}^{n}\|x_k\| \le \left\|\sum_{k=1}^{n} x_k\right\|,$$

where

$$\alpha_1=\min\left\{\frac{\|x_k\|^2-p^2+1}{2\|x_k\|}:1\le k\le n\right\}.$$

The equality holds if and only if

$$\sum_{k=1}^{n} x_k = \alpha_1 \sum_{k=1}^{n} \|x_k\| a.$$

Proof. The proof is similar to Theorem 2.2 in which we use Theorem 2.1 with $r_2 = 0$. □

The next theorem is a generalization of Theorem 2.1. It is a modification of Theorem 3 of [4], however our proof is apparently different.

Theorem 2.5. *Let $a_1, \dots, a_m$ be orthonormal vectors in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$. Suppose that for $1 \le t \le m$, $r_t, \rho_t \in \mathbb{R}$ and that the vectors $x_k \in H$, $k \in \{1, \dots, n\}$ satisfy*

$$0 \le r_t^2 \|x_k\| \le \operatorname{Re}\langle x_k, r_t a_t\rangle, \quad 0 \le \rho_t^2 \|x_k\| \le \operatorname{Im}\langle x_k, \rho_t a_t\rangle, \quad t \in \{1, \dots, m\}. \tag{2.5}$$

Then we have the inequality

$$\left(\sum_{t=1}^{m} r_t^2 + \rho_t^2\right)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| \le \left\| \sum_{k=1}^{n} x_k \right\|. \tag{2.6}$$

The equality holds in (2.7) if and only if

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \|x_k\| \sum_{t=1}^{m} (r_t + i\rho_t) a_t. \tag{2.7}$$

Proof. If $\sum_{t=1}^{m} (r_t^2 + \rho_t^2) = 0$, the theorem is trivial. Assume that $\sum_{t=1}^{m} (r_t^2 + \rho_t^2) \neq 0$. Summing inequalities (2.6) over k from 1 to n and again over t from 1 to m, we get

$$\begin{aligned}
\sum_{t=1}^{m} (r_t^2 + \rho_t^2) \sum_{k=1}^{n} \|x_k\| &\le \operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{t=1}^{m} r_t a_t \right\rangle + \operatorname{Im}\left\langle \sum_{k=1}^{n} x_k, \sum_{t=1}^{m} \rho_t a_t \right\rangle \\
&= \operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{t=1}^{m} r_t a_t \right\rangle + \operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, i\sum_{t=1}^{m} \rho_t a_t \right\rangle \\
&= \operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, \sum_{t=1}^{m} (r_t + i\rho_t) a_t \right\rangle \\
&\le \left| \left\langle \sum_{k=1}^{n} x_k, \sum_{t=1}^{m} (r_t + i\rho_t) a_t \right\rangle \right| \\
&\le \left\| \sum_{k=1}^{n} x_k \right\| \left\| \sum_{t=1}^{m} (r_t + i\rho_t) a_t \right\| \\
&= \left\| \sum_{k=1}^{n} x_k \right\| \left(\sum_{t=1}^{m} (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}}.
\end{aligned}$$

Then

$$\left(\sum_{t=1}^{m} (r_t^2 + \rho_t^2)\right)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| \le \left\| \sum_{k=1}^{n} x_k \right\|. \tag{2.8}$$

If (2.8) holds, then

$$\left\| \sum_{k=1}^{n} x_k \right\| = \left\| \sum_{k=1}^{n} \|x_k\| \sum_{t=1}^{m} (r_t + i\rho_t) a_t \right\| = \sum_{k=1}^{n} \|x_k\| \left(\sum_{t=1}^{m} (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}}.$$

Conversely, if the equality holds in (2.7), we obtain from (2.6) that

$$\begin{aligned}\left(\sum_{t=1}^{m}(r_t^2+\rho_t^2)\right)^{\frac{1}{2}}\left\|\sum_{k=1}^{n}x_k\right\| &= \sum_{t=1}^{m}(r_t^2+\rho_t^2)\sum_{k=1}^{n}\|x_k\| \\ &\le \operatorname{Re}\left\langle\sum_{k=1}^{n}x_k,\sum_{t=1}^{m}(r_t+i\rho_t)a_t\right\rangle \\ &\le \left|\left\langle\sum_{k=1}^{n}x_k,\sum_{t=1}^{m}(r_t+i\rho_t)a_t\right\rangle\right| \\ &\le \left\|\sum_{k=1}^{n}x_k\right\|\left\|\sum_{t=1}^{m}(r_t+i\rho_t)a_t\right\| \\ &= \left\|\sum_{k=1}^{n}x_k\right\|\left(\sum_{t=1}^{m}(r_t^2+\rho_t^2)\right)^{\frac{1}{2}}.\end{aligned}$$

Thus we have

$$\left|\left\langle\sum_{k=1}^{n}x_k,\sum_{t=1}^{m}(r_t+i\rho_t)a_t\right\rangle\right| = \left\|\sum_{k=1}^{n}x_k\right\|\left\|\sum_{t=1}^{m}(r_t+i\rho_t)a_t\right\|.$$

Consequently there exists $\eta \ge 0$ such that

$$\sum_{k=1}^{n}x_k = \eta\sum_{t=1}^{m}(r_t+i\rho_t)a_t$$

from which we have

$$\begin{aligned}\eta\left(\sum_{t=1}^{m}(r_t^2+\rho_t^2)\right)^{\frac{1}{2}} &= \left\|\eta\sum_{t=1}^{m}(r_t+i\rho_t)a_t\right\| \\ &= \left\|\sum_{k=1}^{n}x_k\right\| \\ &= \sum_{k=1}^{n}\|x_k\|\left(\sum_{t=1}^{m}(r_t^2+\rho_t^2)\right)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\eta = \sum_{k=1}^{n}\|x_k\|.$$

□

Corollary 2.6. *Let $a_1,\ldots,a_m$ be orthornormal vectors in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field. Suppose for $1\le t\le m$ that the vectors $x_k\in H$, $k\in\{1,\ldots,n\}$ satisfy*

$$0 \le r_t^2\|x_k\| \le \operatorname{Re}\langle x_k, r_t a_t\rangle.$$

Then we have the inequality

$$\left(\sum_{t=1}^{m}r_t^2\right)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\| \le \left\|\sum_{k=1}^{n}x_k\right\|.$$

The equality holds if and only if

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \|x_k\| \sum_{t=1}^{m} r_t a_t.$$

Proof. Apply Theorem 2.5 for $\rho_t = 0$. □

Theorem 2.7. *Let $a_1, \dots, a_m$ be orthornormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H - \{0\}$, $k \in \{1, \dots, n\}$ satisfy*

$$\|x_k - a_t\| \le p_t, \quad \|x_k - i a_t\| \le q_t, \quad p_t, q_t \in \left(0, \sqrt{\alpha^2 + 1}\right), \quad 1 \le t \le m,$$

where $\alpha = \min_{1 \le k \le n} \|x_k\|$. Let

$$\alpha_t = \min\left\{ \frac{\|x_k\|^2 - p_t^2 + 1}{2\|x_k\|} : 1 \le k \le n \right\},$$
$$\beta_t = \min\left\{ \frac{\|x_k\|^2 - q_t^2 + 1}{2\|x_k\|} : 1 \le k \le n \right\}.$$

Then we have the inequality

$$\left(\sum_{t=1}^{m} \alpha_t^2 + \beta_t^2 \right)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| \le \left\| \sum_{k=1}^{n} x_k \right\|,$$

where equality holds if and only if

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \|x_k\| \sum_{t=1}^{m} (\alpha_t + i\beta_t) a_t.$$

Proof. For $1 \le t \le m$, $1 \le k \le n$ it follows from $\|x_k - a_t\| \le p_t$ that

$$\langle x_k - a_t \rangle, x_k - a_t \rangle \le p_t^2,$$
$$\frac{\|x_k\|^2 - p_t^2 + 1}{2\|x_k\|} \|x_k\| \le \operatorname{Re}\langle x_k, a_t \rangle,$$
$$\alpha_t \|x_k\| \le \operatorname{Re}\langle x_k, a_t \rangle,$$

and similarly

$$\beta_t \|x_k\| \le \operatorname{Re}\langle x_k, i a_t \rangle = \operatorname{Im}\langle x_k, a_t \rangle.$$

Now applying Theorem 2.4 with $r_t = \alpha_t$, $\rho_t = \beta_t$ we deduce the desired inequality. □

Corollary 2.8. *Let $a_1, \dots, a_m$ be orthornormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H$, $k \in \{1, \dots, n\}$ satisfy*

$$\|x_k - a_t\| \le 1, \quad \|x_k - i a_t\| \le 1, \quad 1 \le t \le m.$$

Then

$$\frac{\alpha}{\sqrt{2}} \sqrt{m} \sum_{k=1}^{n} \|x_k\| \le \left\| \sum_{k=1}^{n} x_k \right\|.$$

The equality holds if and only if

$$\sum_{k=1}^{n} x_k = \alpha \frac{(1+i)}{2} \sum_{k=1}^{n} \|x_k\| \sum_{t=1}^{m} a_t.$$

Proof. Apply Theorem 2.7 for $\alpha_t = \frac{\alpha}{2} = \beta_t$. □

Remark 2.9. It is interesting to note that

$$\frac{\alpha}{\sqrt{2}}\sqrt{m} \leq \frac{\|\sum_{k=1}^{n} x_k\|}{\sum_{k=1}^{n}\|x_k\|} \leq 1,$$

where

$$\alpha \leq \sqrt{\frac{2}{m}}.$$

Corollary 2.10. *Let a be a unit vector in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$. Suppose that the vectors $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$ satisfy*

$$\|x_k - a\| \leq p_1, \quad \|x_k - ia\| \leq p_2, \quad p_1, p_2 \in (0,1].$$

Let

$$\alpha_1 = \min\left\{\frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} : 1 \leq k \leq n\right\},$$
$$\alpha_2 = \min\left\{\frac{\|x_k\|^2 - p_2^2 + 1}{2\|x_k\|} : 1 \leq k \leq n\right\}.$$

If $\alpha_1 \neq (1-p_1^2)^{\frac{1}{2}}$, or $\alpha_2 \neq (1-p_2^2)^{\frac{1}{2}}$, then we have the following strictly inequality

$$(2 - p_1^2 - p_2^2)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\| < \left\|\sum_{k=1}^{n} x_k\right\|.$$

Proof. If equality holds, then by Theorem 2.2 we have

$$(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\| \leq \left\|\sum_{k=1}^{n} x_k\right\| = (2 - p_1^2 - p_2^2)^{\frac{1}{2}}\sum_{k=1}^{n}\|x_k\|$$

and so

$$(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} \leq (2 - p_1^2 - p_2^2)^{\frac{1}{2}}.$$

On the other hand for $1 \leq k \leq n$,

$$\frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} \geq (1-p_1^2)^{\frac{1}{2}}$$

and so

$$\alpha_1 \geq (1-p_1^2)^{\frac{1}{2}}.$$

Similarly

$$\alpha_2 \geq (1-p_2^2)^{\frac{1}{2}}.$$

Hence

$$(2 - p_1^2 - p_2^2)^{\frac{1}{2}} \leq (\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}}.$$

Thus

$$\sqrt{\alpha_1^2 + \alpha_2^2} = (2 - p_1^2 - p_2^2)^{\frac{1}{2}}.$$

Therefore

$$\alpha_1 = (1-p_1^2)^{\frac{1}{2}} \quad \text{and} \quad \alpha_2 = (1-p_2^2)^{\frac{1}{2}},$$

a contradiction. □

The following result looks like Corollary 2 of [4].

Theorem 2.11. *Let a be a unit vector in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$, $M \geq m > 0$, $L \geq \ell > 0$ and $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$ such that*

$$\operatorname{Re}\langle Ma - x_k, x_k - ma\rangle \geq 0, \quad \operatorname{Re}\langle Lia - x_k, x_k - \ell ia\rangle \geq 0,$$

or equivalently,

$$\|x_k - \frac{m+M}{2}a\| \leq \frac{M-m}{2}, \quad \|x_k - \frac{L+\ell}{2}ia\| \leq \frac{L-\ell}{2}.$$

Let

$$\alpha_{m,M} = \min\left\{\frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} : 1 \leq k \leq n\right\}$$

and

$$\alpha_{\ell,L} = \min\left\{\frac{\|x_k\|^2 + \ell L}{(\ell+L)\|x_k\|} : 1 \leq k \leq n\right\}.$$

Then we have the inequlity

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| \leq \left\|\sum_{k=1}^{n} x_k\right\|.$$

The equality holds if and only if

$$\sum_{k=1}^{n} x_k = (\alpha_{m,M} + i\alpha_{\ell,L}) \sum_{k=1}^{n} \|x_k\| a.$$

Proof. For each $1 \leq k \leq n$, it follows from

$$\|x_k - \frac{m+M}{2}a\| \leq \frac{M-m}{2}$$

that

$$\left\langle x_k - \frac{m+M}{2}a, x_k - \frac{m+M}{2}\right\rangle \leq \left(\frac{M-m}{2}\right)^2.$$

Hence

$$\|x_k\|^2 + mM \leq (m+M)\operatorname{Re}\langle x_k, a\rangle.$$

Then

$$\frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|}\|x_k\| \leq \operatorname{Re}\langle x_k, a\rangle,$$

and consequently

$$\alpha_{m,M}\|x_k\| \leq \operatorname{Re}\langle x_k, a\rangle.$$

Similarly from the second inequality we deduce

$$\alpha_{\ell,L}\|x_k\| \leq \operatorname{Im}\langle x_k, a\rangle.$$

Applying Theorem 2.1 for $r_1 = \alpha_{m,M}$, $r_2 = \alpha_{\ell,L}$, we infer the desired inequality. □

Theorem 2.12. *Let a be a unit vector in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$, $M \geq m > 0$, $L \geq \ell > 0$ and $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$ such that*

$$\operatorname{Re}\langle Ma - x_k, x_k - ma\rangle \geq 0, \quad \operatorname{Re}\langle Lia - x_k, x_k - \ell ia\rangle \geq 0,$$

or equivalently

$$\left\|x_k - \frac{m+M}{2}a\right\| \leq \frac{M-m}{2}, \qquad \left\|x_k - \frac{L+\ell}{2}ia\right\| \leq \frac{L-\ell}{2}.$$

Let

$$\alpha_{m,M} = \min\left\{\frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} : 1 \leq k \leq n\right\}$$

and

$$\alpha_{\ell,L} = \min\left\{\frac{\|x_k\|^2 + \ell L}{(\ell+L)\|x_k\|} : 1 \leq k \leq n\right\}.$$

If $\alpha_{m,M} \neq 2\frac{\sqrt{mM}}{m+M}$*, or* $\alpha_{\ell,L} \neq 2\frac{\sqrt{\ell L}}{\ell+L}$*, then we have*

$$2\left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2}\right)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| < \left\|\sum_{k=1}^{n} x_k\right\|.$$

Proof. If

$$2\left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2}\right)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| = \left\|\sum_{k=1}^{n} x_k\right\|$$

then by Theorem 2.11 we have

$$\begin{aligned}(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\| &\leq \left\|\sum_{k=1}^{n} x_k\right\| \\ &= 2\left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2}\right)^{\frac{1}{2}} \sum_{k=1}^{n} \|x_k\|.\end{aligned}$$

Consequently

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \leq 2\left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2}\right)^{\frac{1}{2}}.$$

On the other hand for $1 \leq k \leq n$,

$$\frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} \geq 2\frac{\sqrt{mM}}{m+M}, \quad \text{and} \quad \frac{\|x_k\|^2 + \ell L}{(\ell+L)\|x_k\|} \geq 2\frac{\sqrt{\ell L}}{\ell+L},$$

so

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \geq 2\left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2}\right)^{\frac{1}{2}}.$$

Then

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} = 2\left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2}\right)^{\frac{1}{2}}.$$

Hence

$$\alpha_{m,M} = 2\frac{\sqrt{mM}}{m+M}$$

and

$$\alpha_{\ell,L} = 2\frac{\sqrt{\ell L}}{\ell+L}$$

a contradiction. □

Finally we mention two applications of our results to complex numbers.

Corollary 2.13. *Let* $a \in \mathbb{C}$ *with* $|a| = 1$. *Suppose that* $z_k \in \mathbb{C}$, $k \in \{1, \ldots, n\}$ *such that*

$$|z_k - a| \le p_1, \quad |z_k - ia| \le p_2, \quad p_1, p_2 \in \left(0, \sqrt{\alpha^2 + 1}\right),$$

where

$$\alpha = \min\{|z_k| : 1 \le k \le n\}.$$

Let

$$\alpha_1 = \min\left\{\frac{|z_k|^2 - p_1^2 + 1}{2|z_k|} : 1 \le k \le n\right\},$$
$$\alpha_2 = \min\left\{\frac{|z_k|^2 - p_2^2 + 1}{2|z_k|} : 1 \le k \le n\right\}.$$

Then we have the inequality

$$\sqrt{\alpha_1^2 + \alpha_2^2} \sum_{k=1}^{n} |z_k| \le \left|\sum_{k=1}^{n} z_k\right|.$$

The equality holds if and only if

$$\sum_{k=1}^{n} z_k = (\alpha_1 + i\alpha_2)\left(\sum_{k=1}^{n} |z_k|\right) a.$$

Proof. Apply Theorem 2.2 for $H = \mathbb{C}$. □

Corollary 2.14. *Let* $a \in \mathbb{C}$ *with* $|a| = 1$. *Suppose that* $z_k \in \mathbb{C}$, $k \in \{1, \ldots, n\}$ *such that*

$$|z_k - a| \le 1, \quad |z_k - ia| \le 1.$$

If $\alpha = \min\{|z_k| : 1 \le k \le n\}$. *Then we have the inequality*

$$\frac{\alpha}{\sqrt{2}} \sum_{k=1}^{n} |z_k| \le \left|\sum_{k=1}^{n} z_k\right|$$

the equality holds if and only if

$$\sum_{k=1}^{n} z_k = \alpha \frac{(1+i)}{2} \left(\sum_{k=1}^{n} |z_k|\right) a.$$

Proof. Apply Corollary 2.3 for $H = \mathbb{C}$. □

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