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SOME CONVERGENCE RESULTS ON SINC INTERPOLATION

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ABSTRACT. This paper is devoted to the investigation of sinc interpolation properties corresponding to a sequence of functions having the sinc function as a basis, the interpolation is taken over the dyadic partition of the interval $[0, 2\pi]$. In particular, a new class of functions for which the interpolation converge is introduced. The convergence of our interpolation processes is studied and answered in quite a comprehensive way. In fact, the paper aims to provide a guideline towards a large number of problems of interest in applied sciences.

Key words and phrases: Interpolation, Rate of Convergence, Sinc Approximation.

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1. INTRODUCTION

The sinc approximation method is a very promising method for function approximation, for approximation of derivatives, for approximate definite and indefinite integration, for solving initial value problems, for approximation and inversion of Fourier and Laplace transforms. The sinc method is an attractive alternative for numerical solutions to those problems which have no closed form. The theory of sinc series on the whole real line is developed in [8]. There are several reasons to approximate by sinc functions. Firstly, they are easily implemented and give good accuracy for problems with singularities; approximations by sinc function are typified by errors of the form $\mathcal{O}(\exp(-c/h))$ where $c > 0$ is a constant and h is a step size. Secondly, approximation by sinc functions handles singularities in the problem. The effect of any such singularities will appear in some form in any scheme of numerical solution, and it is well known

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that polynomial methods do not perform well near singularities. Finally, these kinds of approximation yield both an effective and rapidly convergent scheme for solving the problem, and so circumvents the instability problems that one typically encounters in some difference methods. Numerical processes of interpolation on the real line, with the help of adroitly selected conformal maps is adapted to handle these same processes on finite intervals, or in general on other subsets of the real line. For more details see, [3, 4, 5]. Also, it is worthy to mention the work by Stenger [9], where he presents practically useful constructive linear methods of approximation of analytic functions by polynomials, sinc functions and rational functions. In [6], the author proves some convergence results on finite intervals, using the linear combination of the basis functions $B_{n,k} = S(k,h) \circ \sin h^{-1}\left(\cos h^{-1}\left(\frac{1}{|x|}\right)\right)$ where $k = -n, \dots, n, h = \log n/n$, and $S(k,h)$ is the sinc translated function, to be defined later.

Although there is no unique choice for the conformal map, and so one will not guarantee an exponential decay of the convergence rate using the sinc method. It should be pointed out that it might be possible that the selection of the conformal mapping does not lead to a symmetric discrete system. While a symmetric approximation system is not necessary for a good approximation, it is computationally efficient and analytically advantageous for solving the discrete system. As a final note on selection availability of the conformal mapping. In problems where two (or more) maps are applicable, the use of either of the maps leads to a smaller size of the discrete system, for example, in the case of the domain $(0,\infty)$ there are available the selections $\ln(x)$ and $\ln(\sinh(x))$. The map $\ln(x)$ often leads to a smaller discrete system that does the map $\ln(\sinh(x))$ for equivalent accuracy. To avoid these difficulties and as an alternative for the extension (using conformal maps) made by Stenger [8], this paper is devoted to the investigation of sinc interpolation on the interval $[0,2\pi]$ (see, [7]). The paper is organized as follows. In Section 2 we define our interpolation processes $S_n(f;x)$, where the nodes are taken to be the diadic partition of the interval $[0,2\pi]$. We then study some basic properties of the interpolating function $S_n(f;x)$. In Section 3 we take up the functional properties of $S_n(f;x)$. Section 4 deals with new classes of functions for which the interpolation processes converges. In the last section of this chapter, we give the most important convergence results in this paper.

2. THE INTERPOLATION PROCESSES

Let $E_1 = \{0,\pi,2\pi\}$, and $E_2 = \{0,\pi/2,\pi,3\pi/2,2\pi\}$. In general let

$$E_n = \left\{ \frac{2k\pi}{2^n}, \ \ 0 \le k \le 2^n \right\}. \tag{2.1}$$

In the following Lemma we state, without proof, some properties of the partition E_n

Lemma 2.1. *For the sets E_n the following holds true*

(1) *The sequence $\{E_n\}$ is an increasing sequence, i.e, $E_1 \subset E_2 \subset \dots$.*
(2) *$E = \cup_{n=1}^{\infty} E_n$ is dense subset of $[0,2\pi]$.*

Definition 2.1. Let $f : [0,2\pi] \to \mathbb{R}$ be any function. For each natural number n we define,

$$S_n(f,x) = \sum_{x_k \in E_n} f(x_k) L_{n,k}(x), \tag{2.2}$$

where

$$L_{n,k}(x) = \begin{cases} \dfrac{\sin[2^{(n-1)}(x-x_k)]}{2^{(n-1)}(x-x_k)}, & x \neq x_k \\ 1, & x = x_k \end{cases} \tag{2.3}$$

and $x_k = \frac{2k\pi}{2^n}$ for $0 \le k \le 2^n$.

In the following sequence of lemmas we will give the basic properties of $S_n(f,x)$ as an interpolating function.

Lemma 2.2. *For any natural numbers* n, k *and* j *where* $0 \le k, j \le 2^n$ *we have*

$$L_{n,k}(x_j) = \delta_{j,k}.$$

Proof. If $j = k$ then $L_{n,k}(x_j) = 1$ by the definition of $L_{n,k}(x)$. Now if $j \neq k$, we have

$$\begin{aligned} L_{n,k}(x_j) &= \frac{\sin[2^{n-1}(x_j - x_k)]}{2^{n-1}(x_j - x_k)} \\ &= \frac{\sin\left[2^{n-1}\left(\frac{2j\pi}{2^n} - \frac{2k\pi}{2^n}\right)\right]}{2^{n-1}(x_j - x_k)} \\ &= \frac{\sin[(j-k)\pi]}{2^{n-1}(x_j - x_k)} \\ &= 0. \end{aligned}$$

This completes the proof. □

Lemma 2.3. $S_n(f,x)$ *interpolates* f *on* E_n *for any function* f *defined on* $[0, 2\pi]$*. i.e.,* $S_n(f, x_k) = f(x_k)$ *for all* $x_k \in E_n$*.*

Proof.

$$\begin{aligned} S_n(f, x_k) &= \sum_{j=0}^{2^n} f(x_j) L_{n,j}(x_k) \\ &= f(x_k) L_{n,k}(x_k) + \sum_{x_j \neq x_k} f(x_j) L_{n,j}(x_k) \\ &= f(x_k). \end{aligned}$$

Since x_k is an arbitrary element of E_n, the result follows. □

We have shown that $S_n(f,x)$ interpolates f on the set E_n, the following lemma is a generalization:

Lemma 2.4. *For any real valued function* f *on* $[0, 2\pi]$*, if* $\lim_{n\to\infty} S_n(f,x) = g(x)$ *then* $g(x) = f(x)$ *for all* $x \in E$*.*

Proof. Let x be arbitrary element of E. Since E is the union of the sets $E_n's$ there must be n_0 such that $x \in E_n$ for all $n \ge n_0$. Now for $n \ge n_0$ we have $S_n(f,x) = f(x)$ therefore $\lim_{n\to\infty} S_n(f,x) = f(x)$, thus $g(x) = f(x)$. Since x is an arbitrary element of E the result follows. □

Lemma 2.5. $S_n(f,x) = S_m(f,x)$ *for all* x *in* E_k*, where* $k = \min(n,m)$*.*

Proof. Let n and m be any two natural numbers, and $k = \min(m,n)$. Let $x_\ell \in E_k$ for some ℓ, then $x_\ell \in E_n$ because $k \le n$. But S_n interpolates f on E_n, so that $S_n(f, x_\ell) = f(x_\ell)$. By the same argument, we can show that $S_m(f, x_\ell) = f(x_\ell)$. □

The following sequence of lemmas give some result on the derivative of the basis of the interpolation, from which one can approximate the solution for some differential equations.

Lemma 2.6. *For any natural numbers* n *and* k *with* $0 \le k \le 2^n$ *we have*

$$L'_{n,k}(x_k) = 0. \tag{2.4}$$

Proof. For $x = x_k$, we have

$$\begin{aligned} L'_{n,k}(x_k) &= \lim_{x\to x_k} \frac{L_{n,k}(x) - L_{n,k}(x_k)}{x - x_k} \\ &= \lim_{x\to x_k} \frac{\sin[2^{n-1}(x-x_k)] - 2^{n-1}(x-x_k)}{2^{n-1}(x-x_k)^2} \\ &= \lim_{x\to x_k} \frac{2^{n-1}\cos[2^{n-1}(x-x_k)] - 2^{n-1}}{22^{n-1}(x-x_k)} \\ &= \lim_{x\to x_k} \frac{-2^{n-1}2^{n-1}\sin[2^{n-1}(x-x_k)]}{22^{n-1}(1)} \\ &= 0 \end{aligned}$$

and the lemma is proved. □

Lemma 2.7. *For any natural numbers n and k with $0 \le k \le 2^n$ we have*

$$L'_{n,k}(x_j) = \frac{2^n \times (-1)^{j-k}}{2\pi(j-k)}. \tag{2.5}$$

Proof.

$$\begin{aligned} L'_{n,k}(x_j) &= \lim_{x\to x_j} \frac{L_{n,k}(x) - L_{n,k}(x_j)}{x - x_k} \\ &= \lim_{x\to x_j} \frac{\sin[2^{n-1}(x-x_k)]}{2^{n-1}(x-x_k)(x-x_j)} \\ &= \lim_{x\to x_j} \frac{2^{n-1}\cos[2^{n-1}(x-x_j)]}{2^{n-1}} \lim_{x\to x_j} \frac{1}{x-x_k} \\ &= \frac{(-1)^{j-k}}{x_j - x_k} \\ &= \frac{2^n \times (-1)^{j-k}}{2\pi(j-k)}. \end{aligned}$$

Now the proof is complete. □

For the second derivative, we have

Lemma 2.8. *For any natural numbers n and k with $0 \le k \le 2^n$ we have*

$$L''_{n,k}(x_k) = \frac{-2^{2n-2}}{3}. \tag{2.6}$$

Proof.

$$\begin{aligned} L''_{n,k}(x_k) &= \lim_{x\to x_k} \frac{L'_{n,k}(x) - L'_{n,k}(x_k)}{x - x_k} \\ &= \lim_{x\to x_k} \frac{2^{n-1}(x-x_k)\cos[2^{n-1}(x-x_k)] - \sin[2^{n-1}(x-x_k)]}{2^{n-1}(x-x_k)^3} \\ &= \lim_{x\to x_j} \frac{-2^{n-1}2^{n-1}(x-x_k)\sin[2^{n-1}(x-x_k)]}{3.2^{n-1}(x-x_k)^2} \\ &= \frac{-2^{n-1}.2^{n-1}}{3} = \frac{-2^{2n-2}}{3}. \end{aligned}$$

□

The following is a connection between our interpolation and the Lagrange interpolation.

Lemma 2.9. *Let f be a real-valued function on $[0, 2\pi]$ and let H_n denote the Lagrange interpolation function on E_n, then $S_n(f, x) = S_n(H_n, x)$.*

Proof. We recall that $H_n(x_k) = f(x_k)$ for all $x_k \in E_n$. Now,

$$\begin{aligned} S_n(f, x) &= \sum_{x_k \in E_n} f(x_k) L_{n,k}(x) \\ &= \sum_{x_k \in E_n} H_n(x_k) L_{n,k}(x) \\ &= S_n(H_n, x). \end{aligned}$$

□

In fact the previous lemma is a particular case of the following lemma.

Lemma 2.10. *Let f and g be two real-valued functions defined on $[0, 2\pi]$ such that $g(x_k) = f(x_k)$ for all $x_k \in E_n$ for some natural number n, then $S_n(f, x) = S_n(g, x)$ for all $x \in [0, 2\pi]$.*

3. THE FUNCTIONAL PROPERTIES OF S_n

Notice that S_n can be considered as an operator on the space of all real-valued functions on $[0, 2\pi]$. So that, it is convenient to study the functional properties of S_n as an operator on the dual space of $[0, 2\pi]$. Note that $S_n(f, x)$ is a linear operator on the space of all functions defined on $[0, 2\pi]$.

Lemma 3.1. *For each natural number n, the operator S_n is a bounded linear operator.*

Proof. In order to show the boundedness of S_n, we find a real number c_n such that $\|S_n(f)\| \leq c_n \|f\|$. This is an immediate result. □

Corollary 3.2. *For each natural number n, S_n is a continuous linear operator.*

We have shown that S_n is a continuous function on $[0, 2\pi]$ and is a continuous linear operator on the dual space of $[0, 2\pi]$. Therefore, S_n is a continuous maping in both its components, i.e., if we consider

$$S_n : X \times [0, 2\pi] \longrightarrow \mathbb{R},$$

then S_n is continuous on $X \times [0, 2\pi]$, where X is the dual space of $[0, 2\pi]$.

The following few results give us some fixed points for S_n.

Lemma 3.3. *For any natural numbers n and k where $0 \leq k \leq 2^n$ we have*

$$S_n(L_{n,k}, x) = L_{n,k}(x). \tag{3.1}$$

Proof.

$$\begin{aligned} S_n(L_{n,k}, x) &= \sum_{x_j \in E_n} L_{n,k}(x_j) L_{n,j}(x) \\ &= L_{n,k}(x_k) L_{n,k}(x) + \sum_{x_j \neq x_k} L_{n,k}(x_j) L_{n,j}(x) \\ &= L_{n,k}(x). \end{aligned}$$

□

Lemma 3.4. *For any natural number n and for any function f on $[0, 2\pi]$ we have $S_n(S_n(f), x) = S_n(f, x)$.*

Proof.

$$\begin{aligned} S_n(S_n(f),x) &= \sum_{x_k\in E_n} S_n(f,x_k)L_{n,k}(x) \\ &= \sum_{x_k\in E_n}\sum_{x_j\in E_n} f(x_j)L_{n,j}(x_k)L_{n,k}(x) \\ &= \sum_{x_k\in E_n}\sum_{x_j\in E_n} f(x_k)\delta_{k,j}L_{n,k}(x) \\ &= \sum_{x_k\in E_n} f(x_k)L_{n,k}(x) = S_n(f,x). \end{aligned}$$

□

Thus S_n as a linear operator on the class of all real valued functions defined on $[0,2\pi]$ has at least 2^n+2 fixed points.

Talking about fixed points of a linear operator leads to talking about the contraction which is considered in the following corollary.

Theorem 3.5. *For any function f on $[0,2\pi]$, the operator $S_n(f,x)$ is not a contraction.*

Proof. We firstly recall that the space of all real valued functions on $[0,2\pi]$ is a complete metric space with respect to the metric

$$d(f,g) = \sup\{f(x)-g(x) : x\in[0,2\pi]\}. \tag{3.2}$$

Now, if we assume on the contrary that $S_n(f,x)$ is a contraction, then S_n will satisfy the requirments of the "Banach fixed point theorem", and, hence, S_n has a unique fixed point. The last statement is a contradiction because S_n has at least 2^n+2 fixed points. Thus, we conclude that S_n is not a contraction. □

Since S_n is not a contraction, one may ask about the relation between $d(S_n(f),S_n(g))$ and $d(f,g)$. The following theorem answers this question.

Theorem 3.6. *Let f and g be any two functions on $[0,2\pi]$. For each natural number n, we have*

$$d(S_n(f),S_n(g)) \le (2^n+1)d(f,g). \tag{3.3}$$

Proof.

$$\begin{aligned} d(S_n(f),S_n(g)) &= \sup_{[0,2\pi]} |S_n(f,x)-S_n(g,x)| \\ &= \sup_{[0,2\pi]} |S_n(f-g)| \\ &= \sup_{[0,2\pi]} \left|\sum_{k=0}^{2^n} [f(x_k-g(x_k)]\, L_{n,k}(x)\right| \\ &\le \sup_{[0,2\pi]} \sum_{k=0}^{2^n} |f(x_k)-g(x_k)||L_{n,k}(x)| \\ &\le \sup_{[0,2\pi]} \sum_{k=0}^{2^n} |f(x_k)-g(x_k)| \end{aligned}$$

$$\leq \sup_{[0,2\pi]} \sum_{k=0}^{2^n} |f(x) - g(x)| = (2^n + 1)d(f, g).$$

□

4. SPECIAL CLASSES OF FUNCTIONS

Since the nodes of the interpolation are of the form $\frac{2k\pi}{2^n}$, one can think about the limit; $\lim_{n\to\infty} \sum_{k=0}^{2^n} \left| f\left(\frac{2k\pi}{2^n}\right)\right|$. In fact this idea introduces the following definition.

Definition 4.1. Let $U[0, 2\pi]$ be the class of all real valued-functions f on $[0, 2\pi]$ and for which

$$\lim_{n\to\infty} \sum_{k=0}^{2^n} \left| f\left(\frac{2k\pi}{2^n}\right)\right| < \infty.$$

Lemma 4.1. *The condition in the last lemma is equivalent to the condition*

$$\sum_{x_k \in E} |f(x_k)| < \infty.$$

Proof. We show that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n} \left| f\left(\frac{2k\pi}{2^n}\right)\right| = \sum_{x_k \in E} |f(x_k)|$$

in order to prove the lemma. For, numerate the countable set E as following:
The elements of E_1 are x_0, x_1 and x_2.
The elements of $E_2 - E_1$ are x_3, x_4.
The elements of $E_3 - E_2$ are $x_5, ..., x_9$.
In general, the elements of $E_{n+1} - E_n$ are $x_{2^n+1}, ..., x_{2^{n+1}}$. Now,

$$\sum_{x_k \in E} |f(x_k)| = \lim_{n\to\infty} \sum_{k=0}^{2^n} |f(x_k)| = \lim_{n\to\infty} \sum_{k=0}^{2^n} \left| f\left(\frac{2k\pi}{2^n}\right)\right|$$

the last equation is valid because of our choice of the numeration. □

The reader must realize that any rearrangement of the above sums is not important because we are dealing with absolute sums.

Example 4.1. Here we give an example of a function that belongs to the class $U[0, 2\pi]$, i.e, we show that $U[0, 2\pi] \neq \emptyset$. For, let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{k^2}, & x = x_k, \\ 1, & x \neq x_k. \end{cases}$$

Here we consider some numeration for the countable set E. It is clear that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n} \left| f\left(\frac{2k\pi}{2^n}\right)\right| = \sum_{x_k \in E} |f(x_k)| = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Therefore, $f \in U[0, 2\pi]$

Example 4.2. In this example we show that the class $U[0, 2\pi]$ doesn't contain any polynomial of the form $f(x) = ax^m$. For, consider

$$a_n = \sum_{k=0}^{2^n} \left| f\left(\frac{2k\pi}{2^n}\right)\right| = \sum_{k=0}^{2^n} \left| a\frac{(2k\pi)^m}{2^{nm}}\right| = |a|\frac{2^m\pi^m}{2^{nm}} \sum_{k=0}^{2^n} k^m.$$

Although the exact formula for the last sum needs more complicated computations, we know that this sum will be a polynomial in 2^n of degree $m+1$. Thus, $a_n = |a|\frac{(2\pi)^m}{2^{nm}} \times g(m+1)$, where $g(m+1)$ is the indicated polynomial. Now it is clear that, $\lim_{n\to\infty} a_n = \infty$. Therefore $f \neq U[0, 2\pi]$.

Lemma 4.2. *If f is any real valued function on $[0, 2\pi]$ such that $|f|$ is integrable in the sense of Reimann and $f \in U[0, 2\pi]$ then $\int_0^{2\pi} |f(x)|dx = 0$.*

Proof. For each natural number n, $E_n = \left\{\frac{2k\pi}{2^n}, 0 \le k \le 2^n\right\}$ is a partition for $[0, 2\pi]$. The subintervals of this partition are

$$\left[0, \frac{2\pi}{2^n}\right], \left[\frac{2\pi}{2^n}, \frac{4\pi}{2^n}\right], \dots, \left[\frac{2(2^n-1)\pi}{2^n}, 2\pi\right].$$

Now consider the Riemann sum of f over this partition, $R_n(f) = \frac{2\pi}{2^n}\sum_{k=1}^{2^n} |f(x_k^*)|$ where x_k^* is any point of the $k-th$ interval of the partition. Since $|f|$ is integrable (in the sense of Riemann) we can take x_k^* to be $x_k = 2k\pi$ and, hence,

$$R_n(|f|) = \frac{2\pi}{2^n}\sum_{k=1}^{2^n} |f(x_k)| = \frac{2\pi}{2^n}\sum_{k=1}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|.$$

Now write the integral of f as the limit of a Riemann sum to get

$$\int_0^{2\pi} |f(x)|dx = \lim_{n\to\infty} R_n(|f|) = \lim_{n\to\infty} \frac{2\pi}{2^n}\sum_{k=1}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|.$$

But since $f \in U[0, 2\pi]$ we have $\lim_{n\to\infty} \sum_{k=1}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|$ is finite. Thus,

$$\int_0^{2\pi} |f(x)|dx = \lim_{n\to\infty} \frac{2\pi}{2^n} \times (\text{finite value}) = 0.$$

□

Lemma 4.3. *If $f \in U[0, 2\pi]$ and $|f|$ is integrable in the sense of Riemann, then $[0, 2\pi]$ does not contain any interval I such that $f(x) \neq 0$ for all $x \in I$.*

Proof. Assume that there is an interval $I \subset [0, 2\pi]$ such that $f(x) \neq 0$ for all $x \in I$, then

$$\begin{aligned}\int_0^{2\pi} |f(x)|dx &= \int_I |f(x)dx + \int_{[0,2\pi]-I} |f(x)|dx \\ &\ge \int_I |f(x)|dx \\ &> 0\end{aligned}$$

but this contradicts the last lemma, and the lemma is proved. □

Lemma 4.4. *The only continuous function in $U[0, 2\pi]$ is the zero function.*

Proof. Let f be a non-zero continuous function on $[0, 2\pi]$, then there is at least one $x \in [0, 2\pi]$ such that $f(x) \neq 0$. Since f is continuous at x, there must be an interval I containing x for which $f(x) \neq 0$ for all $x \in I$. But $m(I) > 0$ and this contradicts the last lemma. □

5. SOME CONVERGENCE RESULTS

Lemma 5.1. *For any natural numbers n and k with $0 \le k \le 2^n$, we have*

$$L_{n,k}(x) = \frac{1}{2^n}\int_{-2^{n-1}}^{2^{n-1}} \exp\left(\frac{2\pi kit}{2^n}\right) e^{-ixt}dt. \tag{5.1}$$

Proof.

$$\begin{aligned}
&\frac{1}{2^n}\int_{-2^{n-1}}^{2^{n-1}} e^{\frac{2\pi kit}{2^n}-ixt}dt \\
&\quad= \frac{1}{2^n}\int_{-2^{n-1}}^{2^{n-1}} e^{i\left(\frac{2k\pi}{2^n}-x\right)t}dt \\
&\quad= \frac{1}{2^n}\frac{1}{i\left(\frac{2\pi k}{2^n}-x\right)}\left[e^{i\left(\frac{2k\pi}{2^n}-x\right)t}\right]_{-2^{n-1}}^{2^{n-1}} \\
&\quad= \frac{1}{2^n}\frac{1}{i\left(\frac{2\pi k}{2^n}-x\right)}\left[e^{i\left(\frac{2k\pi}{2^n}-x\right)2^{n-1}} - e^{-i\left(\frac{2k\pi}{2^n}-x\right)2^{n-1}}\right] \\
&\quad= \frac{1}{2^n}\frac{1}{i\left(\frac{2\pi k}{2^n}-x\right)}\left[\cos[2^{n-1}\left(\frac{2k\pi}{2^n}-x\right)]\right] \\
&\qquad\quad + \frac{1}{2^n}\frac{1}{i\left(\frac{2\pi k}{2^n}-x\right)}\left[i\sin\left[2^{n-1}\left(\frac{2k\pi}{2^n}-x\right)\right] - \cos\left[2^{n-1}\left(\frac{2k\pi}{2^n}-x\right)\right]\right] \\
&\qquad\quad + \frac{1}{2^n}\frac{1}{i\left(\frac{2\pi k}{2^n}-x\right)}\left[i\sin\left[2^{n-1}\left(\frac{2k\pi}{2^n}-x\right)\right]\right] \\
&\quad= \frac{\sin\left[2^{n-1}\left(x-\frac{2k\pi}{2^n}\right)\right]}{2^{n-1}\left(x-\frac{2k\pi}{2^n}\right)} \\
&\quad= L_{n,k}(x).
\end{aligned}$$

The last proof is valid whenever $x \neq x_k$; the case where $x = x_k$ is easy to seen. □

Corollary 5.2. *The Fourier transform of $L_{n,k}(x)$ is*

$$F(t) = \begin{cases} \frac{2\pi}{2^n}\exp\left(\frac{2ki\pi t}{2^n}\right), & |t| < 2^{n-1} \\ 0, & |t| > 2^{n-1} \end{cases} \tag{5.2}$$

Proof. Let F be the Fourier transform of f, then F must satisfy the equation

$$L_{n,k}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixt}F(x)dx.$$

By the last lemma we find that the function defined in equation (5.2) satisfies this condition, and since the Fourier transform is unique, the result follows. □

Corollary 5.3. *For any natural numbers n, k and j where $0 \le k, j \le 2^n$ we have*

$$\int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx = \frac{2\pi}{2^n}\delta_{k,j}. \tag{5.3}$$

Proof. Let F and G denote the Fourier transforms of $L_{n,k}$ and $L_{n,j}$ respectively, then by Parseval's theorem,

$$\int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} F(t)\overline{G(t)}dt.$$

Firstly, if $k = j$, then

$$\int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx = \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} \frac{4\pi^2}{2^{2n}}dt = \frac{2\pi}{2^n} = \frac{2\pi}{2^n}\delta_{k,k}.$$

Secondly, if $k \neq j$ then

$$\begin{aligned}\int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx &= \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} \frac{4\pi^2}{2^{2n}}e^{2\pi it\left(\frac{k-j}{2^n}\right)}dt \\ &= \frac{2\pi}{2^{2n}}\frac{2^n}{2i\pi(k-j)}\left[e^{2\pi it\left(\frac{k-j}{2^n}\right)}\right]_{-2^{n-1}}^{2^{n-1}} \\ &= \frac{1}{2^n i(k-j)}2i\sin[(k-j)\pi] \\ &= 0.\end{aligned}$$

□

Corollary 5.4. *Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$. For each natural nuber n we define*

$$F_n(x) = \begin{cases} \dfrac{2\pi}{2^n}\displaystyle\sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right)\exp\left(\frac{2k\pi ix}{2^n}\right), & |x| < 2^{n-1}, \\ 0, & |x| > 2^{n-1}, \end{cases} \tag{5.4}$$

then for any natural number n, we have

$$S_n(f,t) = \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} F_n(x)e^{-ixt}dx. \tag{5.5}$$

In fact, F_n is the Fourier transform of S_n.

Proof.

$$\begin{aligned}\frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} F_n(x)e^{-itx}dx &= \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} \frac{2\pi}{2^n}\sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right)e^{\frac{2k\pi ix}{2^n}}e^{-itx}dx \\ &= \sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right)\int_{-2^{n-1}}^{2^{n-1}} e^{\frac{2k\pi ix}{2^n}}e^{-itx}dx \\ &= \sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right)L_{n,k}(t) \\ &= S_n(f,t).\end{aligned}$$

□

Corollary 5.5. *Let f be a real-valued function such that both f and $|f|$ are integrable in the sense of Riemann, and such that $f = 0$ outside $[0, 2\pi]$, also let F be the Fourier transform of f, then*

$$\lim_{n\to\infty} F_n(x) = F(x), \tag{5.6}$$

where F_n is defined in Corollary 5.4. Moreover $F_n \longrightarrow F$ uniformly.

Proof. For fix $x \in [-2^{n-1}, 2^{n-1}]$, let

$$g(t) = f(t)\exp(itx), \qquad t \in [0, 2\pi] \tag{5.7}$$

and consider the n-th Reimann sum of g over $[0, 2\pi]$;

$$R_n(g) = \frac{2\pi}{2^n} \sum_{k=1}^{2^n} f\left(\frac{2k\pi}{2^n}\right) \exp\left(\frac{2k\pi ix}{2^n}\right) \tag{5.8}$$

therefore,

$$\begin{aligned} \lim_{n\to\infty} R_n(g) &= \lim_{n\to\infty} \frac{2\pi}{2^n} \sum_{k=1}^{2^n} f\left(\frac{2k\pi}{2^n}\right) \exp\left(\frac{2k\pi ix}{2^n}\right) \\ &= \lim_{n\to\infty} F_n(x), \end{aligned}$$

which implies

$$\int_0^{2\pi} f(t)\exp(itx)dt = \lim_{n\to\infty} F_n(x)$$

thus,

$$F(x) = \lim_{n\to\infty} F_n(x)$$

The uniformly convergence fact follows because

$$\lim_{n\to\infty} \sum_{k=0}^{2^n} \left| \frac{2\pi}{2^n} f\left(\frac{2k\pi}{2^n}\right) \exp\left(\frac{2k\pi ix}{2^n}\right) \right| \le \lim_{n\to\infty} \sum_{k=0}^{2^n} \left| \frac{2\pi}{2^n} f\left(\frac{2k\pi}{2^n}\right) \right| \tag{5.9}$$

and the last series converges to a real number because $|f|$ is integrable. By the $M-$*test* we have the result. And the corollary is proved. □

Corollary 5.6. *For any natural numbers n and k with $0 \le k \le 2^n$, we have*

$$f\left(\frac{2k\pi}{2^n}\right) = \frac{1}{2\pi} \int_{-2^{n-1}}^{2^{n-1}} F_n(x) \exp\left(\frac{-2k\pi ix}{2^n}\right) dx, \tag{5.10}$$

where $\{F_n\}$ as defined in corollary 5.4.

Proof. Consider the Fourier series representation for F_n in $(-2^{n-1}, 2^{n-1})$;

$$F_n(x) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{2k\pi ix}{2^n}\right), \qquad |x| < 2^{n-1}.$$

Compare this with the definition of $F_n(x)$ to get $c_k = 0$ for $k < 0$ and $k > 2^n$ and $c_k = \frac{2\pi}{2^n} f\left(\frac{2k\pi}{2^n}\right)$ for $0 \le k \le 2^n$. Also we know that

$$c_k = \frac{1}{2^n} \int_{-2^{n-1}}^{2^{n-1}} F(x) \exp\left(\frac{-2k\pi ix}{2^n}\right) dx,$$

and hence

$$f\left(\frac{2k\pi}{2^n}\right) = \frac{1}{2\pi} \int_{-2^{n-1}}^{2^{n-1}} F(x) \exp\left(\frac{-2k\pi ix}{2^n}\right) dx.$$

□

The last corollary gives rise to the following new class of functions.

Definition 5.1. Let $J[0, 2\pi]$ be the class of all real-valued functions, f on $[0, 2\pi]$ and 0 outside this interval) for which there is a function F_n satisfying the following conditions:

- $F_n(x) = 0$ for all x outside $(-2^{n-1}, 2^{n-1})$ for some natural number n
- $f(t) = \frac{1}{2\pi} \int_{-2^{n-1}}^{2^{n-1}} F_n(x)e^{-ixt}$ for all $t \in [0, 2\pi]$.

Although the class $J[0,2\pi]$ seems to be very complicated, it has many nice properties. In the following sequence of theorems we give the most important properties for this class.

Theorem 5.7. *For any function $f \in J[0,2\pi]$ we have the series representation*

$$f(x) = S_n(f,x),$$

where n is as in the definition of $J[0,2\pi]$ and

$$S_n(f,x) = \sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right) L_{n,k}(x). \tag{5.11}$$

Proof. Since $f \in J[0,2\pi]$, there exists n_0 such that

$$f(t) = \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} e^{-ixt} F_n(x)dx$$

for all $n \geq n_0$. The function F in the last equation can be represented on the interval $(-2^{n-1}, 2^{n-1})$ by its Fourier series representation, i.e.

$$F_n(x) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{2k\pi i x}{2^n}\right), -2^{n-1} < x < 2^{n-1}$$

with

$$c_k = \frac{1}{2^n}\int_{-2^{n-1}}^{2^{n-1}} F_n(x)\exp\left(-\frac{2k\pi i x}{2^n}\right)dx.$$

By our choice of F_n we have, $c_k = \frac{2\pi}{2^n} f\left(\frac{2k\pi}{2^n}\right)$.

Substitute this value in the Fourier series of F to get

$$F_n(x) = \begin{cases} \dfrac{2\pi}{2^n}\displaystyle\sum_{k=-\infty}^{\infty} f\left(\frac{2k\pi}{2^n}\right)\exp\left(\frac{2ki\pi x}{2^n}\right), & |x| < 2^{n-1}, \\ 0, & |x| > 2^{n-1}. \end{cases}$$

Now,

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} e^{-ixt}F(x)dx \\ &= \frac{1}{2\pi}\int_{-2^{n-1}}^{2^{n-1}} \frac{2\pi}{2^n}\sum_{k=-\infty}^{\infty} f\left(\frac{2k\pi}{2^n}\right)\exp\left(\frac{2ki\pi x}{2^n}\right)e^{-ixt}dx \\ &= \frac{1}{2^n}\sum_{k=-\infty}^{\infty} f\left(\frac{2k\pi}{2^n}\right)\int_{-2^{n-1}}^{2^{n-1}}\exp\left(\frac{2ki\pi x}{2^n} - ixt\right)dx \\ &= \sum_{k=-\infty}^{\infty} f\left(\frac{2k\pi}{2^n}\right)L_{n,k}(t) = \sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right)L_{n,k}(t) = S_n(f,t). \end{aligned}$$

□

Theorem 5.8. *If $f \in J[0,2\pi]$ then for n as in Definition 5.1,*

$$\int_0^{2\pi} |f(x)|^2 dx = \frac{2\pi}{2^n}\sum_{k=0}^{2^n}\left|f\left(\frac{2k\pi}{2^n}\right)\right|^2.$$

Proof. Since $f \in J[0, 2\pi]$ we have

$$f(x) = \sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right) L_{n,k}(x)$$

for n as in Definition 5.1 . Now,

$$\begin{aligned}
&\int_{\mathbb{R}} |f(x)|^2 dx \\
&\qquad = \int_{\mathbb{R}} \left[\sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right) L_{n,k}(x)\right]^2 dx \\
&\qquad = \int_{\mathbb{R}} \left[\sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 L_{n,k}^2(x) + \sum_{k\neq j} f\left(\frac{2k\pi}{2^n}\right) f\left(\frac{2j\pi}{2^n}\right) L_{n,k}(x)L_{n,j}(x)\right] dx \\
&\qquad = \sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 \int_{\mathbb{R}} L_{n,k}^2(x)dx + \sum_{k\neq j} \left(\frac{2k\pi}{2^n}\right) f\left(\frac{2j\pi}{2^n}\right) \int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx \\
&\qquad = \frac{2\pi}{2^n} \sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 .
\end{aligned}$$

□

The following theorem tells us some type of convergence of our interpolation.

Theorem 5.9. *Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ such that f^2 is integrable in the sense of Riemann, then*

$$\int_0^{2\pi} |f(x)|^2 dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} S_n^2(f, x)dx \tag{5.12}$$

Proof. Firstly, we notice that

$$\begin{aligned}
S_n^2(f, x) &= \left(\sum_{k=0}^{2^n} f\left(\frac{2k\pi}{2^n}\right) L_{n,k}(x)\right)^2 \\
&= \sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 L_{n,k}^2(x) + \sum_{k\neq j} f\left(\frac{2k\pi}{2^n}\right) f\left(\frac{2j\pi}{2^n}\right) L_{n,k}(x)L_{n,j}(x).
\end{aligned}$$

Now integrate both sides of the last equation on $\mathbb{R}$, to get

$$\begin{aligned}
&\int_{\mathbb{R}} S_n^2(f, x)dx \\
&\qquad = \int_{\mathbb{R}} \left(\sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 L_{n,k}^2(x) + \sum_{k\neq j} f\left(\frac{2k\pi}{2^n}\right) f\left(\frac{2j\pi}{2^n}\right) L_{n,k}(x)L_{n,j}(x)\right) dx \\
&\qquad = \sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 \int_{\mathbb{R}} L_{n,k}^2 dx + \sum_{k\neq j} f\left(\frac{2k\pi}{2^n}\right) f\left(\frac{2j\pi}{2^n}\right) \int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx \\
&\qquad = \frac{2\pi}{2^n} \sum_{k=0}^{2^n} \left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 .
\end{aligned}$$

The validity of the last equation arises from corollary 5.3. Now take the limit of the last equation as $n \longrightarrow \infty$ to get

$$\lim_{n\to\infty}\int_{\mathbb{R}} S_n^2(f,x)dx = \lim_{n\to\infty}\frac{2\pi}{2^n}\sum_{k=0}^{2^n}\left|f\left(\frac{2k\pi}{2^n}\right)\right|^2$$

but

$$\lim_{n\to\infty}\frac{2\pi}{2^n}\sum_{k=0}^{2^n}\left|f\left(\frac{2k\pi}{2^n}\right)\right|^2 = \int_0^{2\pi}|f(x)|^2dx$$

because it is the limit of the Riemann sum for f^2 on $[0, 2\pi]$, this completes the proof of the theorem. □

Lemma 5.10. *Let f be any real-valued function on $[0, 2\pi]$, then for any natural numbers n and k with $0 \le k \le 2^n$, we have*

$$\int_{\mathbb{R}} S_n(f,x)L_{n,k}(x)dx = \frac{2\pi}{2^n}f\left(\frac{2k\pi}{2^n}\right). \tag{5.13}$$

Proof. We recall that

$$S_n(f,x) = \sum_{j=0}^{2^n} f\left(\frac{2j\pi}{2^n}\right)L_{n,j}(x),$$

multiply both sides by $L_{n,k}(x)$ and integrate on $\mathbb{R}$ to get:

$$\begin{aligned}
\int_{\mathbb{R}} S_n(f,x)L_{n,k}(x)dx &= \int_{\mathbb{R}}\sum_{j=0}^{2^n} f\left(\frac{2j\pi}{2^n}\right)L_{n,j}(x)L_{n,k}(x)dx \\
&= f\left(\frac{2k\pi}{2^n}\right)\int_{\mathbb{R}} L_{n,k}^2(x)dx + \sum_{j\neq k} f\left(\frac{2j\pi}{2^n}\right)\int_{\mathbb{R}} L_{n,k}(x)L_{n,j}(x)dx \\
&= f\left(\frac{2k\pi}{2^n}\right)\frac{2\pi}{2^n} + 0 \\
&= \frac{2\pi}{2^n}f\left(\frac{2k\pi}{2^n}\right).
\end{aligned}$$

□

Theorem 5.11. *Let f be a real-valued function such that f and $|f|$ are integrable in the sense of Riemann on $[0, 2\pi]$, then*

$$\lim_{n\to\infty} S_n(f,x) = f(x) \quad a.e. \tag{5.14}$$

on $[0, 2\pi]$.

Proof. We saw in Corollary 5.4 that F_n is the Fourier transform of S_n, and that in Corollary 5.5,

$$\lim_{n\to\infty} F_n = F$$

uniformly, where F is the Fourier transform of f.

Therefore,

$$\lim_{n\to\infty}\mathcal{F}(S_n) = \mathcal{F}(f)$$

but by our conditions on f and, $\mathcal{F}$ is a continuous linear operator, we have,

$$\mathcal{F}(\lim_{n\to\infty} S_n) = \mathcal{F}(f)$$

So, we conclude that

$$\lim_{n\to\infty} S_n = f \quad \text{a.e.}$$

□

As a notation, Let $P_n(x)$ denote the *Lagrange* interpolating function of $S_n(f,x)$ with the nodes of E_n, and let $H_n(x)$ denote the *Lagrange* interpolating function of $f(x)$ with the nodes of E_n.

Lemma 5.12. *Let f be any real-valued function on $[0, 2\pi]$, then $P_n(x) = H_n(x)$.*

Proof. By the definition of the *Lagrange* interpolation we have,

$$P_n(x) = \sum_{x_k \in E_n} S_n(f, x_k) J_{n,k}(x),$$

where $J_{n,k}(x) = \prod_{j\neq k} \frac{x-x_j}{x_k-x_j}$ and j ranges over the integers between 0 and 2^n, included. But since $S_n(f, x_k) = f(x_k)$ for all $x_k \in E_n$, we would have

$$\begin{aligned} P_n(x) &= \sum_{j\neq k} f(x_k) J_{n,k}(x) \\ &= H_n(x). \end{aligned}$$

□

Theorem 5.13. *Let $f \in C^{2^n+1}[0, 2\pi]$ for any natural number n then for each $x \in [0, 2\pi]$ we have*

$$(f(x) - S_n(f,x)) = \frac{1}{(2^n+1)!} \prod_{x_k\in E_n} (x - x_k) \left\{ S_n^{(2^n+1)}(\zeta(x)) + f^{(2^n+1)}(\xi(x)) \right\},$$

where $\zeta(x)$ and $\xi(x)$ are two numbers in the interval $(0, 2\pi)$ and depend on x only.

Proof. Let H_n and P_n as in the last lemma, then

$$\begin{aligned} (f(x) - S_n(f,x)) &= (f(x) - H_n(x) + H_n(x) - P_n(x) + P_n(x) - S_n(f,x)) \\ &= (f(x) - H_n(x)) + (H_n(x) - P_n(x)) + (P_n(x) - S_n(f,x)) \\ &= \frac{1}{(2^n+1)!} \prod_{x_k\in E_n} (x - x_k) \left(f^{(2^n+1)}(\xi(x)) \right) \\ &\qquad + \frac{1}{(2^n+1)!} \prod_{x_k\in E_n} (x - x_k) \left(S_n^{(2^n+1)}(\zeta(x)) \right) \\ &= \frac{1}{(2^n+1)!} \prod_{x_k\in E_n} (x - x_k) \left\{ S_n^{(2^n+1)}(\zeta(x)) + f^{(2^n+1)}(\xi(x)) \right\} \end{aligned}$$

which completes the proof. □

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