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A WEIGHTED ANALYTIC CENTER FOR LINEAR MATRIX INEQUALITIES

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ABSTRACT. Let $\mathcal{R}$ be the convex subset of $\mathbb{R}^n$ defined by q simultaneous linear matrix inequalities (LMI) $A_0^{(j)} + \sum_{i=1}^n x_i A_i^{(j)} \succ 0, \;\; j = 1, 2, \ldots, q$. Given a strictly positive vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_q)$, the *weighted analytic center* $x_{ac}(\boldsymbol{\omega})$ is the minimizer *argmin* $(\phi_\omega(x))$ of the strictly convex function $\phi_\omega(x) = \sum_{j=1}^q \omega_j \log \det[A^{(j)}(x)]^{-1}$ over $\mathcal{R}$. We give a necessary and sufficient condition for a point of $\mathcal{R}$ to be a weighted analytic center. We study the *argmin* function in this instance and show that it is a continuously differentiable open function.

In the special case of linear constraints, all interior points are weighted analytic centers. We show that the region $\mathcal{W} = \{x_{ac}(\boldsymbol{\omega}) \mid \boldsymbol{\omega} > 0\} \subseteq \mathcal{R}$ of weighted analytic centers for LMI's is not convex and does not generally equal $\mathcal{R}$. These results imply that the techniques in linear programming of following paths of analytic centers may require special consideration when extended to semidefinite programming. We show that the region $\mathcal{W}$ and its boundary are described by real algebraic varieties, and provide slices of a non-trivial real algebraic variety to show that $\mathcal{W}$ isn't convex. Stiemke's Theorem of the alternative provides a practical test of whether a point is in $\mathcal{W}$. Weighted analytic centers are used to improve the location of standing points for the Stand and Hit method of identifying necessary LMI constraints in semidefinite programming.

Key words and phrases: Weighted analytic center, semidefinite programming, LMI, convexity,real algebraic variety.

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1. INTRODUCTION

The study of Linear Matrix Inequalities (LMI's) in Semidefinite Programming (SDP), is important since, as was shown in [26], many classes of optimization problems can be formulated as SDP problems. Interest in weighted analytic centers for feasible regions defined by LMI's arises from the success of interior point methods in solving SDP problems, e.g., Renegar [20].

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In [16], Mizuno, Todd and Ye studied surfaces of analytic centers in linear programming and proved that these form manifolds.

Luo uses weighted analytic centers in a cutting plane method [14] for solving general convex problems defined by a separation oracle. The method of centers for path following is described by Nesterov and Nemirovsky in [17], and Sturm and Zhang [24] use weighted analytic centers to study the central path for semidefinite programming. We extend the notion of weighted analytic center for linear programming ([1], [14], [19]) to semidefinite constraints.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times m$ real symmetric matrices. A is called *positive definite* (*positive semidefinite*) if all its eigenvalues are strictly positive (nonnegative). If A is positive definite (positive semidefinite), we write $A \succ 0$ ($A \succeq 0$). The symbol $\succeq$ is the *Löwner partial order* for real symmetric matrices, i.e., $A \succeq B$ if and only if $A - B$ is positive semidefinite.

Consider the following system of q linear matrix inequality *LMI* constraints:

$$A^{(j)}(x) := A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succ 0, \;\; j = 1, 2, \ldots, q \tag{1.1}$$

where $A_i^{(j)},\ 0 \leq i \leq n$, are all $m_j \times m_j$ symmetric matrices and $x \in \mathbb{R}^n$. Let

$$\mathcal{R} = \left\{ x \mid A^{(j)}(x) \succ 0,\ j = 1, 2, \ldots, q \right\} \tag{1.2}$$

denote the *feasible region*.

Assumption 1.1. We make the following set of assumptions throughout:

- all the constraints hold in an open set , i.e., $\mathcal{R} \neq \emptyset$ (*this is a Slater condition*);
- at every point of $\mathcal{R}$, n of the gradients of these constraints are linearly independent;
- $q > n$, i.e., the number of constraints exceeds the dimension of the space;
- $\mathcal{R}$ is bounded (unless stated otherwise).

Definition 1.1. A strictly positive vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_q)$ is called a weight vector.

Fix a weight vector $\boldsymbol{\omega} > 0$. Define $\phi_\omega(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\phi_\omega(x) = \begin{cases} \displaystyle\sum_{j=1}^{q} \omega_j \log\det[(A^{(j)}(x))^{-1}] & \text{if } x \in \mathcal{R} \\ \infty & \text{otherwise.} \end{cases} \tag{1.3}$$

Note that setting the k^{th} weight to zero is equivalent to removing the k^{th} constraint.

Definition 1.2. The *weighted analytic center* of $\mathcal{R}$ is given by

$$x_{ac}(\boldsymbol{\omega}) = argmin\left\{ \phi_\omega(x) \mid x \in \mathcal{R} \right\}.$$

Let $\mathbf{e} = [1, 1, \cdots, 1]$ be a vector of q *ones*. The *analytic center* of $\mathcal{R}$ is $x_{ac} = x_{ac}(\mathbf{e})$. If each constraint $A^{(j)}(x) \succ 0$ is a linear inequality $(a^{(j)})^T x - b^{(j)} > 0$, then

$$x_{ac}(\boldsymbol{\omega}) = argmax\{\sum_{j=1}^{q} \omega_j \log[(a^{(j)})^T x - b^{(j)}] \mid x \in \mathcal{R}\}.$$

This shows that Definition 1.2 is consistent with the usual definition of weighted analytic centers for linear inequalities [1].

We investigate necessary and sufficient conditions for a point of $\mathcal{R}$ to be a weighted analytic center. We use Stiemke's Theorem [23] of the alternative as a decision tool to decide whether or not a point of $\mathcal{R}$ is a weighted analytic center. We prove that $\phi_\omega(x)$ is a strictly convex function, and that for a given $\boldsymbol{\omega}$, the weighted analytic center is unique and is in $\mathcal{R}$. We give examples showing that a particular point $x \in \mathcal{R}$ can be the weighted analytic center for more than one weight vector.

We give a new proof that, in the special case of linear constraints, all interior points are weighted analytic centers. We then show that, in general, the region

$$\mathcal{W} = \{x_{ac}(\boldsymbol{\omega}) \mid \boldsymbol{\omega} > 0\} \subseteq \mathcal{R}$$

of weighted analytic centers for LMI's does not equal $\mathcal{R}$ and is not convex.

This lack of convexity is clearly seen in Figures 3.3 and 3.4 that show successive horizontal slices of a given region $\mathcal{R}$. This is interesting because there are many analytic center based path following algorithms in the literature ([17], [24]) for problems with linear constraints. It is useful to know that the region of weighted analytic centers $\mathcal{W}$ is not always convex in the case of LMI constraints. Further we establish that $\mathcal{W}$ is a contractible open subset of $\mathcal{R}$, and is the projection of a real algebraic variety. We show that the boundary of $\mathcal{W}$ can be described using other real algebraic varieties. We also show how weighted analytic centers can improve the location of standing points for the semidefinite stand and hit method (SSH) [12] for identifying necessary constraints in semidefinite programming.

For square matrices $\{A_i\}_{i=1}^q$ denote the block-diagonal matrix having $A_1, A_2, \cdots, A_q$ as its block-diagonal elements, in the given order, by $\mathbf{diag}[A_1, A_2, \cdots, A_q]$. Define the inner product $A \bullet B$ of matrices A and B by $A \bullet B = \sum_i \sum_j a_{ij} b_{ij} = Tr(A^T B)$. Fejer's theorem [[8], p. 459] states that $A \bullet B \geq 0$ when $A \succeq 0$ and $B \succeq 0$. The *Frobenius norm* of A is denoted by $\parallel A \parallel_F$, where $\parallel A \parallel_F = [A \bullet A]^{\frac{1}{2}}$. We introduce some notation:

$$\begin{aligned}
A_{<i>} &= \mathbf{diag}[A_i^{(1)}, A_i^{(2)}, \cdots, A_i^{(q)}] \; for \; i = 0, 1, 2, \ldots, n. \\
\mathrm{B}(x) &= \sum_{i=1}^{n} x_i A_{<i>} \; for \; x \in \mathbf{R}^n. \\
\mathcal{A}(x) &= A_{<0>} + B(x) \; for \; x \in \mathbf{R}^n. \\
\hat{\mathcal{A}}_\omega(x) &= \mathbf{diag}[\omega_1 (A^{(1)}(x))^{-1}, \cdots, \omega_q (A^{(q)}(x))^{-1}].
\end{aligned}$$

Set $N = \sum_{j=1}^q m_j$. Note that $\mathcal{A}(x)$ is $N \times N$ and $\hat{\mathcal{A}}_\omega(x) \succ 0$ for all $x \in \mathcal{R}$.

2. THE WEIGHTED ANALYTIC CENTER

In Lemma 2.3 of this section we show that $\phi_\omega(x)$ is strictly convex. This guarantees the existence and uniqueness of the weighted analytic center. This result is already well known when a single LMI is considered ([17], [3]). Albeit our theorem extends their result, the proof is not a direct consequence. We require the following assumption throughout which is equivalent to saying that $\mathbf{B}(\mathbf{x}) = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{0}$. Assumption 2.1 does not imply that the matrices $\{A_1^{(j)}, A_2^{(j)}, \ldots, A_n^{(j)}\}$ are linearly independent for some j.

Assumption 2.1. The matrices $\{A_{<1>}, A_{<2>}, \ldots, A_{<n>}\}$ are linearly independent.

The barrier function $\phi_\omega(x)$ is a linear combination of convex functions, so it is convex. We give a brief independent proof of this below and show that $\phi_\omega(x)$ is strictly convex. Gradient and Hessian are linear operators on the space of continuously differentiable functions [17, 3], and we describe their action on $\phi_\omega(x)$.

Lemma 2.1. *For $\phi_\omega(x)$ defined as in (1.3)*

$$\begin{aligned}
\nabla_i \phi_\omega(x) &= \sum_{j=1}^{q} \omega_j \nabla_i \log\det[(A^{(j)}(x))^{-1}] = -\sum_{j=1}^{q} \omega_j (A^{(j)}(x))^{-1} \bullet A_i^{(j)} = -\hat{\mathcal{A}}_\omega(x) \bullet A_{<i>} \\
H_{ij}(x) &= \sum_{k=1}^{q} \omega_k \nabla_{ij}^2 \log\det[(A^{(k)}(x))^{-1}] = \sum_{k=1}^{q} \omega_k [(A^{(k)}(x))^{-1} A_i^{(k)}] \bullet [(A^{(k)}(x))^{-1} A_j^{(k)}].
\end{aligned}$$

We can describe the gradient acting on a single constraint as:

$$\nabla \log(\det(A^{(j)}(x))^{-1}) \quad = \quad [-(A^{(j)}(x))^{-1} \bullet A_1^{(j)}, \ldots, -(A^{(j)}(x))^{-1} \bullet A_n^{(j)}]^T \tag{2.1}$$

We can rewrite the Hessian as a linear combination of Hessians of each constraint:

$$H(x) = [H_{ij}(x)] = \sum_{k=1}^{q} \omega_k H^{(k)}(x) = \sum_{k=1}^{q} \omega_k \nabla A^{(k)}(x) \bullet \left[\nabla A^{(k)}(x)\right]^T \tag{2.2}$$

Let $adj(B)$ denote the adjugate matrix of matrix B. We have, for each constraint,

$$\nabla \log(\det(A^{(j)}(x))^{-1}) \quad = \quad -\left[\frac{adj(A^{(k)}(x))}{det(A^{(k)}(x))} \bullet A_1^{(j)}, \ldots, \frac{adj(A^{(k)}(x))}{det(A^{(k))}(x)} \bullet A_n^{(j)}\right]^T \tag{2.3}$$

Corollary 2.2. *Each term of* $\nabla \log(\det(A^{(j)}(x))^{-1})$ *is a quotient of polynomials and the denominators are strictly positive in* $\mathcal{R}$. $\nabla \log(\det(A^{(j)}(x))^{-1})$ *is analytic in* $\mathcal{R}$.

Proof. Each coefficient of $A^{(k)}(x)$ has the form $b_0 + b_1 x_1 + \ldots + b_n x_n$ because of the definition of the constraints. Hence every term of the adjugate and determinant are polynomials in $x_1, \ldots, x_n$. The denominators are all determinants of positive definite matrices, so they are strictly positive in $\mathcal{R}$. Since $\nabla \log(\det(A^{(j)}(x))^{-1})$ is a vector of quotients of polynomials with strictly positive denominators in $\mathcal{R}$, all higher derivatives exist also. Hence it is analytic. ☐

Lemma 2.3. $\phi_\omega(x)$ *is strictly convex over* $\mathcal{R}$

Proof. Let $\hat{\mathcal{A}}_{\sqrt{\omega}}(x) = \textbf{diag}[\sqrt{\omega_1}A^{(1)}(x)^{-1}, \cdots, \sqrt{\omega_q}A^{(q)}(x)^{-1}]$. For $s \in \mathbb{R}^n$, using the Hessian matrix $H(x) = \textbf{diag}\left[H^{(1)}(x), \ldots, H^{(q)}(x)\right]$, convexity of $\phi_\omega(x)$ follows from

$$\begin{aligned}
s^T H(x) s &= \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \hat{\mathcal{A}}_{\sqrt{\omega}}(x) A_{<i>} \bullet \hat{\mathcal{A}}_{\sqrt{\omega}}(x) A_{<j>} \\
&= \hat{\mathcal{A}}_{\sqrt{\omega}}(x) \sum_{i=1}^{n} s_i A_{<i>} \bullet \hat{\mathcal{A}}_{\sqrt{\omega}}(x) \sum_{j=1}^{n} s_j A_{<j>} \\
&= \| \hat{\mathcal{A}}_{\sqrt{\omega}}(x) \mathrm{B}(s) \|_F^2 \geq 0.
\end{aligned}$$

Since $\hat{\mathcal{A}}_{\sqrt{\omega}}(x) \succ 0$ in $\mathcal{R}$, then $s^T H(x) s = 0 \;\Leftrightarrow\; \mathrm{B}(s) = \sum_{i=1}^{n} s_i A_{<i>} = 0 \;\Leftrightarrow\; s = 0$ by Assumption 2.1. Hence, $\phi_\omega(x)$ is strictly convex. ☐

Unlike the instance of linear inequalities, not all feasible points can be expressed as a weighted analytic center (cf. Theorem 3.3). Necessary and sufficient conditions for this are given in the next proposition.

Proposition 2.4. $x^* \in \mathcal{R}$ *is a weighted analytic center* $\Leftrightarrow$ *there exists* $\boldsymbol{\omega} > 0$ *such that* $\sum_{j=1}^{q} \omega_j \nabla A^{(j)}(x^*) = \mathbf{0}$*, or equivalently,* $\sum_{j=1}^{q} \omega_j A^{(j)}(x^*)^{-1} \bullet A_i^{(j)} = 0, \;\; i = 1, 2, \ldots, n$.

Proof. By Lemma 2.3, $\phi_\omega(x)$ is strictly convex, hence the gradient is zero at the absolute minumum [15]. Thus for $1 \leq i \leq n$, $\nabla_i \phi_\omega(x^*) = 0 \Leftrightarrow x^*$ is a weighted analytic center. ☐

The following proposition shows that the barrier function $\phi_\omega(x)$ is bounded if and only if the feasible region $\mathcal{R}$ is bounded.

Proposition 2.5. *The following are equivalent:*

$[\,a\,]$ $\mathcal{R}$ *is unbounded*
$[\,b\,]$ *there is a direction* $s \neq 0$ *such that*, $B(s) \succeq 0, \; i.e., \; \sum_{i=1}^{n} s_i A_i^{(j)} \succeq 0, \; 1 \leq j \leq q$
$[\,c\,]$ $\phi_\omega(x)$ *is unbounded below.*

Proof. $[\,a\,] \Rightarrow [\,b\,]$ Suppose that $\mathcal{R}$ is unbounded and $x_0 \in \mathcal{R}$. By the convexity of $\mathcal{R},$ for some direction $s \neq 0$ the ray $\mathcal{R}_\sigma = \{x_0 + \sigma s \mid \sigma > 0\}$ is feasible. Therefore, we have

$$\mathcal{A}(x_0 + \sigma s) = \mathcal{A}(x_0) + \sigma B(s) \succeq 0 \;\forall\, \sigma \geq 0.$$

This means $B(s) \bullet Y + \frac{1}{\sigma}\mathcal{A}(x_0) \bullet Y \geq 0$ for all $\sigma > 0$ and $Y \succeq 0$. Hence $B(s) \bullet Y \geq 0$ for all $Y \succeq 0$. By Fejer's Theorem [8], $B(s) \succeq 0$ and $B(s) \neq 0$ by Assumption 2.1.

$[\,b\,] \Rightarrow [\,c\,]$ Given $x_0 \in \mathcal{R}$ and a nonzero direction s for which $B(s) \succeq 0$, then $\mathcal{A}(x_0 + \sigma s) \succ 0$ for all $\sigma \geq 0$. Therefore $A^{(j)}(x_0) \succ 0$ and $\sum_{i=1}^{n} s_i A_i^{(j)} \succeq 0,\ 1 \leq j \leq q$. By [8, Corollary 7.6.5] we can find a nonsingular matrix C_j such that $C_j A^{(j)}(x_0) C_j^T = I,\ \ C_j(\sum_{i=1}^{n} s_i A_i^{(j)}) C_j^T = \mathbf{diag}[(a_j)_1, (a_j)_2, \ldots, (a_j)_{m_j}]$, with $(a_j)_k \geq\ 0, 1 \leq k \leq m_j$. By Assumption 2.1 *at least* one $(a_j)_k > 0$. Therefore,

$$\begin{aligned}
A^{(j)}(x_0 + \sigma s) &= C_j^{-1}(I + \sigma\, \mathbf{diag}[(a_j)_1, (a_j)_2, \ldots, (a_j)_{m_j}]) C_j^{T-1} \\
\phi_\omega(x_0 + \sigma s) &= \sum_{j=1}^{q} \omega_j \log\det[A^{(j)}(x_0 + \sigma s)^{-1}] \\
&= \sum_{j=1}^{q} 2\log\det(C_j) - \sum_{j=1}^{q}\sum_{k=1}^{m_j} \log(1 + \sigma(a_j)_k) \to -\infty \text{ as } \sigma \to \infty.
\end{aligned}$$

$[\,c\,] \Rightarrow [\,a\,]$ ϕ_ω is bounded below on every bounded region since it is strictly convex by Lemma 2.3. Hence, if $\phi_\omega(x)$ is unbounded, then $\mathcal{R}$ is unbounded. ☐

3. THE REGION $\mathcal{W}$ OF WEIGHTED ANALYTIC CENTERS

3.1. **Theorems of the alternative and the boundary of $\mathcal{W}$.** In this section we investigate properties of the region of weighted analytic centers $\mathcal{W}$ and its boundary. Denote the boundary of a set $\mathcal{S}$ by $\partial(\mathcal{S})$. We recall the standard definition of derivative of a function of several variables [21, p. 216] and apply it to $\phi_\omega(x)$. We define the matrix $M(x)$:

Definition 3.1.

$$M(x) = \left[\frac{-d\phi_\omega(x)}{dx}\right]^T = \begin{bmatrix} A^{(1)}(x)^{-1} \bullet A_1^{(1)} & \cdots & A^{(q)}(x)^{-1} \bullet A_1^{(q)} \\ \cdots & \cdots & \cdots \\ A^{(1)}(x)^{-1} \bullet A_n^{(1)} & \cdots & A^{(q)}(x)^{-1} \bullet A_n^{(q)} \end{bmatrix} \tag{3.1}$$

The j^{th} column of $M(x)$ is $-\nabla \log\det[A^{(j)}(x)^{-1}]$. These columns are the components of the gradient of the barrier term in $\phi_\omega(x)$ for each constraint, i.e., see (2.1). $M(x)$ is an analytic function on $\mathcal{R}$ by Corollary 2.2. For a unit vector s, $s^T M(x)$ gives the *directional derivative* in direction s of $A^j(x)$ for each constraint $A^j(x) \succeq 0$. At each point $x \in \mathcal{R}$, for a weight vector $\boldsymbol{\omega} > 0,\ M(x)\boldsymbol{\omega} = -\nabla\phi_\omega(x)$. The region of weighted analytic centers is

$$\mathcal{W} = \{x :\ x \in \mathcal{R} \text{ and there exists } \boldsymbol{\omega} > 0 \ \text{ such that } M(x)\boldsymbol{\omega} = 0\} \tag{3.2}$$

We recall Stiemke's Theorem of the alternative to obtain another characterization of the region of weighted analytic centers $\mathcal{W}$.

Theorem 3.1. *(Stiemke's Theorem* [23, 15]*) Let M be a $n \times q$ matrix and let $\boldsymbol{\omega} \in \mathbf{R}^q$ and $s \in \mathbf{R}^n$. Exactly one of the following two systems has a solution:*

$$\begin{aligned}
\textbf{System 1}:\ s^T M &\geq 0,\ s^T M \neq 0,\ s \in \mathbf{R}^n \\
\textbf{System 2}:\ M\boldsymbol{\omega} &= 0,\ \boldsymbol{\omega} > 0,\ \boldsymbol{\omega} \in \mathbf{R}^q.
\end{aligned}$$

Corollary 3.2. $\mathcal{W} = \left\{x :\ x \in \mathcal{R} \textit{ such that } s^T M(x) \geq 0,\ s^T M(x) \neq 0 \textit{ is infeasible}\right\}$

The Corollary shows that if there exists a direction s in which $s^T M(x) \geq 0$ and $s^T M(x) \neq 0$, then x isn't a weighted analytic center. The set of weight vectors for any given x in $\mathcal{R}$ is the intersection of the null space of $M(x)$ with the positive orthant in $\mathbf{R}^q$.

In general, $\mathcal{W}$ does not have a simple description. However, in the case of linear constraints, all interior points are weighted analytic centers and $\mathcal{W} = \mathcal{R}$.

Theorem 3.3. *If $\mathcal{R}$ is defined by the linear system $(a^{(j)})^T x - b^{(j)} > 0$ $(1 \leq j \leq q)$, i.e., $\mathcal{R} = \left\{x : (a^{(j)})^T x - b^{(j)} > 0,\ 1 \leq i \leq q\right\}$, then $\mathcal{W} = \mathcal{R}$.*

Proof. We know that $\mathcal{W} \subseteq \mathcal{R}$. By Proposition 2.4, a point x_0 is a weighted analytic center if and only if there exist weights $\boldsymbol{\omega}$ such that

$$\sum_{j=1}^{q} \left(\frac{a_i^{(j)} \omega_j}{(a^{(j)})^T x_0 - b^{(j)}} \right) = 0,\ (1 \leq i \leq n). \tag{3.3}$$

Let $x^* = x^*_{ac}$ be the analytic center of the linear system. By definition, (3.3) holds at x^* with $\boldsymbol{\omega} = \mathbf{e}$, i.e.,

$$\sum_{j=1}^{q} \left(\frac{a_i^{(j)}}{(a^{(j)})^T x^* - b^{(j)}} \right) = 0,\ (1 \leq i \leq n). \tag{3.4}$$

We have x^* is a point of $\mathcal{R}$ and therefore, $(a^{(j)})^T x^* - b^{(j)} > 0$. Given a point x_0 of $\mathcal{R}$, set

$$\omega_j = \frac{(a^{(j)})^T x_0 - b^{(j)}}{(a^{(j)})^T x^* - b^{(j)}}$$

for $1 \leq j \leq q$. These values and (3.4) give

$$\sum_{j=1}^{q} \left(\frac{a_i^{(j)} \omega_j}{(a^{(j)})^T x_0 - b^{(j)}} \right) = \sum_{j=1}^{q} \left(\frac{a_i^{(j)}}{(a^{(j)})^T x^* - b^{(j)}} \right) = 0,\ \text{for } 1 \leq i \leq n. \tag{3.5}$$

Hence, $x_{ac}(\boldsymbol{\omega}) = x_0$. ☐

The next example shows that it is not generally true that every point in $\mathcal{R}$ is a weighted analytic center. We give a precise description of the boundaries $\partial(\mathcal{W})$ of $\mathcal{W}$ and $\partial(\mathcal{R})$ of $\mathcal{R}$. The second constraint is deliberately chosen to be redundant. It is a simple matter to determine the feasible region for each constraint, i.e, for the third it is the set of points for which $x_1 \geq -1$ and $x_2 \geq -2$.

Example 3.1. We have region $\mathcal{R}$ given by $n = 2$ variables and $q = 5$ LMI constraints:

$$\begin{aligned}
A^{(1)}(x) &= \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0 \\
A^{(2)}(x) &= \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0 \\
A^{(3)}(x) &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0 \\
A^{(4)}(x) &= \begin{bmatrix} 3.8 & 0 \\ 0 & 3.8 \end{bmatrix} + x_1 \begin{bmatrix} -0.4 & 0 \\ 0 & -0.4 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0 \\
A^{(5)}(x) &= \begin{bmatrix} 2.6 & 0 \\ 0 & 2.6 \end{bmatrix} + x_1 \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0.
\end{aligned}$$

All entries of $M(x_1, x_2)$ are quotients of polynomials in $x_1,\ x_2$.

$$\begin{bmatrix} \dfrac{-2+x_2}{9-5\,x_2-2\,x_1+x_1x_2} & \dfrac{1}{-5+x_1} & \dfrac{1}{1+x_1} & \dfrac{4}{-19+2\,x_1-5\,x_2} & \dfrac{8}{13+4\,x_1+5\,x_2} \\ \dfrac{-5+x_1}{9-5\,x_2-2\,x_1+x_1x_2} & \dfrac{1}{-2+x_2} & \dfrac{1}{2+x_2} & \dfrac{-10}{-19+2\,x_1-5\,x_2} & \dfrac{10}{13+4\,x_1+5\,x_2} \end{bmatrix}$$

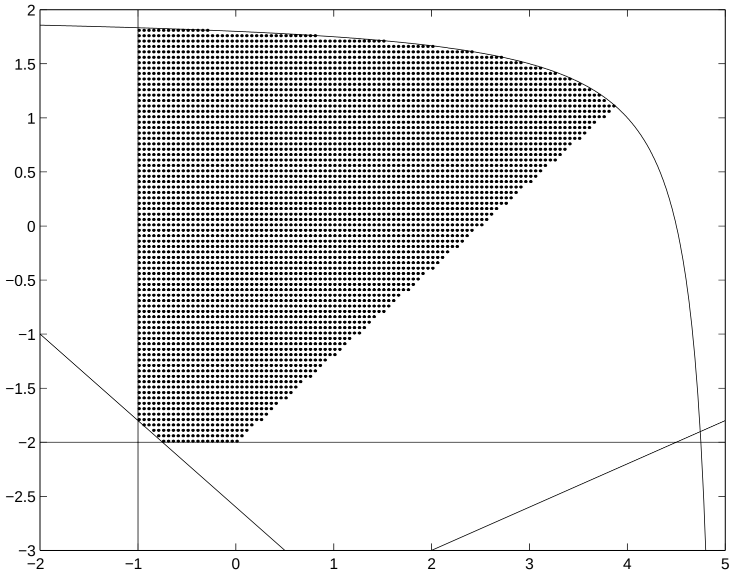

Figure 3.1: The region $\mathcal{W}$ of weighted analytic centers bounded by the 5 constraints.

Figure 3.1 shows the feasible region for Example 3.1, where the shaded region is $\mathcal{W}$. The analytic center is located at $x_1 = 1.3291554838,\ x_2 = 0.4529930537$. We demonstrate that *not every point is a weighted analytic center,* e.g., for $x^* = (4, -1.5)^T, x^* \notin \mathcal{W}$. We first compute the matrix

$$M(x^*) = \begin{bmatrix} -1.4000 & -1.0000 & 0.2000 & -1.1429 & 0.3721 \\ -0.4000 & -0.2857 & 2.0000 & 2.8571 & 0.4651 \end{bmatrix}$$

We note that $[-1\ \ 1]\, M(x^*) = [1.0000\ \ 0.7143\ \ 1.8000\ \ 4.0000\ \ 0.0930] > 0$, so the system $s^T M(x^*) \geq 0, s^T M(x^*) \neq 0$ is feasible. By Corollary 3.2 , x^* is not a weighted analytic center.

The point $(1, 0.5)$ is a weighted analytic center. We evaluate

$$M(1, 0.5) = \begin{bmatrix} -0.3000 & -0.2500 & 0.5000 & -0.2051 & 0.4103 \\ -0.8000 & -0.6667 & 0.4000 & 0.5128 & 0.5128 \end{bmatrix}.$$

The null space of $M(1, 0.5)$ is spanned by the 3 column vectors of the matrix N:

$$N = \begin{bmatrix} -0.8333 & 1.2088 & 0.3297 \\ 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.1355 & -0.6227 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

We form 2 linear combinations of these columns and transpose to get weight vectors

$$\begin{aligned}\boldsymbol{\omega}_1 &= ([1,1,1]N)^T = [0.7051,\ 1,\ 0.5128,\ 1,\ 1]\\ \boldsymbol{\omega}_2 &= ([2,1,1.5]N)^T = [0.0366,\ 2,\ 0.2015,\ 1,\ 1.5000].\end{aligned}$$

Any convex combination $(1-t)\times\boldsymbol{\omega}_1 + t\times\boldsymbol{\omega}_2,\ 0\le t\le 1$, is another weight vector corresponding to $(x_1,x_2)=(1,0.5)$. The underlying mechanism is described in Theorem 3.7.

To gain a better understanding of the region $\mathcal{W}$, a mesh with grid size 0.05 was formed over the feasible region. By (3.2), a mesh point x^* is in $\mathcal{W}$ if and only if there is a weight vector $\boldsymbol{\omega}>0$ so that for $M=M(x^*)$, $M\omega=0$ holds. In this case, there is an optimal solution of the Linear Programming problem, where $M=M(x^*)$.

$$\textit{Max} \quad \sum_{j=1}^{q}\omega_j \tag{3.6}$$

$$\textit{subject to} \quad M\omega = 0 \tag{3.7}$$

$$\sum_{j=1}^{q}\omega_j = 1 \tag{3.8}$$

$$\omega > 0.$$

If the LPP has a solution, the optimal objective value must be 1. If the problem is infeasible, then x isn't a weighted analytic center, i.e., $x\notin\mathcal{W}$. We can determine the region of weighted analytic centers by solving the LPP at each mesh point.

Lemma 3.4. *The optimal objective value of LP (3.6) equals* $1 \Leftrightarrow x^*\in\mathcal{W}$. $\square$

The diagonal part of the boundary of the shaded region in Figure 3.1 in the interior of the feasible region $\mathcal{R}$ has the appearance of a straight line. By scaling with positive values *(multiplication by a positive definite diagonal matrix on the right)* - we convert $M(x)$ to

$$\begin{bmatrix} -2+x_2 & -2+x_2 & 2+x_2 & 4 & 8\\ -5+x_1 & -5+x_1 & 1+x_1 & -10 & 10\end{bmatrix}$$

To make the 1^{st} and 5^{th} columns dependant, eliminate k from $8k=-2+x_2$ and $10k=-5+x_1$ to get $x_2=\frac{4}{5}x_1-2$. This means the 1^{st} and 5^{th} columns are linearly dependent along the line $\mathcal{L}=\{(x_1,x_2): 5x_2-4x_1+10=0\}$, which is the diagonal in Figure 3.1. The line $\mathcal{L}$ has normal $s=[5,-4]^T$. Multiply $M(x)$ by the normal vector s^T to get:

$$\left[\begin{array}{ccccc}\dfrac{10+5\,x_2-4\,x_1}{9-5\,x_2-2\,x_1(1+x_2)} & \dfrac{10+5\,x_2-4\,x_1}{(-5+x_1)\,(-2+x_2)} & \dfrac{6+5\,x_2-4\,x_1}{(1+x_1)\,(2+x_2)} & \dfrac{60}{(-19+2\,x_1-5\,x_2)} & 0\end{array}\right].$$

The first 2 entries are also zero on the line $\mathcal{L}$. Hence $\mathcal{L}$ is a line *(this is a real algebraic variety!)* on which the directional derivative in direction $s=[5,-4]^T$ for the first 2 constraints is zero. Substitute $x_2=\frac{4}{5}\,x_1-2$ in $s^TM(x)$ and obtain:

$$s^T\,M_{bdry} = \left[\begin{array}{ccccc}0 & 0 & \dfrac{5}{x_1\,(1+x_1)} & \dfrac{60}{9+2\,x_1} & 0\end{array}\right]$$

which is non-negative and not zero on $0<x_1<3.881966$. By Corollary 3.2, the line segment is not in $\mathcal{W}$, so the line $\mathcal{L}$ demarks the boundary.

Consider the point $(1,\ -\frac{6}{5})$ on the line $\mathcal{L}$. Evaluate

$$M\left(1,\ -\frac{6}{5}\right) = \begin{bmatrix} -\frac{16}{59} & -\frac{1}{4} & \frac{1}{2} & -\frac{4}{11} & \frac{8}{11} \\ -\frac{20}{59} & -\frac{5}{16} & \frac{5}{4} & \frac{10}{11} & \frac{10}{11} \end{bmatrix}.$$

Columns $1, 2, 5$ are parallel and $\omega(t) = \frac{1}{3}[t, 2-2t, 0, 0, 1+t]$ is a non-zero solution of $M\omega = 0$ for all $0 \le t \le 1$. It is easy to check that there is no solution using columns 3 and 4, except with negative weights! It isn't generally easy to determine the boundary of every region of weighted analytic centers.

To understand more of the region $\mathcal{W}$, consider the augmented matrix obtained from the constraints (3.7) and (3.8):

$$\begin{bmatrix} \frac{-2+x_2}{9-5\,x_2-2\,x_1+x_1x_2} & \frac{1}{-5+x_1} & \frac{1}{1+x_1} & \frac{4}{-19+2\,x_1-5\,x_2} & \frac{8}{13+4\,x_1+5\,x_2} & 0 \\ \frac{-5+x_1}{9-5\,x_2-2\,x_1+x_1x_2} & \frac{1}{-2+x_2} & \frac{1}{2+x_2} & \frac{-10}{-19+2\,x_1-5\,x_2} & \frac{10}{13+4\,x_1+5\,x_2} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The general solution of the augmented system is the vector $\omega(u, v)$=

$$\begin{bmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \\ \underline{\omega}_3 \\ \underline{\omega}_4 \\ \underline{\omega}_5 \end{bmatrix} = \begin{bmatrix} & u & \\ \frac{(-12-5\,x_2-2\,x_1)(-5+x_1)(-2+x_2)}{\beta} - & u\frac{(-5+x_1)(-2+x_2)\gamma}{\beta(9-5\,x_2-2\,x_1+x_1\,x_2)} - & v\frac{\eta(30-6\,x_1)(-2+x_2)}{\beta(13+4\,x_1+5\,x_2)} \\ \frac{(-20+5\,x_2+2\,x_1)(2+x_2)(1+x_1)}{\beta} - & u\frac{(20-5\,x_2-2\,x_1)(2+x_2)(1+x_1)}{\beta(9-5\,x_2-2\,x_1+x_1\,x_2)} - & v\frac{\chi(12+6\,x_2)(1+x_1)}{\beta(13+4\,x_1+5\,x_2)} \\ \frac{(4+3\,x_2-2\,x_1)(-19+2\,x_1-5\,x_2)}{\beta} - & u\frac{(-4-3\,x_2+2\,x_1)(-19+2\,x_1-5\,x_2)}{\beta(9-5\,x_2-2\,x_1+x_1\,x_2)} - & v\frac{\alpha(-19+2\,x_1-5\,x_2)}{\beta(13+4\,x_1+5\,x_2)} \\ & & v \end{bmatrix}$$

with $\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3, \underline{\omega}_4, \underline{\omega}_5 \ge 0$ and where we have made the following set of substitutions for notational convenience:

$$\begin{aligned} \alpha &= 8x_1^2 + 2\,x_1x_2 - 74x_1 - 15x_2^2 + 59\,x_2 + 92 \\ &= \left(2x_1 + 3x_2 - \frac{170}{11}\right)\left(4x_1 - 5x_2 - \frac{67}{11}\right) - \frac{258}{121}, \\ \beta &= \left(-236 + 14x_1 - 77x_2 + 16x_1x_2 + 4x_1^2 + 15x_2^2\right) \\ &= (2x_1 + 5x_2 + 56)(2x_1 + 3x_2 - 49) + 2508, \\ \gamma &= 4x_1^2 + 16x_1x_2 + 16x_1 + 15x_2^2 - 72x_2 - 224 \\ &= (2x_1 + 5x_2 + 56)(2x_1 + 3x_2 - 48) + 2464, \\ \chi &= (-175 + 27x_1 + 20x_2), \\ \eta &= (-13 + 7x_1 - 10x_2). \end{aligned}$$

We note that the denominators are never zero in the shaded region $\mathcal{W}$. At every point $x = (x_1, x_2) \in \mathcal{W}$ we have a non-trivial solution and an open neighborhood where there is a set of weight vectors with the same point x for the optimal solution of $\phi_\omega(x)$. For example, when

$x_1 = 1,\ x_2 = 1$, any point $(u, v) > 0$ in the interior of the triangle constrained by the lines

$$\frac{19}{66} - \frac{245}{198}\,u + \frac{8}{121}\,v > 0$$

$$\frac{13}{44} + \frac{13}{132}\,u - \frac{96}{121}\,v > 0$$

$$\frac{5}{12} + \frac{5}{36}\,u - \frac{3}{11}\,v > 0$$

gives rise to an $\omega(u, v)$ for which $\phi_\omega(x)$ has optimal value at $x_1 = 1,\ x_2 = 1$. This is generalized in Theorem 3.7.

In the table below we give the location of $argmin(\phi_\omega(x))$ when some $\omega_i = 0$. Note that the optimal value can be in the interior of $\mathcal{W}$. Constraints 1 and 3 (for instance) bound a region containing $\mathcal{R}$, but the constraints $\{3, 4, 5\}$ give an unbounded region. This is useful in understanding repelling paths in section 3.3. We list six cases with a zero value for at least one ω_i:

ω	x_1^{opt}	x_2^{opt}
$[0, 1, 1, 1, 1]$	2.370	1.147
$[1, 0, 1, 1, 0]$	0.988	0.614
$[1, 0, 1, 0, 1]$	2.684	0.346
$[1, 0, 0, 1, 1]$	0.324	0.784
$[1, 0, 1, 0, 0]$	1.767	-0.155
$[0, 0, 1, 1, 1]$	∞	∞

In the next section we generalize these observations and show that $\mathcal{W}$ is the projective image of a real *algebraic variety*.

3.2. **Algebraic varieties and the region of weighted analytic centers.** We define matrices $D(x), D^{(j)}(x), P(x)$ and polynomials $P_{i,j}(x)$

$$\begin{aligned}
D(x) &= \prod_{j=1}^{n} det[A^{(j)}(x)] \\
D^{(j)}(x) &= D(x)A^{(j)}(x)^{-1} = \left(\prod_{j \neq i} \det(A^{(j)}(x))\right) adj(A^{(j)}(x)) \\
P_{i,j}(x) &= D^{(j)}(x) \bullet A_i^{(j)} \\
P(x) &= [P_{i,j}(x)]\,. \qquad (3.9)
\end{aligned}$$

Note that $D^{(j)}(x) \succ 0$ and $D(x) > 0$ for all $x \in \mathcal{R}$ since all the $A^{(j)}(x)$ matrices are positive definite there. The idea of using the product of the determinants is analogous to the method of Sonnevand [22] of taking the products of the distances from a set of linear constraints in order to find the analytic center of a polyhedron. We note that $P(x)$ is a $n \times q$ matrix of *polynomials* in x, where $M(x)$ is given by (3.1). $P(x)$ has rank n since it has n linearly independent columns at every point of $\mathcal{R}$ by Assumption 1.1. The optimal value problem can be restated as a problem in polynomials.

Theorem 3.5. (a) $\nabla_i \phi_\omega(x) = 0$ *for* $1 \le i \le n \Leftrightarrow \sum_{j=1}^{q} \omega_j P_{i,j}(x) = 0.$

(b) *The solution set of* $\sum_{j=1}^{q} \omega_j P_{i,j}(x) = 0,\ 1 \le i \le n,$ *is a real algebraic variety* $\mathcal{V}$ *in* $x = (x_1, x_2, \cdots, x_n)$ *and* $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_q)$.

(c) $\mathcal{W} \subseteq$ *the projection of* $\mathcal{V}$ *into* $\mathbf{R}^n$ *given by* $(x, \omega) \to x$

Proof. Since both $(A^{(j)}(x))^{-1} \succ 0$ and $D(x) > 0$ in $\mathcal{R}$, by Corollary (2.2),

$$\hat{\mathcal{A}}_\omega(x) \bullet A_{<i>} = 0 \Leftrightarrow \sum_{j=1}^{q} \frac{\omega_j}{det[A^{(j)}(x)]} adj[A^{(j)}(x)] \bullet A_i^{(j)} = 0 \Leftrightarrow \sum_{j=1}^{q} \omega_j P_{i,j}(x) = 0.$$

The last equivalence is obtained by multiplying the right-hand side by the product $D(x)$. Every entry of $D^{(j)}(x)$ is a polynomial in x, so the solution set

$$\mathcal{V} = \left\{ (x, \boldsymbol{\omega}) : \sum_{j=1}^{q} \omega_j P_{i,j}(x) = 0,\ 1 \le j \le n \right\} \tag{3.10}$$

is a real algebraic variety in x and $\boldsymbol{\omega}$.

From Proposition 2.4 ,

$$\mathcal{W} = \left\{ x \in \mathcal{R} \mid \sum_{j=1}^{q} \omega_j D^{(j)}(x) \bullet A_i^{(j)} = 0,\ 1 \le i \le n, \text{ for some } (\omega_1, \cdots, \omega_q) > 0 \right\}$$

where $\mathcal{R}$ is the feasible region of the system of LMI's. Thus, the region $\mathcal{W}$ is a subset of the projection of $\mathcal{V}$. □

We now have a system $P\boldsymbol{\omega} = 0$ of n equations in q variables $\omega_1, \ldots, \omega_q$:

$$\sum_{j=1}^{q} P_{i,j}(x)\omega_j = 0, \quad 1 \le i \le n. \tag{3.11}$$

Notation: Let $\triangle^{q-1} = \{\boldsymbol{\omega} \in \mathbf{R}^q : \sum_{j=1}^{q} \omega_j = 1,\ \omega_j > 0\}$ denote the standard open ($q-1$)-simplex. Denote the normalized vector $\boldsymbol{\omega}$ by $\underline{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \in \triangle^{q-1} \subset \mathbf{R}^q$.

We apply the implicit function theorem [21, Theorem 9.28] to the vector-valued function $\boldsymbol{F} = [\boldsymbol{F_1}, \boldsymbol{F_2}, \ldots, \boldsymbol{F_n}]$

$$\boldsymbol{F}(x, \underline{\omega}) : \mathcal{W} \times \triangle^{q-1} \to \mathbf{R}^n \text{ where } \boldsymbol{F}(x, \underline{\omega}) = \nabla(\phi_{\underline{\boldsymbol{\omega}}}(x))$$

whose domain is a subset of $\mathbf{R}^{n+q}$. Gradient is with respect to x only.

Theorem 3.6. *There is a unique continuously differentiable function* $\psi : \triangle^{q-1} \to \mathbf{R}^n$ *such that* $\psi(\underline{\omega}) = argmin(\phi_{\underline{\boldsymbol{\omega}}}(x))$ *which satisfies* $M(\psi(\underline{\boldsymbol{\omega}}))\underline{\boldsymbol{\omega}} = 0$. *The derivative of* $\psi(\underline{\omega})$ *is* $\psi'(\underline{\boldsymbol{\omega}}) = -J(\psi(\underline{\boldsymbol{\omega}}), \underline{\boldsymbol{\omega}})^{-1} M(\psi(\underline{\boldsymbol{\omega}}))$, *where* J *is the Jacobian of* $\boldsymbol{F}$.

Proof. We confirm that the conditions of the implicit function theorem are satisfied:

(i) the functions $A^{(i)}(x)^{-1} \bullet A_i^{(j)} = \frac{adj(A^{(i)}(x))}{|A^{(i)}(x)|} \bullet A_i^{(j)}$ are continuous, and

(ii) the partial derivatives $\frac{\partial \boldsymbol{F}_i}{\partial x_j}$ and $\frac{\partial \boldsymbol{F}_i}{\partial \omega_j}$ are continuous in a neighborhood of $(\hat{x}, \underline{\boldsymbol{\omega}}) \in \mathbf{R}^{n+k}$.

We must show that the $n \times n$ matrix $J = [D_j \boldsymbol{F}_i] = \left[\frac{\partial \boldsymbol{F}_i}{\partial x_j}\right]$ with respect to $x_1, \ldots, x_n$ is invertible in $\mathcal{W}$. By Lemma 2.1 we have the continuous partial derivatives

$$\begin{aligned}\frac{\partial \boldsymbol{F}_i}{\partial x_j} &= \frac{\partial(-\sum_{k=1}^{q} \underline{\omega}_k (A^{(k)}(x))^{-1} \bullet A_i^{(k)})}{\partial x_j} \\ &= -\sum_{k=1}^{q} \underline{\omega}_k [(A^{(k)}(x))^{-1} A_i^{(k)}] \bullet [(A^{(k)}(x))^{-1} A_j^{(k)}].\end{aligned}$$

However, by Lemma 2.1 and (2.2), the Jacobian matrix $J = \sum_{k=1}^{q} \underline{\omega}_k H^{(k)}(x)$ is a linear combination of the Hessians of the $A^{(k)}(x)$. Since each of the Hessians is positive definite at all $x \in \mathcal{W}$, and $\underline{\omega}_k > 0$, it follows that J is positive definite in $\mathcal{W}$ and J is invertible.

By the implicit function theorem, there is a unique continuously differentiable function $\boldsymbol{\psi}$: $\triangle^{q-1} \to \mathbb{R}^n$ such that $\psi(\hat{\underline{\boldsymbol{\omega}}}) = \hat{x}$ and $M(\psi(\underline{\boldsymbol{\omega}}))\underline{\boldsymbol{\omega}} = 0$. This is precisely the condition for $\hat{x}$ to be the absolute minimum of $\phi_{\underline{\omega}}(x)$, so $\psi(\underline{\boldsymbol{\omega}}) = argmin(\phi_{\underline{\omega}}(x))$ is unique. The derivative is given by $\psi'(\underline{\boldsymbol{\omega}}) = -J(\psi(\underline{\boldsymbol{\omega}}), \underline{\boldsymbol{\omega}})^{-1} M(\psi(\underline{\boldsymbol{\omega}}))$ [21, Theorem 9.28]. □

We next examine the mapping ψ and use it to obtain some properties of $\mathcal{W}$.

Theorem 3.7. *$\psi : \triangle^{q-1} \to \mathcal{W}$ is a continuous open onto mapping. $\mathcal{W}$ is a connected open contractible subset of $\mathcal{R}$. The preimages $\psi^{-1}(x)$ are convex and are homeomorphic to either $\triangle^{q-(n+1)}$ or $\triangle^{q-n}$.*

Proof. Every point $\hat{x} \in \mathcal{W}$ is a weighted analytic center for some $\hat{\boldsymbol{\omega}} > 0$, so ψ is an onto mapping. Since $\mathcal{W}$ is the continuous image of the contractible set $\triangle^{q-1}$, it is connected and contractible too. Since ψ is continuously differentiable, by [21, Theorem 9.25] ψ is an open function, and $\mathcal{W}$ is an open set.

If $\psi(\underline{\boldsymbol{\omega}}^1) = \psi(\underline{\boldsymbol{\omega}}^2) = \hat{x}$, then $M(\hat{x})\underline{\boldsymbol{\omega}}^1 = M(\hat{x})\underline{\boldsymbol{\omega}}^2 = 0$. Clearly $(1-t) \times \underline{\boldsymbol{\omega}}^1 + t \times \underline{\boldsymbol{\omega}}^2 > 0$ for $0 \le t \le 1$, and $M(\hat{x})((1-t) \times \underline{\boldsymbol{\omega}}^1 + t \times \underline{\boldsymbol{\omega}}^2) = 0$. Thus, $\psi((1-t) \times \underline{\boldsymbol{\omega}}^1 + t \times \underline{\boldsymbol{\omega}}^2) = \hat{x}$, since the condition for optimality is satisfied.

Select a point $x_0 = \psi(\underline{\omega}_0)$. The pair $\{x_0, \underline{\boldsymbol{\omega}}_{\mathbf{0}}\}$ satisfies the system of $(n+1)$ linear equations in q unknowns $\mathcal{E}(x) = \{(3.7), (3.8)\}$ which we write as $E\omega = [0, \ldots, 0, 1]^T$. By Assumption 1.1, E has rank $\ge n$. If rank(E) $= n+1$ we can solve for ω to get the general solution, the affine space of vectors $v_0(x) + Span\{v_1(x), \ldots, v_{q-(n+1)}(x)\}$. The solution set is $\{\underline{\boldsymbol{\omega}} : \psi(\underline{\boldsymbol{\omega}}) = x_0\}$. This is the non-empty intersection of a $(q-(n+1))$ affine space, $\{v_0 + \sum_{j=1}^{q-(n+1)} \rho_j v_j : \rho_j \in \mathbb{R}\}$ with the standard open simplex $\triangle^{q-1}$, and is homeomorphic to $\triangle^{(q-(n+1))}$.

In the case that rank(E) $= n$ we get an affine space of dimension one higher. The intersection of the affine space with $\triangle^{q-1}$ is homeomorphic to $\triangle^{(q-n)}$. These are the only 2 cases. □

We note that $M(x)$ and $P(x)$ are not defined on $\partial(\mathcal{R})$ since $\log(\det(A^{(j)}(x)^{-1}))$ is not defined on the boundary of the feasible region where the j^{th} constraint is active. We next study $\partial(\mathcal{W}) - \partial(\mathcal{R})$. We use the matrix $P(x)$ (3.9) and borrow the results obtained for $M(x)$ here.

Theorem 3.8. *If $x_\beta \in \partial(\mathcal{W})$, the boundary of $\mathcal{W}$, then for some k there is a direction s for which the directional derivative of $A^{(k)}(x)$ is zero. The boundary $\partial(\mathcal{W})$ is contained in the union of the real algebraic varieties determined by the zero directional derivatives $\nabla_s \log\left(\det(A^{(j)}(x))\right) = s^T \nabla \log\left(\det(A^{(j)}(x))\right) = \frac{1}{D(x)} \times D^{(k)}(x) \bullet (A^{(k)}(s) - A^{(k)}(0)) = 0,\ 1 \le j \le q.$*

Proof. Let $x_\beta \in \partial(\mathcal{W}) - \partial(\mathcal{R})$ be a boundary point. By Stiemke's Theorem 3.1 there is a direction s such that $s^T P(x_\beta) \ge 0,$ and $s^T P(x_\beta) \ne 0.$

If $s^T P(x_\beta) > 0,$ then by continuity $s^T P(x) > 0$ is true for all x in a neighborhood N of x_β because $s^T P(x)$ is a linear combination of polynomials in x. However, if this were the case, then for some $\omega_y > 0$, $y = \psi(\omega_y) \in \mathcal{W} \cap N$, and we would have $0 = s^T\left(P(y)\omega_y\right) = \left(s^T P(y)\right)\omega_y > 0$ which is a *contradiction*. Hence we *never* have $s^T P(x_\beta) > 0$.

This means that for at least one k, the directional derivative of $A^{(k)}(x)$ in direction s is zero, at every point of the boundary, i.e.,

$$\nabla_s \log\left(\det(A^{(k)}(x))\right) = 0 \Leftrightarrow \sum_{i=1}^{n} D^{(k)}(x) \bullet s_i A_i^{(k)} = D^{(k)} \bullet \left(A^{(k)}(s) - A^{(k)}(0)\right) = 0.$$

This is a polynomial equation which defines a real algebraic variety in x. By Stiemke's Theorem, these varieties are disjoint from $\mathcal{W}$. □

An algebraic variety is generally nonconvex. We give an example where $\mathcal{W}$ isn't convex. We use Lemma 3.4 to test grid points along slices or cross-sections of a 3-dimensional region $\mathcal{R}$. We construct a convex region bounded by 4 semidefinite constraints designed so that the sphere centered at the origin of radius 1.01 contains the intersection of their feasible regions. They are all of the type $A^{(j)}(x) = I + B^{(j)}(x)$, where I is the 5×5 identity matrix.

Example 3.2.

$$A^{(1)}(x) = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} + x_1 \begin{bmatrix} 0.45 & -0.38 & -0.01 & 0.05 & -0.20 \\ -0.38 & 0.47 & -0.30 & -0.07 & 0.13 \\ -0.01 & -0.30 & 0.59 & 0.13 & 0.21 \\ 0.05 & -0.07 & 0.13 & 0.24 & 0.32 \\ -0.20 & 0.13 & 0.21 & 0.32 & 0.60 \end{bmatrix}$$
$$+x_2 \begin{bmatrix} 0.04 & 0.04 & -0.23 & -0.12 & -0.12 \\ 0.04 & 0.02 & -0.17 & -0.07 & -0.12 \\ -0.23 & -0.17 & 1.03 & 0.49 & 0.65 \\ -0.12 & -0.07 & 0.49 & 0.22 & 0.33 \\ -0.12 & -0.12 & 0.65 & 0.33 & 0.37 \end{bmatrix} + x_3 \begin{bmatrix} 0.11 & 0.03 & -0.35 & -0.14 & -0.24 \\ 0.03 & -0.01 & -0.13 & -0.07 & -0.10 \\ -0.35 & -0.13 & 1.04 & 0.41 & 0.75 \\ -0.14 & -0.07 & 0.41 & 0.12 & 0.29 \\ -0.24 & -0.10 & 0.75 & 0.29 & 0.52 \end{bmatrix}$$

$$A^{(2)}(x) = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} + x_1 \begin{bmatrix} -0.84 & -1.12 & 0.96 & 0.63 & 0.03 \\ -1.12 & -1.85 & 1.13 & 1.00 & 0.21 \\ 0.96 & 1.13 & -1.44 & -0.88 & 0.11 \\ 0.63 & 1.00 & -0.88 & -0.73 & -0.04 \\ 0.03 & 0.21 & 0.11 & -0.04 & -0.10 \end{bmatrix}$$
$$+x_2 \begin{bmatrix} 0.16 & -0.04 & -0.53 & -0.21 & 0.15 \\ -0.04 & -0.24 & -0.25 & -0.10 & 0.15 \\ -0.53 & -0.25 & 1.11 & 0.44 & -0.18 \\ -0.21 & -0.10 & 0.44 & 0.17 & -0.07 \\ 0.15 & 0.15 & -0.18 & -0.07 & -0.01 \end{bmatrix} + x_3 \begin{bmatrix} -0.11 & -0.24 & -0.10 & -0.10 & -0.03 \\ -0.24 & -0.20 & 0.29 & 0.13 & -0.05 \\ -0.10 & 0.29 & 0.73 & 0.44 & -0.07 \\ -0.10 & 0.13 & 0.44 & 0.26 & -0.03 \\ -0.03 & -0.05 & -0.07 & -0.03 & 0.05 \end{bmatrix}$$

$$A^{(3)}(x) = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} + x_1 \begin{bmatrix} 0.04 & 0.04 & -0.23 & -0.12 & -0.12 \\ 0.04 & 0.02 & -0.17 & -0.07 & -0.12 \\ -0.23 & -0.17 & 1.03 & 0.49 & 0.65 \\ -0.12 & -0.07 & 0.49 & 0.22 & 0.33 \\ -0.12 & -0.12 & 0.65 & 0.33 & 0.37 \end{bmatrix}$$
$$+x_2 \begin{bmatrix} -0.74 & -0.10 & 0.74 & 0.30 & -0.79 \\ -0.10 & 0.39 & 0.26 & -0.24 & -1.08 \\ 0.74 & 0.26 & -2.26 & -2.66 & -1.24 \\ 0.30 & -0.24 & -2.66 & -0.36 & -0.95 \\ -0.79 & -1.08 & -1.24 & -0.95 & -0.21 \end{bmatrix} + x_3 \begin{bmatrix} 0.22 & 0.06 & -0.69 & -0.28 & -0.47 \\ 0.06 & -0.01 & -0.26 & -0.14 & -0.21 \\ -0.69 & -0.26 & 2.08 & 0.83 & 1.50 \\ -0.28 & -0.14 & 0.83 & 0.25 & 0.57 \\ -0.47 & -0.21 & 1.50 & 0.57 & 1.00 \end{bmatrix}$$

$$A^{(4)}(x) = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} + x_1 \begin{bmatrix} -2.42 & -0.45 & 0.98 & -1.69 & -2.07 \\ -0.45 & 0.15 & 0.35 & -0.95 & -0.96 \\ 0.98 & 0.35 & -0.29 & 0.25 & 0.45 \\ -1.69 & -0.95 & 0.25 & 0.50 & 0.07 \\ -2.07 & -0.96 & 0.45 & 0.07 & -0.40 \end{bmatrix}$$
$$+x_2 \begin{bmatrix} -1.31 & -0.03 & 0.56 & 0.12 & -2.08 \\ -0.03 & 0.88 & -0.14 & -0.78 & -2.62 \\ 0.56 & -0.14 & -0.39 & -3.36 & 0.10 \\ 0.12 & -0.78 & -3.36 & 0.16 & -0.59 \\ -2.08 & -2.62 & 0.10 & -0.59 & 1.05 \end{bmatrix} + x_3 \begin{bmatrix} 0.39 & 0.63 & 1.05 & 0.99 & 0.75 \\ 0.63 & 0.49 & 0.70 & 0.87 & 1.02 \\ 1.05 & 0.70 & 0.08 & 0.82 & 0.69 \\ 0.99 & 0.87 & 0.82 & -0.03 & 0.94 \\ 0.75 & 1.02 & 0.69 & 0.94 & 0.04 \end{bmatrix}.$$

The constraints $A^{(j)}(x),\ 1 \leq j \leq 4$ define a convex region $\mathcal{R} \subseteq \mathbb{R}^3$. We check every point in the grid of the cross-sections of $\mathcal{R}$ taken on planes with constant height x_3. Recall that by Lemma 3.4 a grid point x^* is a weighted analytic center if and only if for $M = M(x^*)$, LP (3.6) has optimal value 1.

The analytic center of $\mathcal{R}$ was computed by the MATLAB function *fminunc* to be at

$$[-0.11091949382933, 0.06861587677354, 0.43344995850369].$$

The top of the region $\mathcal{R}$ occurs between $x_3 = 1$ and $x_3 = 1.01$, as the constraint $A^{(4)}(x) \succ 0$ is infeasible when $x_3 = 1.01$.

We now show a progression of five slices moving down from the top of $\mathcal{R}$, with the same magnification and centers. These slices of $\mathcal{R}$ in $\mathbb{R}^3$ show its intersection with $\mathcal{W}$. In each slice we have $-.075 \leq x_1 \leq +.075$ in 61 steps of $.0025$ and $-.015 \leq x_2 \leq +.015$ in 61 steps of $.0005$. The slices demonstrate the lack of convexity of $\mathcal{W}$. The feasible region $\mathcal{R}$ is shaded, and the set of weighted analytic centers from $\mathcal{W}$ on each slice is shaded dark.

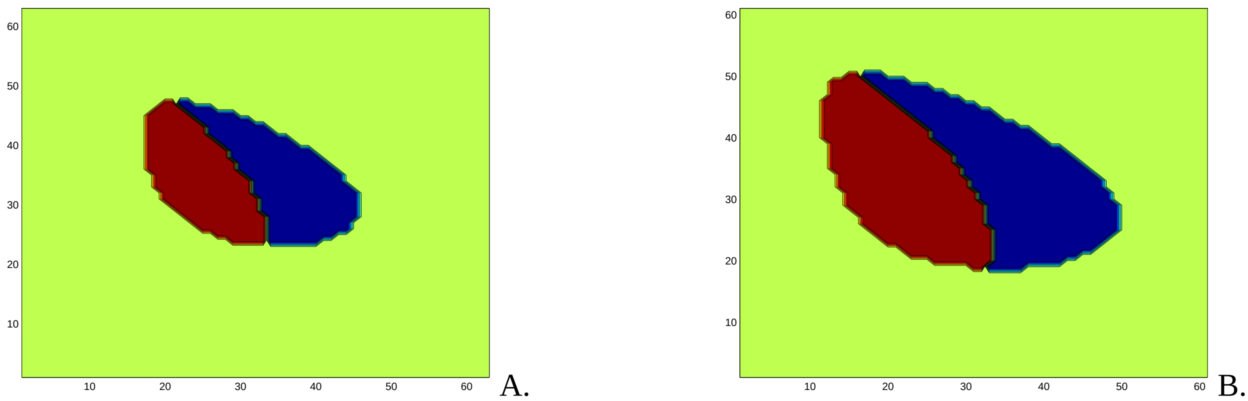

Figure 3.2: A) Slice of $\mathcal{R}$ at height $x_3 = 1$. B) Slice of $\mathcal{R}$ at height $x_3 = .9975$

It is clear that from the diagrams that there is a lack of convexity in the x_1 direction. We confirm this by choosing three points $P_1 = [-0.060,\ 0.004,\ 0.975]$, $P_2 = [0,\ 0.004,\ 0.975]$ and $P_3 = [-.030,\ 0.004,\ 0.975]$. The point P_3 is the midpoint of P_1 and P_2. We have,

$$M(P_1) = \begin{bmatrix} 1.8463 & -4.7567 & 0.3450 & 198.7089 \\ 0.5834 & 0.5157 & -0.1612 & 2.0731 \\ 0.5857 & -0.0173 & 0.6764 & -227.7666 \end{bmatrix}$$

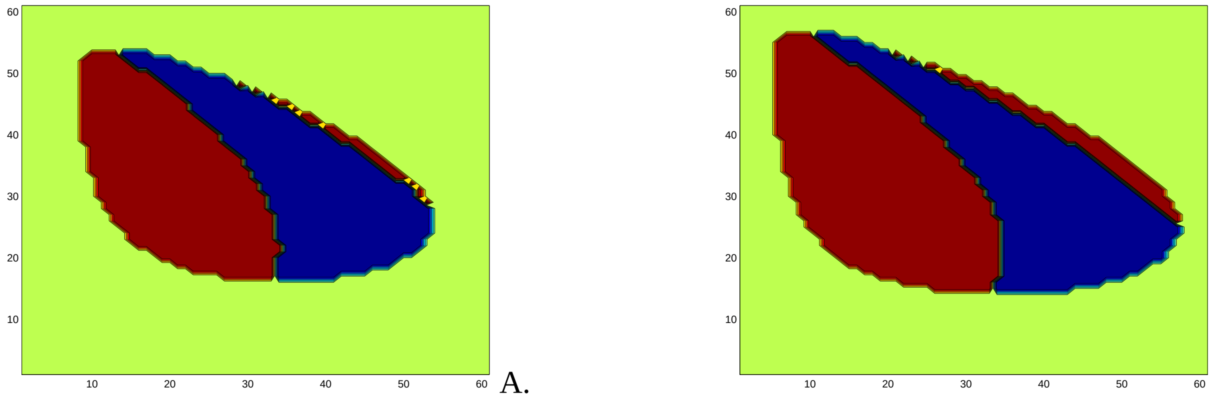

Figure 3.3: A)Slice of $\mathcal{R}$ at height $x_3 = .995$. B)Slice of $\mathcal{R}$ at height $x_3 = .9925$

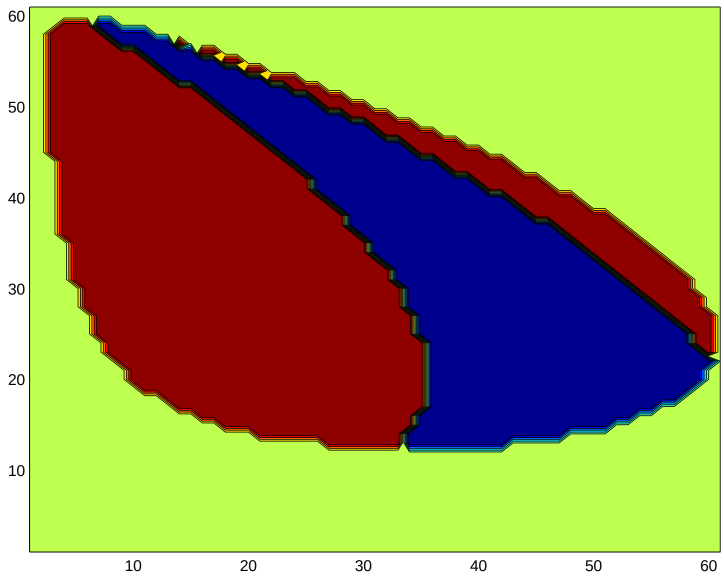

Figure 3.4: Slice of $\mathcal{R}$ at height $x_3 = .99$

$$M(P_2) = \begin{bmatrix} 1.8028 & -5.3775 & 0.3399 & 16.5750 \\ 0.5774 & 0.5641 & -0.1460 & 9.0872 \\ 0.5795 & -0.0776 & 0.6670 & -51.7798 \end{bmatrix}$$

$$M(P_3) = \begin{bmatrix} 1.7616 & -6.2081 & 0.3349 & -3.7207 \\ 0.5716 & 0.6283 & -0.1312 & -5.5310 \\ 0.5735 & -0.1596 & 0.6579 & -43.9561 \end{bmatrix}$$

At P_1, by solving LPP (3.6) with $M = M(x^*)$, we find weights $\omega_1 = [0.0494,\ 0.1758,\ 0.7724,\ 0.0024]$ so that we have a weighted analytic center at P_1. For P_2 we have the weights $\omega_2 = [0.1929,\ 0.0853,\ 0.7089,\ 0.0128]$ which makes $P_2 = x_{ac}(\omega_2)$. At their midpoint, P_3, we have, $[0.1028,\ 1.4601,\ 0.2814]M(P_3) = [1.1915,\ 0.2487,\ 0.0095,\ 0.4033]$. Thus, by Corollary 3.2 there is no feasible set of weight vectors $\boldsymbol{\omega}$ making P_3 a weighted analytic center. Hence we see that $\mathcal{W}$ is not convex.

3.3. **Repelling paths.** In the case of linear constraints repelling forces directed away from the planes and inversely proportional to distance are described ([26, p. 71]). Repelling paths traced by a particle in the interior of $\mathcal{R}$ under the influence of the force field generated by a subset of the constraints are discussed in [6]. We consider repelling path determined by LMI constraints.

Definition 3.2. Let $\alpha^{(k)}$ be a positive weight vector in $\mathbb{R}^q$ with the k^{th} component α and all other components 1. The repelling path associated with the k^{th} LMI constraint is the trajectory of $s^{(k)}(\alpha) = x_{ac}(\alpha^{(k)})$ as $\alpha \to \infty$ or $\alpha \to 0$.

The *central path* in semidefinite programming ([17], [26]) is an example of a repelling path and it is well-known that these are smooth. The following theorem shows that the limit of a repelling path associated with the k^{th} constraint, as $\alpha \to \infty$, is an interior point of the region determined by the k^{th} constraint. The limit as $\alpha \to 0$ is an interior point of the region determined by the other constraints. The following theorem shows that the repelling path limit as $\alpha \to \infty$ is in the interior of the feasible region of the k^{th} repelling constraint. The limit as $\alpha \to 0$ is in the interior of the feasible region of the other constraints.

Theorem 3.9. *If $s^{(k)}(\alpha) \to x_\infty$ as $\alpha \to \infty$, then $A^{(k)}(x_\infty) \succ 0$. Furthermore, if $s^{(k)}(\alpha) \to x_\infty$ as $\alpha \to 0$, then $A^{(j)}(x_\infty) \succ 0 \; for \; all \; j \neq k$.*

Proof. For $\hat{x} \in \mathcal{R}$, $A^{(j)}(\hat{x}) \succ 0$ for all j. By Definition 1.3, $s^{(k)}(\alpha) = x_{ac}(\alpha^{(k)})$ is the minimizer of the barrier function

$$\phi_{\alpha^{(k)}}(x) = \alpha \log\det[A^{(k)}(x)]^{-1} + \sum_{j=1, j\neq k}^{q} \log\det[A^{(j)}(x)]^{-1}, \tag{3.12}$$

and $\phi_{\alpha^{(k)}}(s^{(k)}(\alpha)) \leq \phi_{\alpha^{(k)}}(\hat{x})$. By using (3.12) and dividing both sides of the inequality by α we get

$$\log\det[A^{(k)}(s^{(k)}(\alpha))]^{-1} + \frac{1}{\alpha}\sum_{j=1, j\neq k}^{q} \log\det[A^{(j)}(s^{(k)}(\alpha))]^{-1}$$
$$\leq \log\det[A^{(k)}(\hat{x})]^{-1} + \frac{1}{\alpha}\sum_{j=1, j\neq k}^{q} \log\det[A^{(j)}(\hat{x})]^{-1}$$

The term $\frac{1}{\alpha}\sum_{j=1, j\neq k}^{q} \log\det[A^{(j)}(\hat{x})]^{-1}$ converges to zero as $\alpha \to \infty$. If x_∞ is on the boundary of $A^{(k)}(x) \succeq 0$, then $\log\det[A^{(k)}(s^{(k)}(\alpha))]^{-1} \to \infty$ as $\alpha \to \infty$. Hence, if x_∞ were on the boundary of $A^{(k)}(x) \succeq 0$, then $\frac{1}{\alpha}\sum_{j=1, j\neq k}^{q} \log\det[A^{(j)}(s^{(k)}(\alpha))]^{-1} \to -\infty$ as $\alpha \to \infty$. This is not possible, since $\sum_{j=1, j\neq k}^{q} \log\det[A^{(j)}(s^{(k)}(\alpha))]^{-1} > 0$ over $\mathcal{R}_k$. The proof of the case $\alpha \to 0$ is similar. □

Figure 3.5 gives the repelling paths determined by each constraint of Example 3.1. Each constraint was given weights from 0.0001 to 1000 while the other constraints were each given fixed weight 1. The paths intersect at the analytic center and two of them (1,2) overlap. It shows that limit of a repelling path is not necessarily a boundary point of the feasible region. In Section 5 we discuss how limits of repelling paths can be used with the stand-and-hit algorithm to determine necessary LMI constraints.

4. THE WEIGHTED ANALYTIC CENTER AND REDUNDANCY DETECTION

Weighted analytic centers are used in the cutting plane method by Luo [14] for solving general convex problems defined by a separation oracle. Ramaswany and Mitchell [19] have also used them for studying multiple cuts. The method of centers for path following is described in [17]. Sturm and Zhang [24] use analytic centers to study the central path in semidefinite

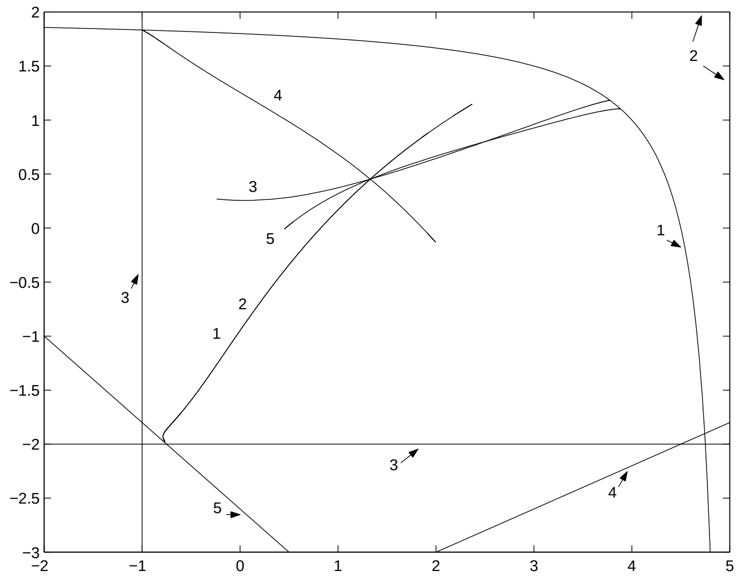

Figure 3.5: Repelling Paths of Example 3.1

programming. We present another application of weighted analytic centers to the probabilistic Stand-and-Hit method [12] for identifying necessary constraints in semidefinite programming. For $k = 1, 2, \ldots, q$, define the regions $\mathcal{R}_k$ ($\mathcal{R} \subset R_k$) by

$$\mathcal{R}_k = \{x \mid A^{(j)}(x) \succeq 0, \;\; j \in \{1, 2, \ldots, k-1, k+1, \ldots, q\}\}.$$

Definition 4.1. $\mathrm{A}^{(\mathrm{k})}(\mathbf{x}) \succ 0$ is called redundant with respect to the set $\{A^{(j)}(x) \succeq 0\}_{j=1}^{q}$ if $\mathcal{R} = R_k$, and is called necessary if $\mathcal{R} \neq \mathcal{R}_k$.

A LMI constraint is called *necessary* if its removal changes the feasible region of the problem, otherwise it is called *redundant*. The *semidefinite redundancy problem* is to decide whether or not the k^{th} constraint $\mathrm{A}^{(\mathrm{k})}(\mathbf{x}) \succ 0$ is redundant with respect to the set $\{A^{(j)}(x) \succeq 0\}_{j=1}^{q}$.

The significance of this for the linear case is discussed in [5] and for SDP's in [12]. This is important as the running times of SDP algorithms grows nonlinearly with the number of constraints [26]. The Semidefinite Stand-and-Hit (SSH) method [11, 12] starts by selecting a point $\hat{x} \in \mathcal{R}$ called the *standing point*. We generate a sequence of search vectors $\{s_i\}$ from a uniform distribution over the surface of the unit hypersphere $S^{n-1} = \{x \in \mathbf{R}^n \mid \parallel x \parallel_2 = 1\}$. Each s_i determines a feasible line segment $\{\hat{x} + \sigma s_i \,|\, 0 \leq \sigma \leq \sigma_i\}$ such that if $det[A^{(j)}(\hat{x} + \sigma_i s_i)] = 0$ for index j and $A^{(k)}(\hat{x} + \sigma_i s_i) \succ 0$ for $k \neq j$, then the j^{th} constraint is necessary.

The SSH algorithm:

Initialization: Denote the index set of identified constraints by $\mathcal{J}$ and set $\mathcal{J} = \emptyset$.
Choose a standing point $\hat{x}$ of $\mathcal{R}$. Calculate $A^j(\hat{x})^{-1/2}$ for $1 \leq j \leq q$.

Repeat

Search Direction: From N(0,1) choose n entries u to generate a random point

$s = u/\|u\|_2$ uniformly on the unit hypersphere S^{n-1}

Hitting Step: Calculate $B_j(s, \hat{x}) = -A^{(j)}(\hat{x})^{-\frac{1}{2}}(\sum_{i=1}^{n} s_i A_i^{(j)})A^{(j)}(\hat{x})^{-\frac{1}{2}}$,

$\sigma_+^{(j)} = 1/\lambda_{\max}^+(B_j(s, \hat{x}))$ and $\sigma_-^{(j)} = -1/\lambda_{\min}^-(B_j(s, \hat{x}))$ for $1 \leq j \leq q$.

Calculate $\sigma_+ = \min\{\sigma_+^{(j)} \mid 1 \leq j \leq q\}$ and $\sigma_- = \min\{\sigma_-^{(j)} \mid 1 \leq j \leq q\}$.

For $1 \leq k \leq q$, if $\sigma_+^{(k)} = \sigma_+$ or $\sigma_-^{(k)} = \sigma_-$ and $k \notin \mathcal{J}$, set $\mathcal{J} = \mathcal{J} \cup \{k\}$.

Until a stopping rule holds.

Since $\hat{x}$ is fixed, we only compute $A^j(\hat{x})^{-\frac{1}{2}}$ once throughout the detection process. After the termination of the SSH algorithm, all LMI's in the set $\mathcal{J}$ are declared necessary.

Let $p_j(\hat{x})$ be the probability that constraint j is detected in an iteration of the SSH algorithm from the standing point $\hat{x}$. It has been shown [7] that the expected number of iterations required to detect all necessary constraints is minimized if $\hat{x}$ can be found so that the detection probabilities $p_j(\hat{x})$ of all the constraints are equal. In general, such a point $\hat{x}$ does not exist, even for the linear case. We pursue the more modest goal of trying to find a point $\hat{x}$ that will minimize the standard deviation of the probabilities.

The strategy proposed in [10] is to initially choose $\hat{x}$ as the analytic center. We run the SSH algorithm from $\hat{x}$ and for each constraint, determine the number of times the constraint is detected, i.e., its *hit frequency*. The hit frequencies are directly related to the probabilities $p_j(\hat{x})$. We use these frequencies to determine a new weighted analytic center, and repeat the SSH algorithm. We continue with such repetitions, each from a newly calculated weighted analytic center, until a stopping criteria has been satisfied. The detection probability is directly proportional to the *angle of sight* of the constraint from the standing point. We next apply our SSH strategy.

Example 4.1. Consider Example 3.1. The analytic center of this problem is $x_{ac} = (1.3292, 0.4530)$. After 50 iterations of SSH at x_{ac}, the number of hits of constraints (1,2,3,4,5) were (50,0,48,0,2) respectively. Figure 4.1 shows how $x_{ac} = (1.3292, 0.4530)$ moves to $x_{ac}(\omega) = (1.7308, -0.1470)$ under the influence of the weight vector $\omega = (50, 1, 48, 1, 2)$. The undetected constraints, i.e., (2) and (4) are given weights 1 so that the feasible region $\mathcal{R}$ remains unchanged. The weighted center $x_{ac}(\omega)$ moved closer to undetected constraint (4).

Example 4.2. We consider a problem with $q = 101$ constraints and $n = 5$ variables. The analytic center was found using the Damped Newton's Method of [17]. From the analytic center, we generated 1000 search directions to get the following nonzero hit frequencies

$$(493, 308, 29, 526, 23, 22, 642, 376, 498, 965, 150, 2026, 71, 13871).$$

Note that only 14 constraints were detected. The standard deviation is 3620.

Using the weights equal to the frequencies for the detected constraints, and weights of one for the undetected constraints, we determined the corresponding weighted analytic center, and used it as the next standing point. From this point, we found the new frequencies of

$$(6363, 3194, 8, 1328, 1, 4, 872, 92, 145, 3052, 279, 4384, 3, 275),$$

which correspond to the same detected constraints. The standard deviation is reduced to 2026. We see that from the new point, the detection probabilities (frequencies) are more "balanced", and we have a better point for the SSH method.

Limits of repelling paths can be used with SSH to determine necessary constraints. The idea is to assign weight α to one constraint and weight 1 to the others. The corresponding

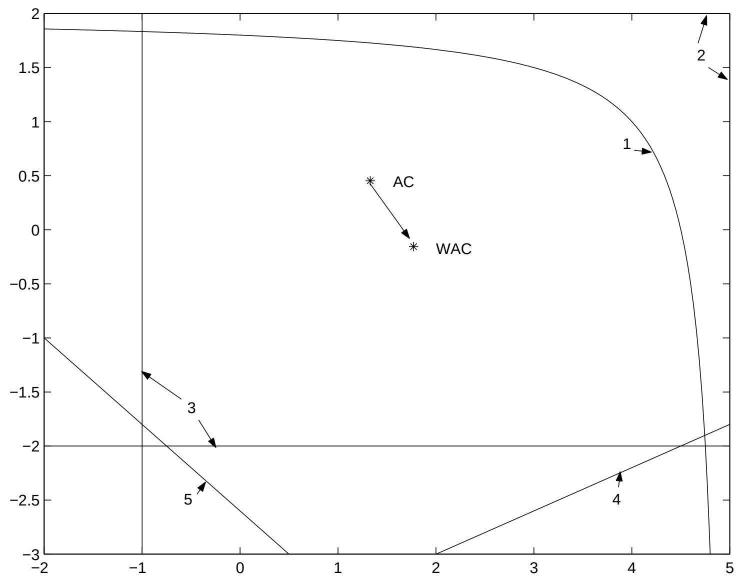

Figure 4.1:

This figure shows how $x_{ac} = (1.3292, 0.4530)$ moves to $x_{ac}(\omega) = (1.7308, -0.1470)$ under the influence of the weight vector $\omega = (50, 1, 48, 1, 2)$.

weighted analytic center can be used as a standing point for SSH to detect constraints close to the boundary. Repeating this procedure over each constraint provides *useful* standing points in some cases. For example, in Figure 3.5, there are 8 repelling limits; 4 are interior points and 4 are boundary points. The boundary points identify constraints 1 and 5. The limits in the interior are *good* standing points for identifying the remaining constraints. This is still under investigation.

5. CONCLUSION

We extended the notion of weighted analytic center from linear constraints to semidefinite programming and have shown that the region of weighted analytic centers $\mathcal{W} \subset \mathcal{R}$. We have studied the geometry and topology of $\mathcal{W}$, both theoretically and through comprehensive examples. We have proven that $\mathcal{W} = \mathcal{R}$ in the case of linear constraints, but in the semidefinite situation $\mathcal{R} \not\subset \mathcal{W}$, i.e., there exist feasible points $\hat{x}$ in $\mathcal{R}$ which are not weighted analytic centers.

We have shown both analytically and by graphical means that $\mathcal{W}$ is an open contractible subset of $\mathbb{R}^n$, but is not convex. We have given cross-sections of $\mathcal{W}$, by making extensive use of Stiemke's Theorem of the alternative while solving small linear programming problems at each point of a grid. In the course of this, we have provided a graphical representation of a non-trivial 3-dimensional real algebraic variety. We have demonstrated how one can use the varieties to describe $\mathcal{W}$ and its boundary. We have also shown by an example, a potential application of weighted analytic centers to improve the standing point of the Stand-and-Hit method (SSH) for identifying necessary constraints in semidefinite programming.

We have also studied the *argmin* function extensively and have proven it to be continuously differentiable and open for the functions used here.

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