Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 1, Issue 2, Article 14, 2000

REGULARITY RESULTS FOR VECTOR FIELDS OF BOUNDED DISTORTION AND APPLICATIONS

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Received 17 January, 2000; accepted 4 April, 2000
Communicated by S. Saitoh

ABSTRACT. In this paper we prove higher integrability results for vector fields B, E, $(B,E) \in L^{2-\epsilon}(\Omega,\mathbb{R}^n) \times L^{2-\varepsilon}(\Omega,\mathbb{R}^n),$ ε small, such that $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$ satisfying a "reverse" inequality of the type

$$|B|^2 + |E|^2 \leq \left(K + \frac{1}{K}\right) \langle B, E\rangle + |F|^2$$

with $K \geq 1$ and $F \in L^r(\Omega,\mathbb{R}^n),$ $r > 2-\varepsilon.$ Applications to the theory of quasiconformal mappings and partial differential equations are given. In particular, we prove regularity results for very weak solutions of equations of the type

$$\operatorname{div} a(x,\nabla u) = \operatorname{div} F.$$

If $|a(x,z)|^2 + |z|^2 \leq (K + 1/K)\,\langle a(x,z), z\rangle$, in the homogeneous case, our method provides a new proof of the regularity result

$$u \in W^{1,2-\varepsilon}_{loc}(\Omega) \Rightarrow u \in W^{1,2+\varepsilon}_{loc}(\Omega)$$

where ε is sufficiently small. A result of higher integrability for functions verifying a reverse integral inequality is used, and its optimality is proved.

Key words and phrases: Reverse Inequalities, Finite Distortion Vector Fields, Div-Curl Vector Fields, Elliptic Partial Differential Equations.

2000 *Mathematics Subject Classification.* 35J60, 26D15.

ISSN (electronic): 1443-5756

We wish to thank Prof. T. Iwaniec for stimulating discussions on this subject.

This work has been performed as a part of a National Research Project supported by M.U.R.S.T..

022-99

1. INTRODUCTION

The usual way to establish the $W^{1,2+\varepsilon}_{loc}(\Omega)$, $\varepsilon > 0$, regularity of solutions $u \in W^{1,2}_{loc}(\Omega)$ of equations of the type

$$\operatorname{div} a(x, \nabla u) = \operatorname{div} F \qquad \text{in} \quad \Omega,$$

where $|a(x,z)|^2 + |z|^2 \le (K + 1/K)\,\langle a(x,z), z\rangle$, is to combine the Caccioppoli inequality

$$⨍_Q |\nabla u|^2 dx \le c \left[⨍_{2Q} \left| \frac{u - u_{2R}}{2R} \right|^2 dx + ⨍_{2Q} |F|^2 dx \right],$$

where R is the sidelength of the cube $Q \subset 2Q \subset \Omega$, with the Poincaré-Sobolev inequality

$$\left(⨍_Q \left| \frac{u - u_R}{R} \right|^2 dx \right)^{\frac{1}{2}} \le \left(⨍_Q |\nabla u|^{\frac{2n}{n+2}} dx \right)^{\frac{n+2}{2n}},$$

to obtain the nonhomogeneous reverse Hölder inequality

$$⨍_Q |\nabla u|^2 dx \le c \left\{ \left[⨍_{2Q} (|\nabla u|^2)^{\frac{n}{n+2}} dx \right]^{\frac{n+2}{n}} + ⨍_{2Q} |F|^2 dx \right\}. \tag{1.1}$$

The higher integrability result then arises by using the well-known Giaquinta-Modica technique [3, 2].

The aim of this paper is to provide a different way to get regularity results, based on inequalities for div-curl vector fields (see Theorem 2.1, [5, 10]). Starting from these inequalities, under the assumption of bounded distortion, we get directly a family of reverse type inequalities, namely

$$\begin{aligned} (1.2)\quad & ⨍_Q (|\nabla u|^2)^{1-\varepsilon} dx \\ & \le c_1 \varepsilon ⨍_{2Q} (|\nabla u|^2)^{1-\varepsilon} dx + c_2 \left[⨍_{2Q} (|\nabla u|^2)^{(1-\varepsilon)\frac{n}{n+1}} dx \right]^{\frac{n+1}{n}} + c_3 ⨍_{2Q} (|F|^2)^{(1-\varepsilon)} dx. \end{aligned}$$

Notice that, even if inequality (1.2) contains an extra term, by using our method we are able to obtain a higher integrability result also for *very weak* solutions of some nonlinear elliptic equations by just assuming an integrability on the gradient below the natural exponent (see [8]). Let us observe also that if $\varepsilon = 0$ the exponent $n/(n+1)$ in inequality (1.2) is larger than the exponent $n/(n+2)$ in inequality (1.1). Actually, inequality (1.2) follows from a more general argument about vector fields of bounded distortion, which includes an analogous result of the theory of quasiregular mappings (with the same exponent we get in (1.2), see [7]).

After recalling known results in Section 2, we prove a higher integrability result for functions verifying a reverse-type inequality (Theorem 3.1) in Section 3. In Section 4 we give a counterexample showing that generally the assumptions in Theorem 3.1 cannot be weakened. In Section 5 we prove a higher integrability result for finite distortion vector fields (see Proposition 5.1), and we give some applications to the theory of quasiconformal mappings and to the theory of regularity for very weak solutions of homogeneous nonlinear elliptic equations in divergence form. Finally, in Section 6, we extend our method to the case of more general vector fields in order to study the case of nonhomogeneous equations (see Theorem 6.1).

2. PRELIMINARY RESULTS

In the following we will consider div-curl vector fields $B = (B_1, \ldots, B_n) \in L^q(\mathbb{R}^n, \mathbb{R}^n)$, $E = (E_1, \ldots, E_n) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, i.e.

$$\begin{aligned} \operatorname{curl} E &= \left(\frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \right)_{i,j=1,\ldots,n} = 0 \\ \operatorname{div} B &= \sum_{i=1}^n \frac{\partial B_i}{\partial x_i} = 0 \end{aligned} \tag{2.1}$$

in the sense of distributions.

The following basic estimates are established in [5] (see also [10] for the present formulation). We denote by Q_0, Q open cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axis, and by $2Q$ the cube with the same center of Q and double side-length.

Theorem 2.1. *Let $1 < p, q < \infty$ be a Hölder conjugate pair, $\frac{1}{p} + \frac{1}{q} = 1$, and let $1 < r, s < \infty$ be a Sobolev conjugate pair, $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{n}$. Then there exists a constant $c_n = c_n(p, s)$ such that for each cube Q such that $2Q \subset Q_0 \subset \mathbb{R}^n$ we have*

$$\begin{aligned} \left| \fint_Q \frac{\langle B, E \rangle}{|B|^\varepsilon |E|^\varepsilon} dx \right| &\le c_n \varepsilon \left[\fint_{2Q} |E|^{(1-\varepsilon)p} dx \right]^{\frac{1}{p}} \left[\fint_{2Q} |B|^{(1-\varepsilon)q} dx \right]^{\frac{1}{q}} \\ &\quad + c_n \left[\fint_{2Q} |E|^{(1-\varepsilon)s} dx \right]^{\frac{1}{s}} \left[\fint_{2Q} |B|^{(1-\varepsilon)r} dx \right]^{\frac{1}{r}}, \end{aligned} \tag{2.2}$$

whenever $0 \le 2\varepsilon \le \min\left\{ \frac{p-1}{p}, \frac{q-1}{q}, \frac{r-1}{r}, \frac{s-1}{s} \right\}$ and $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$.

The following proposition by Giaquinta-Modica [3, 2] will be useful in the sequel.

Proposition 2.2. *Let $g \in L^\alpha(Q_0), \alpha > 1$ and $f \in L^r(Q_0), r > \alpha$ be two non-negative functions and suppose that for every cube Q such that $2Q \subset Q_0$ the following estimate holds*

$$\fint_Q g^\alpha dx \le b \left\{ \left(\fint_{2Q} g dx \right)^\alpha + \fint_{2Q} f^\alpha dx \right\} + \theta \fint_{2Q} g^\alpha dx \tag{2.3}$$

with $b > 1$. There exist constants $\theta_0 = \theta_0(\alpha, n)$, $\sigma_0 = \sigma_0(b, \theta, \alpha, r, n)$ such that if $\theta < \theta_0$, then $g \in L^{\alpha+\sigma}_{loc}(Q_0)$ for all $0 < \sigma < \sigma_0$ and

$$\left(\fint_Q g^{\alpha+\sigma} dx \right)^{\frac{1}{\alpha+\sigma}} \le c \left\{ \left(\fint_{2Q} g^\alpha dx \right)^{\frac{1}{\alpha}} + \left(\fint_{2Q} f^{\alpha+\sigma} dx \right)^{\frac{1}{\alpha+\sigma}} \right\}, \tag{2.4}$$

where c is a positive constant depending on b, θ, α, r, n.

3. REVERSE HÖLDER INEQUALITIES AND HIGHER INTEGRABILITY

This section is concerned with a variant of the result established in Proposition 2.2. We remark that in our assumption (3.1) we will consider a family of inequalities of the type (2.3) in which both the exponent of integrability of the function g and a coefficient in the right hand side depend on ε. Nevertheless, even if Proposition 2.2 cannot be applied *a priori*, in the theorem we will prove that we can obtain a higher integrability result for g and an estimate of the type (2.4).

Theorem 3.1. *Let $g \in L^{2(1-\varepsilon)}(Q_0)$ and $f \in L^r(Q_0)$, $0 \le \varepsilon < \frac{1}{2}, r > 2(1-\varepsilon)$, be nonnegative functions such that*

$$\fint_Q g^{2(1-\varepsilon)}dx \le c_1\varepsilon \fint_{2Q} g^{2(1-\varepsilon)}dx + c_2 \left\{ \left(\fint_{2Q} g^{2(1-\varepsilon)\frac{n}{n+1}}dx \right)^{\frac{n+1}{n}} + \left(\fint_{2Q} f^{2(1-\varepsilon)}dx \right) \right\} \tag{3.1}$$

for every cube $Q \subset 2Q \subset Q_0$, for some constants $c_1 \ge 0$, $c_2 > 0$.

Then there exist $\bar\varepsilon = \bar\varepsilon(c_1, n)$ and $\bar\eta = \bar\eta(c_1, c_2, r, \varepsilon, n)$ such that if $0 \le \varepsilon < \bar\varepsilon$, then $g \in L^{2(1-\varepsilon)+\eta}_{loc}(Q_0)$, $\forall\ 0 \le \eta < \bar\eta$ and

$$\left(\fint_Q g^{2(1-\varepsilon)+\eta}dx \right)^{\frac{1}{2(1-\varepsilon)+\eta}} \le c \left\{ \left(\fint_{2Q} g^{2(1-\varepsilon)}dx \right)^{\frac{1}{2(1-\varepsilon)}} + \left(\fint_{2Q} f^{2(1-\varepsilon)+\eta}dx \right)^{\frac{1}{2(1-\varepsilon)+\eta}} \right\},$$

where c is a positive constant depending on c_2, r, ε, n.

Proof. Since the functions $g_\varepsilon = g^{2(1-\varepsilon)\frac{n}{n+1}}$, $f_\varepsilon = f^{2(1-\varepsilon)\frac{n}{n+1}}$ verify the inequality

$$\fint_Q g_\varepsilon^{\frac{n+1}{n}} dx \le c_2 \left\{ \left(\fint_{2Q} g_\varepsilon dx \right)^{\frac{n+1}{n}} + \left(\fint_{2Q} f_\varepsilon^{\frac{n+1}{n}} dx \right) \right\} + c_1\varepsilon \fint_{2Q} g_\varepsilon^{\frac{n+1}{n}} dx \tag{3.2}$$

we can apply Proposition 2.2 with $\alpha = \frac{n+1}{n}$ and $b = c_2$. We get $\theta_0 = \theta_0(n)$ and $\sigma_0 = \sigma_0(c_2, r, \varepsilon, n)$ such that, if (3.2) holds with $c_1\varepsilon < \frac{\theta_0}{2}$, then $g_\varepsilon \in L^{\alpha+\sigma}_{loc}(Q_0)$ for every $0 < \sigma < \sigma_0$, i.e.

$$\left[g^{2(1-\varepsilon)\frac{n}{n+1}} \right]^{\frac{n+1}{n}+\sigma} \in L^1_{loc}(Q_0) \qquad \forall\ 0 < \sigma < \sigma_0$$

and

$$\left(\fint_Q g_\varepsilon^{\frac{n+1}{n}+\sigma} dx \right)^{\frac{1}{\frac{n+1}{n}+\sigma}} \le c \left\{ \left(\fint_{2Q} g_\varepsilon^{\frac{n+1}{n}} dx \right)^{\frac{n}{n+1}} + \left(\fint_{2Q} f_\varepsilon^{\frac{n+1}{n}+\sigma} dx \right)^{\frac{n}{n+1}+\sigma} \right\} \tag{3.3}$$

with c depending on c_2, r, ε, n.

Set

$$0 < \bar\varepsilon < \frac{\theta_0}{2c_1}, \qquad 0 < \bar\eta < (1-\bar\varepsilon)\frac{2n\sigma_0}{n+1}.$$

If $0 \le \varepsilon < \bar\varepsilon$ and $0 \le \eta < \bar\eta$, we have

$$\varepsilon < \bar\varepsilon < 1 - \bar\eta\frac{n+1}{2n\sigma_0} < 1 - \eta\frac{n+1}{2n\sigma_0}$$

or, equivalently,

$$2(1-\varepsilon) + \eta < 2(1-\varepsilon)\frac{n}{n+1}\left[\frac{n+1}{n} + \sigma_0 \right].$$

Therefore we get

$$g \in L^{2(1-\varepsilon)+\eta}_{loc}(Q_0)$$

and inequality (3.3) becomes

$$\left(\fint_Q g^{2(1-\varepsilon)+\eta}dx \right)^{\frac{1}{2(1-\varepsilon)+\eta}} \le c \left\{ \left(\fint_{2Q} g^{2(1-\varepsilon)}dx \right)^{\frac{1}{2(1-\varepsilon)}} + \left(\fint_{2Q} f^{2(1-\varepsilon)+\eta}dx \right)^{\frac{1}{2(1-\varepsilon)+\eta}} \right\}.$$

□

Let us observe that upon closer inspection of the proof of Theorem 3.1, one can note that the gain of integrability given by $\sigma_0 = \sigma_0(c_2, r, \varepsilon, n)$ is actually dependent only on $c_2, \frac{r}{2(1-\varepsilon)}, n$. Nevertheless, if $f \equiv 0$ a.e. in Q_0, the number σ_0, and therefore also $\bar\eta$ and c, do not depend on ε. This remark is crucial to prove the following.

Corollary 3.2. *Let* $0 \le \varepsilon < \frac{1}{2}$ *and* $g \in L^{2(1-\varepsilon)}(Q_0)$, $Q_0 \subset \mathbb{R}^n$, *be such that*

$$\fint_Q g^{2(1-\varepsilon)} dx \le c_1 \varepsilon \fint_{2Q} g^{2(1-\varepsilon)} dx + c_2 \left(\fint_{2Q} g^{2(1-\varepsilon)\frac{n}{n+1}} dx \right)^{\frac{n+1}{n}}$$

for every cube $Q \subset 2Q \subset Q_0$.

Then there exists $\bar{\varepsilon} = \bar{\varepsilon}(c_1, n)$ *such that if* $0 \le \varepsilon < \bar{\varepsilon}$, *then* $g \in L^{2+2\varepsilon}_{loc}(Q_0)$ *and*

$$\left(\fint_Q g^{2(1+\varepsilon)} dx \right)^{\frac{1}{2(1+\varepsilon)}} \le c \left(\fint_{2Q} g^{2(1-\varepsilon)} dx \right)^{\frac{1}{2(1-\varepsilon)}} \tag{3.4}$$

where c *is a positive constant depending on* c_2, n.

Proof. Let us apply Theorem 3.1 with $f \equiv 0$ a.e. in Q_0. If $\varepsilon < \min(\bar{\varepsilon}, \frac{\bar{\eta}}{4})$, choosing $\eta = 4\varepsilon$, from inequality (3.1) we get $g \in L^{2+2\varepsilon}_{loc}(Q_0)$ and inequality (3.4) holds. ☐

4. A COUNTEREXAMPLE

Let us consider f, g non-negative functions on a cube Q_0 satisfying assumptions of the type of Theorem 3.1 with $c_1 = 0$, namely, f, g are such that $g \in L^{\alpha}(Q_0)$, $f \in L^{\lambda\alpha}(Q_0)$ for some $\alpha > 1$, $\lambda > 1$ and

$$\left(\fint_Q g^{\alpha} dx \right)^{\frac{1}{\alpha}} \le a \fint_{2Q} g dx + b \left(\fint_{2Q} f^{\alpha} dx \right)^{\frac{1}{\alpha}} \qquad \forall Q,\ 2Q \subset Q_0. \tag{4.1}$$

In this case it is known, [6], that if λ is sufficiently close to 1, $g \in L^{\lambda\alpha}_{loc}(Q_0)$ and

$$\left(\fint_Q g^{\lambda\alpha} dx \right)^{\frac{1}{\lambda\alpha}} \le a_\lambda \left(\fint_{2Q} g^{\lambda} dx \right)^{\frac{1}{\lambda}} + b_\lambda \left(\fint_{2Q} f^{\lambda\alpha} dx \right)^{\frac{1}{\lambda\alpha}}, \tag{4.2}$$

where a_λ and b_λ are constants depending only on n, α, a, b.

In the following we show that, even if it is still true that $g \in L^{\lambda\alpha}_{loc}(Q_0)$ for any $\lambda < 1$ (sufficiently small), one cannot find any $\lambda < 1$, $a_\lambda > 0$, $b_\lambda > 0$ such that estimate (4.2) holds for any $g \in L^{\alpha}(Q_0)$, $f \in L^{\alpha}(Q_0)$ verifying (4.1). If we consider a function $f \in \bigcap_{1 \le p < \alpha} L^p(Q_0)$ such that $\int_Q f^{\alpha} dx = +\infty$ $\forall Q \subset Q_0$, of course we have $f \in L^{\lambda\alpha}(Q_0)$ for $\lambda < 1$, and it is possible to show immediately that there are no $a_\lambda, b_\lambda > 0$ such that (4.2) holds for any $g \in L^{\alpha}(Q_0)$, *for any* $f \in L^{\lambda\alpha}(Q_0)$ verifying (4.1).

We will proceed as follows: by a contradiction argument, we will be able to prove that there exists $\lambda < 1$ such that *any* function $g_0 \in L^{\lambda\alpha}(Q_0)$, $g_0 > 0$, satisfies a certain reverse inequality, which is generally false.

Let $g \in C(\overline{Q_0})$, $g > 0$. Then there exists $\delta > 0$ such that

$$\left(\fint_{2Q} g^{\alpha} dx \right)^{\frac{1}{\alpha}} \le 2 \fint_{2Q} g dx \qquad \forall Q,\ 2Q \subset Q_0,\ \text{diam}\ 2Q < \delta.$$

In fact,

$$\sup_{x \in 2Q} g(x) \le \sup_{x,y \in 2Q} |g(x) - g(y)| + g(y) \quad \forall y \in 2Q$$

and then, because of the uniform continuity of g, we have

$$\sup_{2Q} g \le 2 \inf_{2Q} g.$$

Let us divide Q_0 into a finite number of disjoint cubes

$$Q_0 = \bigcup_{j=1}^{N} Q_j$$

such that

$$2Q \subset Q_0, \text{ diam } 2Q \geq \delta \quad \Rightarrow \quad \exists Q_j : Q_j \subset 2Q.$$

Now let us point out that if f is any function in $L^r(Q_0)$ for every $1 \leq r < \alpha$, but

$$\int_{Q_j} f^\alpha dx = +\infty \qquad \forall j = 1, \ldots, N$$

then there exist

$$f_k = \max_{j=1,\ldots,N} \frac{k \min\{f,k\}}{\left(\fint_{Q_j} (\min\{f,k\})^\alpha dx \right)^{\frac{1}{2\alpha}}}$$

such that

$$\left(\fint_{2Q} \left(\frac{1}{k} f_k \right)^\alpha \right)^{\frac{1}{\alpha}} \geq \max_{\overline{Q_0}} g \geq \left(\fint_{2Q} g^\alpha dx \right)^{\frac{1}{\alpha}} \qquad \forall Q, \ 2Q \subset Q_0, \text{ diam } 2Q \geq \delta.$$

Therefore

$$\left(\fint_{2Q} g^\alpha dx \right)^{\frac{1}{\alpha}} \leq 2 \fint_{2Q} g dx + \left(\fint_{2Q} \left(\frac{1}{k} f_k \right)^\alpha \right)^{\frac{1}{\alpha}} \qquad \forall Q, \ 2Q \subset Q_0,$$

i.e. $g, \frac{1}{k} f_k$ satisfy inequality (4.1).

Let us suppose to the contrary that for some $\lambda < 1$

$$\left(\fint_{Q} g^{\lambda\alpha} dx \right)^{\frac{1}{\lambda\alpha}} \leq a_\lambda \left(\fint_{2Q} g^\lambda dx \right)^{\frac{1}{\lambda}} + b_\lambda \left(\fint_{2Q} \left(\frac{1}{k} f_k \right)^{\lambda\alpha} \right)^{\frac{1}{\lambda\alpha}} \qquad \forall Q, \ 2Q \subset Q_0.$$

Letting k tend to infinity, we have the inequality

$$\left(\fint_{Q} g^{\lambda\alpha} dx \right)^{\frac{1}{\lambda\alpha}} \leq a_\lambda \left(\fint_{2Q} g^\lambda dx \right)^{\frac{1}{\lambda}} \qquad \forall Q, \ 2Q \subset Q_0$$

for every continuous function g.

This inequality, by an approximation argument, extends to every function $g_0 \in L^{\lambda\alpha}(Q_0)$, $g_0 > 0$

$$\left(\fint_{Q} g_0^{\lambda\alpha} dx \right)^{\frac{1}{\lambda\alpha}} \leq a_\lambda \left(\fint_{2Q} g_0^\lambda dx \right)^{\frac{1}{\lambda}} \qquad \forall Q, \ 2Q \subset Q_0,$$

which is absurd, since this inequality implies a higher integrability for g_0.

5. Higher Integrability Results and Applications

We start with the following regularity result for vector fields of bounded distortion

Proposition 5.1. *Let* $\Omega \subset \mathbb{R}^n$, $0 < \epsilon < 1$ *and* $\Phi = (E, B) \in L^{2-\epsilon}(\Omega, \mathbb{R}^n) \times L^{2-\varepsilon}(\Omega, \mathbb{R}^n)$ *be such that* $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$ *and*

$$|B(x)|^2 + |E(x)|^2 \leq \left(K + \frac{1}{K} \right) \langle B(x), E(x) \rangle \qquad \textit{a.e. in } \Omega, \tag{5.1}$$

where $K \geq 1$. Then there exists $\bar{\varepsilon} = \bar{\varepsilon}(K,n)$ such that $\Phi \in L^{2+\varepsilon}_{loc}(\Omega, \mathbb{R}^n) \times L^{2+\varepsilon}_{loc}(\Omega, \mathbb{R}^n)$ for all $0 < \varepsilon < \bar{\varepsilon}$ and

$$\left(\fint_Q |\Phi|^{2+\varepsilon} dx \right)^{\frac{1}{2+\varepsilon}} \leq c \left(\fint_{2Q} |\Phi|^{2-\varepsilon} dx \right)^{\frac{1}{2-\varepsilon}} \qquad \forall Q,\ 2Q \subset \Omega,$$

where c is a positive constant depending on K, n.

Proof. Let us fix the cube Q such that $2Q \subset \Omega$. Applying Theorem 2.1 with $p = q = 2$ and $r = s = \frac{2n}{n+1}$, from inequality (5.1) we get

$$\fint_Q (|B|^2 + |E|^2)^{1-\varepsilon} dx \leq c_{n,K}\, \varepsilon \fint_{2Q} (|B|^2 + |E|^2)^{1-\varepsilon} dx + c_{n,K} \left(\fint_{2Q} (|B|^2 + |E|^2)^{(1-\varepsilon)\frac{n}{n+1}} dx \right)^{\frac{n+1}{n}}$$

for ε sufficiently small. Substituting g^2 for $|B|^2 + |E|^2$ in the last inequality gives

$$\fint_Q g^{2-2\varepsilon} dx \leq c_{n,K}\, \varepsilon \fint_{2Q} g^{2-2\varepsilon} dx + c_{n,K} \left(\fint_{2Q} g^{(2-2\varepsilon)\frac{n}{n+1}} dx \right)^{\frac{n+1}{n}}.$$

By Corollary 3.2 there exists $\bar{\varepsilon} = \bar{\varepsilon}(K,n)$ such that if $0 \leq \varepsilon < \bar{\varepsilon}$, then $g \in L^{2+2\varepsilon}_{loc}(\Omega)$ and

$$\left(\fint_Q g^{2+2\varepsilon} dx \right)^{\frac{1}{2+2\varepsilon}} \leq c \left(\fint_{2Q} g^{2-2\varepsilon} dx \right)^{\frac{1}{2-2\varepsilon}}$$

and then the assertion. □

Now we consider the equation on a bounded open set $\Omega \subset \mathbb{R}^n$,

$$Au = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n,$$

where A is a differential operator defined by

$$Au = \operatorname{div} a(x, \nabla u).$$

Here $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that $x \to a(x,z)$ is measurable for all $z \in \mathbb{R}^n$ and $z \to a(x,z)$ is continuous for almost every $x \in \Omega$. Furthermore, we assume that there exists $K \geq 1$ such that for almost every $x \in \Omega$ we have

$$|a(x,z)|^2 + |z|^2 \leq \left(K + \frac{1}{K} \right) \langle a(x,z), z \rangle, \tag{5.2}$$

where x, z are arbitrary vectors in $\mathbb{R}^n$.

Let us prove the following result (originally proved in [9], in the linear case, by using a duality technique).

Corollary 5.2. *Let $0 < \epsilon < \frac{1}{2}$ and $u \in W^{1,2-2\varepsilon}_{loc}(\Omega)$ be a very weak solution of*

$$\operatorname{div} a(x, \nabla u) = 0.$$

Then there exists $\bar{\varepsilon} = \bar{\varepsilon}(K,n)$ such that $u \in W^{1,2+2\varepsilon}_{loc}(\Omega)$ for all $0 < \varepsilon < \bar{\varepsilon}$ and

$$\left(\fint_Q |\nabla u|^{2+2\varepsilon} dx \right)^{\frac{1}{2+2\varepsilon}} \leq c \left(\fint_{2Q} |\nabla u|^{2-2\varepsilon} dx \right)^{\frac{1}{2-2\varepsilon}},$$

where c is a positive constant depending on K, n.

Proof. Setting

$$E = \nabla u, \qquad B = a(x, \nabla u)$$

we have $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$ and, by (5.2),

$$|E|^2 + |B|^2 = |\nabla u|^2 + |a(x,\nabla u)|^2 \leq \left(K + \frac{1}{K}\right)\langle a(x,\nabla u), \nabla u\rangle$$

so that E, B are div-curl fields of bounded distortion. Applying Proposition 5.1 we get the assertion. □

Another interesting case to which Proposition 5.1 applies is when one considers a homeomorphism

$$\begin{aligned} &f = (f^1, f^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \\ &f^i \in W^{1,2-\varepsilon}(\Omega) \qquad i = 1, 2, \\ &f\ K\text{-quasiregular}, \quad K \geq 1, \qquad \text{i.e.} \qquad |Df(x)|^2 \leq \left(K + \frac{1}{K}\right) J(x, f), \end{aligned}$$

where $|Df(x)|$ denotes the norm of the distributional differential $Df(x)$ and $J(x, f)$ is the Jacobian determinant $J(x, f) = \det Df(x)$.

Then, writing the Jacobian $J(x, f)$ as $\langle B, E\rangle$, where $E = \nabla f^1 = (f^1_x, f^1_y)$ and $B = (f^2_y, -f^2_x)$ we have $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$ and that (5.1) holds. It follows that for ε sufficiently small $f \in W^{1,2+\varepsilon}(\Omega, \mathbb{R}^2)$, giving back in this way the celebrated theorem by Bojarski [1]. Significant results about the Jacobian determinant are in [7].

6. Regularity Results for Nonhomogeneous Equations

In this section we consider $\Phi = (E, B) \in L^{2-2\epsilon}(\Omega, \mathbb{R}^n) \times L^{2-2\varepsilon}(\Omega, \mathbb{R}^n)$ such that

$$\operatorname{div} B = 0, \quad \operatorname{curl} E = 0, \tag{6.1}$$

$$|B(x)|^2 + |E(x)|^2 \leq \left(K + \frac{1}{K}\right)\langle B(x), E(x)\rangle + |F|^2, \tag{6.2}$$

where F is a function in $L^r(\Omega, \mathbb{R}^n)$, $r > 2 - 2\varepsilon$, for ε sufficiently small.

Theorem 6.1. *Let $0 \leq \varepsilon < \frac{1}{2}$ and E, B vector fields as in (6.1),(6.2). Then there exist $\bar\varepsilon = \bar\varepsilon(K, n)$ and $\bar\eta = \bar\eta(K, r, \varepsilon, n)$ such that if $0 \leq \varepsilon < \bar\varepsilon$, then $\Phi = (E, B) \in L^{2-2\epsilon+\eta}_{loc}(\Omega, \mathbb{R}^n) \times L^{2-2\varepsilon+\eta}_{loc}(\Omega, \mathbb{R}^n)$ for all $0 \leq \eta < \bar\eta$ and*

$$\left(\fint_Q |\Phi|^{2-2\varepsilon+\eta} dx\right)^{\frac{1}{2-2\varepsilon+\eta}} \leq c\left\{\left(\fint_{2Q} |\Phi|^{2-2\varepsilon} dx\right)^{\frac{1}{2-2\varepsilon}} + \left(\fint_{2Q} (|F|^2)^{\frac{2-2\varepsilon+\eta}{2}} dx\right)^{\frac{1}{2-2\varepsilon+\eta}}\right\}, \tag{6.3}$$

where c is a positive constant depending on K, r, ε, n.

Proof. Let us fix Q a cube such that $2Q \subset \Omega$ and set

$$Q^+ = \{x \in Q \mid \langle B, E\rangle \geq 0\ a.e.\}$$

$$Q^- = \{x \in Q \mid \langle B, E\rangle \leq 0\ a.e.\}$$

Let us observe that by (6.2), replacing $|F|$ with f, we have

$$\begin{aligned}\int_{Q^-} \frac{-\langle B,E\rangle}{|B|^\varepsilon |E|^\varepsilon}dx &\le \int_{Q^-} (|B||E|)^{1-\varepsilon}dx \\ &\le \int_{Q^-} (|B|^2+|E|^2)^{1-\varepsilon}dx \\ &\le \int_{Q^-} \left[\left(K+\frac{1}{K}\right)\langle B,E\rangle + f^2\right]^{1-\varepsilon} dx \\ &\le \int_{Q^-} f^{2-2\varepsilon}dx \le \int_Q f^{2-2\varepsilon}dx\end{aligned}$$

and therefore

$$\begin{aligned}\int_Q \frac{\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}dx &= \int_{Q^+} \frac{\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}dx + \int_{Q^-} \frac{\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}dx \\ &\ge \int_{Q^+} \frac{\langle B,E\rangle}{(|B|^2+|E|^2+f^2)^\varepsilon}dx - \int_Q f^{2-2\varepsilon}dx\end{aligned}$$

Applying Theorem 2.1 with $p=q=2$ and $r=s=\frac{2n}{n+1}$, for ε sufficiently small, we get

$$\begin{aligned}⨍_Q \frac{\langle B,E\rangle}{(|B|^2+|E|^2+f^2)^\varepsilon}dx \le c_n\varepsilon ⨍_{2Q}(|B|^2+|E|^2+f^2)^{1-\varepsilon}dx \\ + c_n\left(⨍_{2Q}(|B|^2+|E|^2+f^2)^{(1-\varepsilon)\frac{n}{n+1}}dx\right)^{\frac{n+1}{n}} + ⨍_{2Q} f^{2-2\varepsilon}dx.\end{aligned}$$

By (6.2)

$$\langle B,E\rangle \ge c_K(|B|^2+|E|^2-f^2) = c_K(|B|^2+|E|^2+f^2) - 2c_K f^2$$

and therefore

$$\begin{aligned}&⨍_Q (|B|^2+|E|^2+f^2)^{1-\varepsilon}dx \\ &\quad\le c_{n,K}\,\varepsilon ⨍_{2Q}(|B|^2+|E|^2+f^2)^{1-\varepsilon}dx + c_{n,K}\left(⨍_{2Q}(|B|^2+|E|^2+f^2)^{(1-\varepsilon)\frac{n}{n+1}}dx\right)^{\frac{n+1}{n}} \\ &\qquad + c_K ⨍_{2Q} \frac{f^2}{(|B|^2+|E|^2+f^2)^\varepsilon}dx + ⨍_{2Q} f^{2-2\varepsilon}dx \\ &\quad\le c_{n,K}\,\varepsilon ⨍_{2Q}(|B|^2+|E|^2+f^2)^{1-\varepsilon}dx + c_{n,K}\left(⨍_{2Q}(|B|^2+|E|^2+f^2)^{(1-\varepsilon)\frac{n}{n+1}}dx\right)^{\frac{n+1}{n}} \\ &\qquad + (c_K+1)⨍_{2Q} f^{2-2\varepsilon}dx.\end{aligned}$$

Setting $g^2=|B|^2+|E|^2+f^2$, the last inequality implies

$$⨍_Q g^{2-2\varepsilon}dx \le c_{n,K}\,\varepsilon ⨍_{2Q} g^{2-2\varepsilon}dx + c_{n,K}\left(⨍_{2Q} g^{(2-2\varepsilon)\frac{n}{n+1}}dx\right)^{\frac{n+1}{n}} + (c_K+1)⨍_{2Q} f^{2-2\varepsilon}dx.$$

By Theorem 3.1 there exist $\bar{\varepsilon} = \bar{\varepsilon}(K,n)$ and $\bar{\eta} = \bar{\eta}(K,r,\varepsilon,n)$ such that if $0 \le \varepsilon < \bar{\varepsilon}$, then $g \in L^{2-2\varepsilon+\eta}_{loc}(\Omega)$ for all $0 \le \eta < \bar{\eta}$ and

$$\left(\fint_Q g^{2-2\varepsilon+\eta} dx\right)^{\frac{1}{2-2\varepsilon+\eta}} \le c\left\{\left(\fint_{2Q} g^{2-2\varepsilon} dx\right)^{\frac{1}{2-2\varepsilon}} + \left(\fint_{2Q} (f^2)^{\frac{2-2\varepsilon+\eta}{2}} dx\right)^{\frac{1}{2-2\varepsilon+\eta}}\right\}$$

and then the assertion. □

Let us consider now the following equation on a bounded open set $\Omega \subset \mathbb{R}^n$

$$\text{div } a(x,\nabla u) = \text{div } F, \tag{6.4}$$

where $F \in L^r(\Omega)$, $r > 2-2\varepsilon$, for ε sufficiently small and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the assumptions of Section 5.

Corollary 6.2. *Let $0 \le \varepsilon < \frac{1}{2}$ and $u \in W^{1,2-2\varepsilon}_{loc}(\Omega)$ be a very weak solution of the equation (6.4). Then there exist $\bar{\varepsilon} = \bar{\varepsilon}(K,n)$ and $\bar{\eta} = \bar{\eta}(K,r,\varepsilon,n)$ such that if $0 \le \varepsilon < \bar{\epsilon}$, then $u \in W^{1,2-2\varepsilon+\eta}_{loc}(\Omega)$ for all $0 \le \eta < \bar{\eta}$ and*

$$\left(\fint_Q |\nabla u|^{2-2\varepsilon+\eta} dx\right)^{\frac{1}{2-2\varepsilon+\eta}} \le c\left\{\left(\fint_{2Q} |\nabla u|^{2-2\varepsilon} dx\right)^{\frac{1}{2-2\varepsilon}} + \left(\fint_{2Q} |F|^{2-2\varepsilon+\eta} dx\right)^{\frac{1}{2-2\varepsilon+\eta}}\right\}$$

for all cubes Q such that $2Q \subset \Omega$ and where c is a positive constant depending on K, r, ε, n.

Proof. Setting

$$E = \nabla u, \qquad B = a(x,\nabla u) - F,$$

we have

$$\begin{aligned}
|E|^2 + |B|^2 &\le |\nabla u|^2 + (|a(x,\nabla u)| + |F|)^2 \\
&\le 2(|a(x,\nabla u)|^2 + |\nabla u|^2) + 2|F|^2 \\
&\le 2\left(K + \frac{1}{K}\right)\langle a(x,\nabla u), \nabla u\rangle + 2|F|^2 \\
&= 2\left(K + \frac{1}{K}\right)\langle a(x,\nabla u) - F, \nabla u\rangle + 2\left(K + \frac{1}{K}\right)\langle F, \nabla u\rangle + 2|F|^2.
\end{aligned}$$

Since by Young's inequality

$$\begin{aligned}
\langle F, \nabla u\rangle = 2\langle F, \nabla u\rangle - \langle F, \nabla u\rangle &\le 2\cdot 2\left(K + \frac{1}{K}\right)|F|^2 + 2\frac{1}{2\left(K + \frac{1}{K}\right)}|\nabla u|^2 - \langle F, \nabla u\rangle \\
&\le 4\left(K + \frac{1}{K}\right)|F|^2 + \frac{1}{K + \frac{1}{K}}(|a(x,\nabla u)|^2 + |\nabla u|^2) - \langle F, \nabla u\rangle \\
&\le 4\left(K + \frac{1}{K}\right)|F|^2 + \langle a(x,\nabla u), \nabla u\rangle - \langle F, \nabla u\rangle \\
&= 4\left(K + \frac{1}{K}\right)|F|^2 + \langle a(x,\nabla u) - F, \nabla u\rangle
\end{aligned}$$

we get

$$|E|^2 + |B|^2 \le 4\left(K + \frac{1}{K}\right)\langle B, E\rangle + \left[8\left(K + \frac{1}{K}\right)^2 + 2\right]|F|^2,$$

i.e. E, B are vector fields satisfying (6.1) and (6.2). From Theorem 6.1 we get the higher integrability for $|E|^2 + |B|^2$ and then for $|\nabla u|$; the estimate follows directly from (6.3). □

Remark 6.3. Let $u \in W^{1,2}_{loc}(\Omega)$ be a solution of the equation (6.4). Corollary 6.2 asserts that the function $g = |\nabla u|$ verifies inequality (4.1) and, surprisingly, satisfies also inequality (4.2) with $\lambda < 1$ sufficiently small.

Remark 6.4. We note that, arguing as in the end of Section 5, our result of higher integrability applies also to (K, K')-quasiregular mappings (see [4]), i.e. functions f verifying

$$f = (f^1, f^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2,$$
$$f^i \in W^{1,2-\varepsilon}(\Omega) \qquad i = 1, 2,$$
$$|Df(x)|^2 \le \left(K + \frac{1}{K}\right) J(x, f) + K'.$$

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