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A RECENT NOTE ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS

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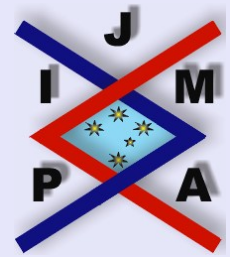
Abstract

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Abstract

The purpose of this note is to present a theorem having conditions of new type and to weaken some assumptions given in two previous papers simultaneously.

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1. Introduction

Recently there have been a number of papers written dealing with absolute summability factors of infinite series, see e.g. [3] – [9]. Among others in [6] we also proved a theorem of this type improving a result of H. Bor [3]. Very recently H. Bor and L. Debnath [5] enhanced a theorem of S.M. Mazhar [9] considering a quasi β-power increasing sequence $\{X_n\}$ for some $0 < \beta < 1$ instead of the case $\beta = 0$.

The purpose of this note is to moderate the conditions of the theorems of Bor-Debnath and ours.

To recall these theorems we need some definitions.

A positive sequence $\mathbf{a} := \{a_n\}$ is said to be *quasi β- power increasing* if there exists a constant $K = K(\beta, \mathbf{a}) \geq 1$ such that

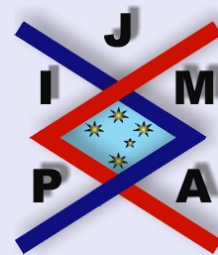$$(1.1) \qquad K\, n^\beta\, a_n \geq m^\beta\, a_m$$

holds for all $n \geq m$. If (1.1) stays with $\beta = 0$ then $\mathbf{a}$ is simply called a *quasi increasing* sequence. In [6] we showed that this latter class is equivalent to the class of *almost increasing* sequences.

A series $\sum a_n$ with partial sums s_n is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where $\{p_n\}$ is a sequence of positive numbers such that

$$P_n := \sum_{\nu=0}^{n} p_\nu \to \infty$$

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and
$$t_n := \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \, s_\nu.$$

First we recall the theorem of Bor and Debnath.

Theorem 1.1. *Let* $\mathbf{X} := \{X_n\}$ *be a quasi β-power increasing sequence for some $0 < \beta < 1$, and $\lambda := \{\lambda_n\}$ be a real sequence. If the conditions*

$$(1.2) \qquad \sum_{n=1}^{m} \frac{1}{n} P_n = O(P_m),$$
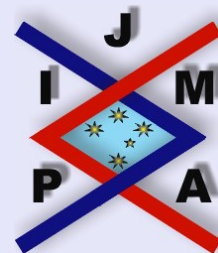
$$(1.3) \qquad \lambda_n \, X_n = O(1),$$

$$(1.4) \qquad \sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m),$$

$$(1.5) \qquad \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m),$$

and

$$(1.6) \qquad \sum_{n=1}^{\infty} n \, X_n |\Delta^2 \, \lambda_n| < \infty, \quad (\Delta^2 \, \lambda_n = \Delta \, \lambda_n - \Delta \, \lambda_{n+1})$$

are satisfied, then the series $\sum a_n \, \lambda_n$ is summable $|\overline{N}, p_n|_k, \ k \geq 1$.

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In my view, the proof of Theorem 1.1 has a little gap, but the assertion is true.

Our mentioned theorem [7] reads as follows.

Theorem 1.2. *If* $\mathbf{X}$ *is a quasi increasing sequence and the conditions (1.4), (1.5),*
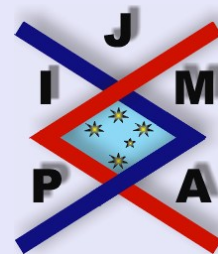
$$(1.7) \qquad \sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n| < \infty,$$

$$(1.8) \qquad \sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty$$

and

$$(1.9) \qquad \sum_{n=1}^{\infty} n \, X_n |\Delta|\Delta \lambda_n|| < \infty$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n|_k, \; k \geq 1$.

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2. Result

Now we prove the following theorem.

Theorem 2.1. *If the sequence $\mathbf{X}$ is quasi β-power increasing for some $0 \leq \beta < 1$, λ satisfies the conditions*

$$(2.1) \qquad \sum_{n=1}^{m} \lambda_n = o(m)$$

and

$$(2.2) \qquad \sum_{n=1}^{m} |\Delta\, \lambda_n| = o(m),$$

furthermore (1.4), (1.5) and

$$(2.3) \qquad \sum_{n=1}^{\infty} n\, X_n(\beta)|\Delta|\Delta\, \lambda_n|| < \infty$$

hold, where $X_n(\beta) := \max(n^\beta\, X_n, \log n)$, then the series $\sum a_n\, \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

Remark 1. *It seems to be worth comparing the assumptions of these theorems.*
  By Lemma 3.3 it is clear that $(1.7) \Rightarrow (2.1)$, furthermore if $\mathbf{X}$ is quasi increasing then $(1.8) \Rightarrow (2.2)$. It is true that (2.3) in the case $\beta = 0$ claims a little bit more than (1.9) does, but only if $X_n < K \log n$. However, in general, $X_n \geq K \log n$ holds, see (1.4) and (1.5). In the latter case, Theorem 2.1 under weaker conditions provides the same conclusion as Theorem 1.2.
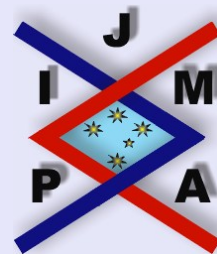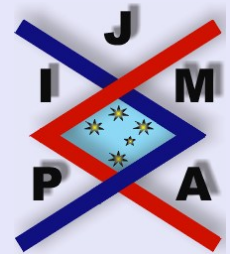
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If we analyze the proofs of Theorem 1.1 and Theorem 1.2, it is easy to see that condition (1.2) *replaces* (1.7), (1.3) *and* (1.6) *jointly imply* (1.8), *finally* (1.9) *requires less than* (1.6). *Thus we can say that the conditions of Theorem 2.1 also claim less than that of Theorem 1.1.*

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3. Lemmas

Later on we shall use the notation $L \ll R$ if there exists a positive constant K such that $L \leq K R$ holds.

To avoid needless repetition we collect the relevant partial results proved in [3] into a lemma.

In [3] the following inequality is verified implicitly.

Lemma 3.1. *Let* T_n *denote the* n*-th* $(\overline{N}, p_n)$ *mean of the series* $\sum a_n \lambda_n$*. If* $\{X_n\}$ *is a sequence of positive numbers, and* $\lambda_n \to 0$*, plus (1.7) and (1.5) hold, then*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k \ll |\lambda_m| X_m + \sum_{n=1}^{m} |\Delta \lambda_n| X_n + \sum_{n=1}^{m} |t_n|^k |\Delta \lambda_n|.$$
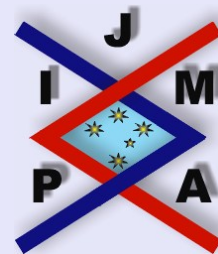
Lemma 3.2 ([7]). *Let* $\{\gamma_n\}$ *be a sequence of real numbers and denote*

$$\Gamma_n := \sum_{k=1}^{n} \gamma_k \qquad and \qquad R_n := \sum_{k=n}^{\infty} |\Delta \gamma_k|.$$

If $\Gamma_n = o(n)$ *then there exists a natural number* $\mathbb{N}$ *such that*

$$|\gamma_n| \leq 2R_n$$

for all $n \geq \mathbb{N}$.

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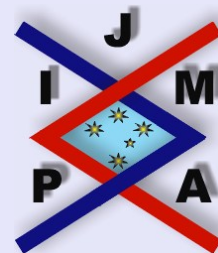
Lemma 3.3 ([1, 2.2.2., p. 72]). *If $\{\mu_n\}$ is a positive, monotone increasing and tending to infinity sequence, then the convergence of the series $\sum a_n \mu_n^{-1}$ implies the estimate*

$$\sum_{i=1}^{n} a_i = o(\mu_n).$$

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4.   Proof of Theorem 2.1

In order to use Lemma 3.1 we first have to show that its conditions follow from the assumptions of Theorem 2.1. Thus we must show that

$$(4.1) \qquad \lambda_n \to 0.$$

By Lemma 3.2, condition (2.1) implies that

$$|\lambda_n| \le 2 \sum_{k=n}^{\infty} |\Delta \lambda_k|,$$

and by (2.2)

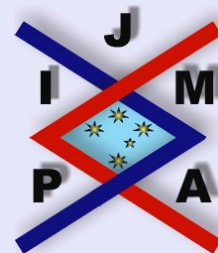$$(4.2) \qquad |\Delta \lambda_n| \le 2 \sum_{k=n}^{\infty} |\Delta|\Delta \lambda_k||,$$

whence

$$(4.3) \qquad |\lambda_n| \ll \sum_{k=n}^{\infty} n|\Delta|\Delta \lambda_n||$$

holds. Thus (2.3) and (4.3) clearly prove (4.1).

Next we verify (1.7). In view of (4.3) and (2.3)

$$\sum_{n=1}^{\infty} \frac{1}{n}|\lambda_n| \ll \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} k|\Delta|\Delta \lambda_k|| \ll \sum_{k=1}^{\infty} k|\Delta|\Delta \lambda_k|| \log k < \infty,$$

that is, (1.7) is satisfied.

In the following steps we show that

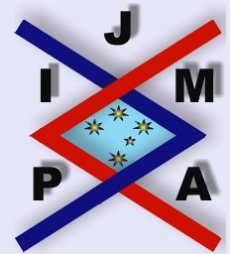$$(4.4) \qquad |\lambda_n| X_n \ll 1,$$

$$(4.5) \qquad \sum_{n=1}^{\infty} |\Delta \lambda_n| X_n \ll 1$$

and

$$(4.6) \qquad \sum_{n=1}^{\infty} |t_n|^k |\Delta \lambda_k| \ll 1.$$

Utilizing the quasi monotonicity of $\{n^\beta X_n\}$, (2.3) and (4.3) we get that

$$(4.7) \qquad |\lambda_n| X_n \leq n^\beta |\lambda_n| X_n$$
$$\ll \sum_{k=n}^{\infty} k^\beta |X_k| |k| \Delta |\Delta \lambda_k||$$
$$\ll \sum_{k=n}^{\infty} k\, X_k(\beta) |\Delta |\Delta \lambda_k||$$
$$< \infty.$$

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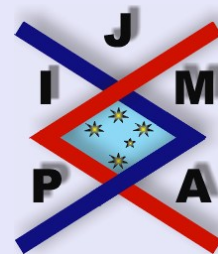*J. Ineq. Pure and Appl. Math.* **7**(2) Art. 44, 2006

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Similar arguments give that

$$
(4.8) \qquad \sum_{n=1}^{\infty} |\Delta \, \lambda_n| X_n \ll \sum_{n=1}^{\infty} X_n \sum_{k=n}^{\infty} |\Delta |\Delta \, \lambda_k||
$$

$$
= \sum_{k=1}^{\infty} |\Delta |\Delta \, \lambda_k|| \sum_{n=1}^{k} n^{\beta} X_n \, n^{-\beta}
$$

$$
\ll \sum_{k=1}^{\infty} k^{\beta} X_k |\Delta |\Delta \, \lambda_k|| \sum_{n=1}^{k} n^{-\beta}
$$

$$
\ll \sum_{k=1}^{\infty} k \, X_k |\Delta |\Delta \, \lambda_k|| < \infty.
$$

Finally to verify (4.6) we apply Abel transformation as follows:

$$
\sum_{n=1}^{m} |t_n|^k |\Delta \, \lambda_n|
$$

$$
\ll \sum_{n=1}^{m-1} |\Delta(n |\Delta \, \lambda_n|)| \sum_{i=1}^{n} \frac{1}{i} |t_i|^k + m |\Delta \, \lambda_m| \sum_{n=1}^{m} \frac{1}{n} |t_n|^k
$$

$$
(4.9) \qquad \ll \sum_{n=1}^{m-1} n |\Delta |\Delta \, \lambda_n|| X_n + \sum_{n=1}^{m-1} |\Delta \, \lambda_{n+1}| X_{n+1} + m |\Delta \, \lambda_m| X_m.
$$

Here the first term is bounded by (2.3), the second one by (4.5), and the third
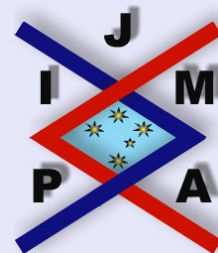
term by (2.3) and (4.2), namely

$$(4.10) \quad m|\Delta\,\lambda_m|X_m \ll m\,X_m \sum_{n=m}^{\infty} |\Delta|\Delta\,\lambda_n|| \ll \sum_{n=m}^{\infty} n\,X_n|\Delta|\Delta\,\lambda_n|| < \infty.$$

Herewith (4.6) is also verified.

Consequently Lemma 3.1 exhibits that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty,$$

and this completes the proof of our theorem.

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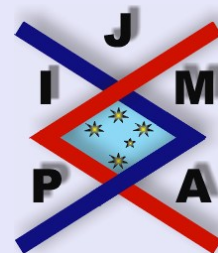
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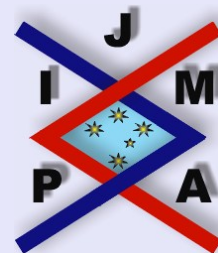
[9] S.M. MAZHAR, Absolute summability factors of infinite series, *Kyung-pook Math. J.*, **39**(1) (1999), 67–73.

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