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SPATIAL BEHAVIOUR FOR THE HARMONIC VIBRATIONS IN PLATES OF KIRCHHOFF TYPE

CIRO D'APICE AND STAN CHIRIŢĂ

DIIMA, UNIVERSITY OF SALERNO,
VIA PONTE DON MELILLO,
84084 FISCIANO (SA), ITALY
dapice@diima.unisa.it

FACULTY OF MATHEMATICS,
UNIVERSITY OF IAŞI,
6600-IAŞI, ROMANIA

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ABSTRACT. In this paper the spatial behaviour of the steady-state solutions for an equation of Kirchhoff type describing the motion of thin plates is investigated. Growth and decay estimates are established associating some appropriate cross-sectional line and area integral measures with the amplitude of the harmonic vibrations, provided the excited frequency is lower than a certain critical value. The method of proof is based on a second–order differential inequality leading to an alternative of Phragmèn–Lindelöf type in terms of an area measure of the amplitude in question. The critical frequency is individuated by using some Wirtinger and Knowles inequalities.

Key words and phrases: Kirchhoff plates, Spatial behaviour, Harmonic vibrations.

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1. INTRODUCTION

The biharmonic equation has essential applications in the static Kirchhoff theory of thin elastic plates. Many studies and various methods have been proposed for researching the spatial behaviour for the solutions of the biharmonic equation in a semi–infinite strip in $\mathbb{R}^2$. We mention here the studies by Knowles [11, 12], Flavin [4], Flavin and Knops [5], Horgan [6] and Payne and Schaefer [16]. Additional references may be found in the review papers by Horgan and Knowles [7] and Horgan [8, 9].

There is no information in the literature about the spatial behaviour of dynamical solutions in the Kirchhoff theory of thin elastic plates. We try to cover this gap by starting in this paper with the study of the spatial behaviour for the harmonic vibrations of thin elastic plates, while the transient solutions will be treated in a future study. It has to be outlined that the interest in the construction of theories of plates grew from the desire to treat vibrations of plates aimed at

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deducing the tones of vibrating bells. Thus, in the present paper we consider a semi–infinite strip for which the lateral boundary is fixed, while its end is subjected to a given harmonic vibration of a prescribed frequency ω. Our approach is based on a differential equation proposed by Lagnese and Lions [13] for modelling thin plates and generalising the Kirchhoff equation of classical thin plates (see, for example, Naghdi [15]). We associate with the amplitude of the harmonic oscillation an appropriate cross–sectional line–integral measure. We individuate a critical frequency in the sense that for all vibration frequencies lower than this one, we can establish a second–order differential inequality giving information upon the spatial behaviour of the amplitude. In this aim we use some Wirtinger and Knowles inequalities. Then we establish an alternative of Phragmèn–Lindelöf type: The measure associated with the amplitude of the oscillation either grows at infinity faster than an increasing exponential or decays toward zero faster than a decreasing exponential when the distance to the end goes to infinity.

We have to note that some time–dependent problems concerning the biharmonic operator are considered in the literature, but these are different from those furnished by the theories of plates. Thus, we mention the papers by Lin [14], Knops and Lupoli [10] and Chiriţă and Ciarletta [1] in connection with the spatial behaviour of solutions for a fourth–order transformed problem associated with the slow flow of an incompressible viscous fluid along a semi–infinite strip, and a paper by Chiriţă and D'Apice [2] concerning the solutions of a fourth–order initial boundary value problem describing the flow of heat in a non–simple heat conductor.

2. BASIC FORMULATION

Throughout this paper Greek and Latin subscripts take the values $1, 2$, summation is carried out over repeated indices, $x = (x_1, x_2)$ is a generic point referred to orthogonal Cartesian coordinates in $\mathbb{R}^2$. The suffix ", ρ" denotes $\frac{\partial}{\partial x_\varrho}$, that is, the derivative with respect to x_ϱ. We consider a semi–infinite strip S in the plane $x_1 O x_2$ defined by

$$S = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \ 0 < x_2 < l, 0 < x_1 \right\}, \quad l > 0. \tag{2.1}$$

In what follows we will consider the following differential equation

$$\alpha^2 \ddot{u} - \beta^2 \Delta \ddot{u} + \gamma^2 \Delta \Delta u = 0, \tag{2.2}$$

where $\Delta u = u_{,\rho\rho}$ is the ordinary two–dimensional Laplacian, α, β and γ are positive constants and a superposed dot denotes the time derivative. If we set $\alpha^2 = \varrho h$, $\beta^2 = \frac{\varrho h^3}{12}$ and $\gamma^2 = D$, where ϱ is the mass density, h is the uniform thickness of the plate and D is the flexural rigidity, then we obtain the approach of plate proposed by Lagnese and Lions [13]. We recall that the flexural rigidity is given by the relation $D = \frac{Eh^3}{12(1-\nu^2)}$, where $E > 0$ is the Young's modulus and ν is the Poisson's ratio ranging over $\left(-1, \frac{1}{2}\right)$. If we set $\alpha^2 = \varrho h$, $\beta^2 = 0$ and $\gamma^2 = D$ in (2.2), then we obtain the equation occurring in the Kirchhoff theory of thin plates (see [15]). The reader is referred to [13, Chapter I] for a heuristic derivation of the present plate model.

We further assume that the lateral sides of the plate are fixed, while its end is subjected to an excited vibration. Then we study the spatial behaviour of the harmonic vibrations of the plate, that is we study the solution of the equation (2.2) of the type $u(x, t) = v(x)e^{i\omega t}$, where $\omega > 0$ is the constant prescribed frequency of the excited vibration on the end of the strip.

More precisely, we consider in the strip S the following boundary value problem $\mathcal{P}$ defined by the equation:

$$-\omega^2 \alpha^2 v + \beta^2 \omega^2 \Delta v + \gamma^2 \Delta \Delta v = 0, \quad \text{in } S, \tag{2.3}$$

the lateral boundary conditions:

$$\begin{cases} v(x_1,0)=0, & v_{,2}(x_1,0)=0, \\ v(x_1,l)=0, & v_{,2}\,(x_1,l)=0, \quad x_1\in[0,\infty), \end{cases} \tag{2.4}$$

and the end conditions:

$$v(0,x_2)=g_1(x_2), \quad v_{,1}(0,x_2)=g_2(x_2), \quad x_2\in[0,l], \tag{2.5}$$

where g_1 and g_2 are prescribed continuous differentiable functions.

For future convenience we introduce the following notations:

$$D_{x_1^* x_1}=\left\{y=(y_1,y_2)\in\mathbb{R}^2 : 0\le x_1^*<y_1<x_1, \quad 0<y_2<l\right\}, \tag{2.6}$$

$$D_{x_1}=\left\{y=(y_1,y_2)\in\mathbb{R}^2 : 0\le x_1<y_1, \quad 0<y_2<l\right\}. \tag{2.7}$$

3. A SECOND–ORDER DIFFERENTIAL INEQUALITY

Throughout the following we shall assume that the constant coefficients α, β and γ are strictly positive. A discussion will be made at the end for the limit case when β tends to zero, that is for the Kirchhoff model of thin elastic plates.

We start our analysis by establishing a fundamental identity concerning the solution $v(x)$ of the considered boundary value problem $\mathcal{P}$. This identity will give us an idea on the measure to be introduced.

Thus, in view of the equation (2.3), we have

$$\begin{aligned} &-\omega^2\alpha^2v^2+\beta^2\omega^2\left[\left(vv_{,1}\right)_{,1}-v_{,1}^2+\left(vv_{,2}\right)_{,2}-v_{,2}^2\right] \\ &\quad+\gamma^2\left[\left(vv_{,111}\right)_{,1}-v_{,1}v_{,111}+2\left(vv_{,112}\right)_{,2}-2v_{,2}v_{,112}+\left(vv_{,222}\right)_{,2}-v_{,2}v_{,222}\right]=0 \end{aligned} \tag{3.1}$$

from which we obtain

$$\begin{aligned} &-\omega^2\left[\alpha^2v^2+\beta^2(v_{,1}^2+v_{,2}^2)\right]+\beta^2\omega^2\left[\left(vv_{,1}\right)_{,1}+\left(vv_{,2}\right)_{,2}\right] \\ &\quad+\gamma^2\left[\left(vv_{,111}\right)_{,1}+2\left(vv_{,112}\right)_{,2}+\left(vv_{,222}\right)_{,2}\right] \\ &\quad-\gamma^2\left[\left(v_{,1}v_{,11}\right)_{,1}-v_{,11}^2+2\left(v_{,2}v_{,12}\right)_{,1}-2v_{,12}^2+\left(v_{,2}v_{,22}\right)_{,2}-v_{,22}^2\right]=0, \end{aligned} \tag{3.2}$$

and hence, we get

$$\begin{aligned} &-\omega^2\left[\alpha^2v^2+\beta^2(v_{,1}^2+v_{,2}^2)\right]+\gamma^2\left(v_{,11}^2+2v_{,12}^2+v_{,22}^2\right) \\ &\quad+\left\{\beta^2\omega^2vv_{,1}+\gamma^2vv_{,111}-\gamma^2v_{,1}v_{,11}-2\gamma^2v_{,2}v_{,12}\right\}_{,1} \\ &\quad+\left\{\beta^2\omega^2vv_{,2}+2\gamma^2vv_{,112}+\gamma^2vv_{,222}-\gamma^2v_{,2}v_{,22}\right\}_{,2}=0. \end{aligned} \tag{3.3}$$

By integrating the relation (3.3) over $[0,l]$ and by using the lateral boundary conditions described in (2.4), we get the following identity

$$\begin{aligned} &-\omega^2\int_0^l\left[\alpha^2v^2+\beta^2(v_{,1}^2+v_{,2}^2)\right]dx_2+\gamma^2\int_0^l\left(v_{,11}^2+2v_{,12}^2+v_{,22}^2\right)dx_2 \\ &\quad+\int_0^l\left[\frac{1}{2}\beta^2\omega^2v^2+\gamma^2(vv_{,11}-v_{,1}^2-v_{,2}^2)\right]_{,11}dx_2=0. \end{aligned} \tag{3.4}$$

Before deriving our growth and decay estimates, we proceed to establish a second–order differential inequality in terms of a cross–sectional line integral measure which is fundamental in our analysis on the spatial behaviour. In this aim we associate with the solution $v(x)$ of the considered boundary value problem $\mathcal{P}$ the following cross–sectional line integral measure

$$\mathcal{I}(x_1) = \int_0^l \left[\gamma^2(v_{,1}^2 + v_{,2}^2 - vv_{,11}) - \frac{1}{2}\beta^2\omega^2 v^2\right] dx_2, \quad x_1 > 0, \tag{3.5}$$

so that the identity (3.4) furnishes

$$\mathcal{I}^{''}(x_1) = \gamma^2 \int_0^l \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) dx_2 - \omega^2 \int_0^l \left[\alpha^2 v^2 + \beta^2(v_{,1}^2 + v_{,2}^2)\right] dx_2, \ \ x_1 > 0. \tag{3.6}$$

Further, we use the lateral boundary conditions described by (2.4) in order to write the following Wirtinger type inequalities

$$\int_0^l v_{,1}^2 dx_2 \le \frac{l^2}{\pi^2}\int_0^l v_{,12}^2 dx_2, \tag{3.7}$$

$$\int_0^l v_{,2}^2 dx_2 \le \frac{l^2}{4\pi^2}\int_0^l v_{,22}^2 dx_2, \tag{3.8}$$

$$\int_0^l v^2 dx_2 \le \left(\frac{2}{3}\right)^4 \frac{l^4}{\pi^4}\int_0^l v_{,22}^2 dx_2. \tag{3.9}$$

On the other hand, by using the same lateral boundary conditions in the inequality established by Knowles [12] (see the Appendix), we deduce that

$$\int_0^l \left(\beta^2 v_{,2}^2 + \alpha^2 v^2\right) dx_2 \le \frac{\beta^2}{\Lambda(\alpha,\beta)}\int_0^l v_{,22}^2 dx_2, \tag{3.10}$$

where $\Lambda(\alpha, \beta)$ is defined by

$$\Lambda(\alpha,\beta) = \lambda\left(\frac{\alpha^2}{\beta^2}\right), \tag{3.11}$$

and $\lambda(t)$ is as defined in the Appendix. Therefore, we have

$$\Lambda(\alpha,\beta) = \frac{4}{l^2}\frac{r^4(\tau)}{\tau + r^2(\tau)}, \quad \tau = \frac{\alpha^2 l^2}{4\beta^2}, \tag{3.12}$$

and $r(\tau)$ is the smallest positive root of the equation

$$\tan r = -\sqrt{\frac{\tau}{\tau + r^2}}\tanh\left(r\sqrt{\frac{\tau}{\tau + r^2}}\right), \quad \tau \ge 0. \tag{3.13}$$

Thus, on the basis of the relations (3.7) and (3.10), we can conclude that

$$\int_0^l \left[\alpha^2 v^2 + \beta^2(v_{,1}^2 + v_{,2}^2)\right] dx_2 \le \frac{\gamma^2}{\omega_m^2}\int_0^l (2v_{,12}^2 + v_{,22}^2) dx_2, \tag{3.14}$$

where $\omega_m = \omega_m(\alpha, \beta, \gamma)$ is defined by

$$\frac{1}{\omega_m^2} = \frac{1}{\gamma^2}\max\left\{\frac{l^2\beta^2}{2\pi^2}, \frac{\beta^2}{\Lambda(\alpha,\beta)}\right\}. \tag{3.15}$$

By taking into account the relations (3.6) and (3.14), we obtain the following estimate

$$\mathcal{I}''(x_1) \ge \gamma^2\left(1 - \frac{\omega^2}{\omega_m^2}\right)\int_0^l \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) dx_2, \quad x_1 > 0. \tag{3.16}$$

Throughout in this paper we shall assume that the prescribed frequency ω of the excited vibration is lower than the critical value ω_m defined by the relation (3.15), that is we assume that

$$0 < \omega < \omega_m. \tag{3.17}$$

This assumption then implies that

$$\mathcal{I}''(x_1) \geq 0 \quad \text{for all} \quad x_1 > 0. \tag{3.18}$$

We proceed now to estimate the term $I(x_1)$ as defined by the relation (3.5). We first note that

$$|\mathcal{I}(x_1)| \leq \gamma^2 \left| \int_0^l (v_{,1}^2 + v_{,2}^2 - vv_{,11}) dx_2 \right| + \frac{1}{2}\beta^2\omega^2 \int_0^l v^2 dx_2. \tag{3.19}$$

Further, we use an idea of Payne and Schaefer [16] for estimating the first integral in (3.19). Thus, by means of the Cauchy–Schwarz and arithmetic–geometric mean inequalities and by using the Wirtinger type inequalities (3.7), (3.8) and (3.9), we deduce

$$\begin{aligned} &\left| \int_0^l (v_{,1}^2 + v_{,2}^2 - vv_{,11}) dx_2 \right| \\ &\quad \leq \int_0^l (v_{,1}^2 + v_{,2}^2) dx_2 + \left(\int_0^l v^2 dx_2 \int_0^l v_{,11}^2 dx_2 \right)^{\frac{1}{2}} \\ &\quad \leq \frac{l^2}{2\pi^2} \left\{ \int_0^l \left(2v_{,12}^2 + \frac{1}{2} v_{,22}^2 \right) dx_2 + \frac{8}{9} \left(\int_0^l v_{,22}^2 dx_2 \int_0^l v_{,11}^2 dx_2 \right)^{\frac{1}{2}} \right\} \\ &\quad \leq \frac{l^2}{2\pi^2} \int_0^l \left\{ \frac{4}{9\varepsilon} v_{,11}^2 + 2v_{,12}^2 + \left(\frac{1}{2} + \frac{4\varepsilon}{9} \right) v_{,22}^2 \right\} dx_2, \end{aligned} \tag{3.20}$$

for some positive constant ε. We now choose $\varepsilon = \frac{4}{9}$ and note that $\frac{1}{2} + \frac{4\varepsilon}{9} = \frac{113}{162} < 1$. With this choice the relations (3.19) and (3.20) give

$$|\mathcal{I}(x_1)| \leq m_0^2 \int_0^l \gamma^2 \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2 \right) dx_2 + \frac{1}{2}\beta^2\omega^2 \int_0^l v^2 dx_2, \tag{3.21}$$

where

$$m_0^2 = \frac{l^2}{2\pi^2}. \tag{3.22}$$

On the basis of the inequality (3.9), we further deduce that

$$|\mathcal{I}(x_1)| \leq \tilde{m}_0^2 \int_0^l \gamma^2 \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2 \right) dx_2, \quad x_1 > 0, \tag{3.23}$$

where

$$\tilde{m}_0^2 = m_0^2 + \frac{\beta^2\omega^2}{2\gamma^2} \left(\frac{2}{3} \right)^4 \frac{l^4}{\pi^4}. \tag{3.24}$$

Finally, the relations (3.16) and (3.23) lead to the following estimate

$$\tilde{m}^2 |\mathcal{I}(x_1)| \leq \mathcal{I}''(x_1), \quad x_1 > 0, \tag{3.25}$$

where $\tilde{m}$ is defined by

$$\tilde{m}^2 = \frac{1}{\tilde{m}_0^2} \left(1 - \frac{\omega^2}{\omega_m^2} \right). \tag{3.26}$$

Consequently, we have established the following two second–order differential inequalities

$$\mathcal{I}^{''}(x_1) + \tilde{m}^2 \mathcal{I}(x_1) \geq 0, \tag{3.27}$$

$$\mathcal{I}^{''}(x_1) - \tilde{m}^2 \mathcal{I}(x_1) \geq 0, \tag{3.28}$$

which will be used in the derivation of the alternatives that we will consider, always under the condition that (3.17) holds true.

4. SPATIAL BEHAVIOUR

In this section we will analyse the consequences of the second–order differential inequalities on the spatial behaviour of the measure $\mathcal{I}(x_1)$. In fact, in view of the relation (3.18), it follows that we have only the two cases:

i) there exist a value $z_1 \in [0, \infty)$ such that $\mathcal{I}^{'}(z_1) > 0$,
ii) $\mathcal{I}^{'}(x_1) \leq 0, \quad \forall\, x_1 \in [0, \infty)$.

4.1. Discussion of the Case i).

Since we have $\mathcal{I}^{''}(x_1) \geq 0$ for all $x_1 > 0$, we deduce that

$$\mathcal{I}(x_1) \geq \mathcal{I}(z_1) + \mathcal{I}'(z_1)(x_1 - z_1) \quad \text{for all } \; x_1 \geq z_1, \tag{4.1}$$

and hence it follows that, at least for sufficiently large values of x_1, $\mathcal{I}(x_1)$ must become strictly positive. That means there exists a value $z_2 \in [z_1, \infty)$ so that $\mathcal{I}(z_2) > 0$. Because we have $\mathcal{I}'(x_1) \geq \mathcal{I}'(z_2) > 0$ for all $x_1 \in [z_2, \infty)$, it results that $\mathcal{I}(x_1)$ is a non–decreasing function on $[z_2, \infty)$ and therefore, we have $\mathcal{I}(x_1) \geq \mathcal{I}(z_2) > 0$ for all $x_1 \in [z_2, \infty)$. Further, the relation (3.25) implies

$$\frac{d}{dx_1}\left\{ e^{-\tilde{m}x_1} \left[\mathcal{I}^{'}(x_1) + \tilde{m}\mathcal{I}(x_1) \right] \right\} \geq 0, \quad x_1 \in [z_2, \infty), \tag{4.2}$$

$$\frac{d}{dx_1}\left\{ e^{\tilde{m}x_1} \left[\mathcal{I}^{'}(x_1) - \tilde{m}\mathcal{I}(x_1) \right] \right\} \geq 0, \quad x_1 \in [z_2, \infty). \tag{4.3}$$

By an integration over $[z_2, x_1]$, $x_1 > z_2$, the relations (4.2) and (4.3) give

$$\mathcal{I}^{'}(x_1) + \tilde{m}\mathcal{I}(x_1) \geq \left[\mathcal{I}^{'}(z_2) + \tilde{m}\mathcal{I}(z_2) \right] e^{\tilde{m}(x_1 - z_2)}, \quad x_1 \geq z_2, \tag{4.4}$$

$$\mathcal{I}^{'}(x_1) - \tilde{m}\mathcal{I}(x_1) \geq \left[\mathcal{I}^{'}(z_2) - \tilde{m}\mathcal{I}(z_2) \right] e^{-\tilde{m}(x_1 - z_2)}, \;\; x_1 \geq z_2, \tag{4.5}$$

and therefore, we get

$$\mathcal{I}^{'}(x_1) \geq \mathcal{I}^{'}(z_2) \cosh[\tilde{m}(x_1 - z_2)] + \tilde{m}\mathcal{I}(z_2) \sinh[\tilde{m}(x_1 - z_2)], \;\; x_1 \geq z_2. \tag{4.6}$$

On the other hand, by taking into account the notation (2.6) and by integrating the relation (3.6) over $[z_2, x_1]$, $x_1 > z_2$, we obtain

$$\mathcal{I}^{'}(x_1) = \mathcal{I}^{'}(z_2) + \gamma^2 \int_{D_{z_2 x_1}} \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2 \right) da - \omega^2 \int_{D_{z_2 x_1}} \left[\alpha^2 v^2 + \beta^2 (v_{,1}^2 + v_{,2}^2) \right] da. \tag{4.7}$$

Consequently, the relations (4.6) and (4.7) give

$$\gamma^2 \int_{D_{z_2 x_1}} \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da \geq \omega^2 \int_{D_{z_2 x_1}} \left[\alpha^2 v^2 + \beta^2 (v_{,1}^2 + v_{,2}^2)\right] da + \mathcal{I}'(z_2) \left\{\cosh\left[\tilde{m}(x_1 - z_2)\right] - 1\right\} + \tilde{m}\mathcal{I}(z_2) \sinh\left[\tilde{m}(x_1 - z_2)\right], \quad x_1 > z_2, \tag{4.8}$$

and hence

$$\lim_{x_1 \to \infty} \left\{ e^{-\tilde{m}x_1} \int_{D_{z_2 x_1}} \gamma^2 \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da \right\} \geq \frac{1}{2} e^{-\tilde{m}z_2} \left[\mathcal{I}'(z_2) + \tilde{m}\mathcal{I}(z_2)\right] > 0. \tag{4.9}$$

Thus, we can conclude that, within the class of amplitudes $v(x)$ for which there exists $z_1 \geq 0$ so that $\mathcal{I}'(z_1) > 0$, the following measure

$$\mathcal{E}(x_1) = \int_{D_{x_1}^*} \left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da, \quad D_{x_1}^* = [0, x_1] \times [0, l], \tag{4.10}$$

grows to infinity faster than the exponential $e^{\tilde{m}x_1}$ when x_1 goes to infinity.

4.2. **Discussion of the Case ii).** In this case we have

$$\mathcal{I}'(x_1) \leq 0 \quad \text{for all} \quad x_1 \in [0, \infty), \tag{4.11}$$

and therefore, $\mathcal{I}(x_1)$ is a non–increasing function on $[0, \infty)$. We prove then that

$$\mathcal{I}(x_1) \geq 0 \quad \text{for all} \quad x_1 \in [0, \infty). \tag{4.12}$$

To verify this relation we consider some z_0 arbitrary fixed in $[0, \infty)$ and note that, by means of the relation (4.11), we have

$$\mathcal{I}(x_1) \leq \mathcal{I}(z_0) \quad \text{for all} \quad x_1 \geq z_0. \tag{4.13}$$

On the other hand, the relation (3.27), when integrated over $[z_0, x_1]$, $x_1 > z_0$, gives

$$\begin{aligned} 0 &\leq \mathcal{I}'(z_0) - \mathcal{I}'(x_1) \\ &\leq \tilde{m}^2 \int_{z_0}^{x_1} I(\xi) d\xi \\ &\leq \tilde{m}^2 \int_{z_0}^{x_1} \mathcal{I}(z_0) d\xi = \tilde{m}^2 \mathcal{I}(z_0)(x_1 - z_0), \end{aligned} \tag{4.14}$$

and hence it results that $\mathcal{I}(z_0) \geq 0$. This proves that the relation (4.12) holds true.

Now, on the basis of the relation (4.12) and by using the relations (3.5) and (3.20) (with the appropriate choice for ε), we deduce that

$$\begin{aligned} 0 &\leq \mathcal{I}(x_1) \\ &= \gamma^2 \int_0^l (v_{,1}^2 + v_{,2}^2 - v v_{,11}) dx_2 - \frac{1}{2}\beta^2\omega^2 \int_0^l v^2 dx_2 \\ &\leq \gamma^2 \int_0^h (v_{,1}^2 + v_{,2}^2 - v v_{,11}) dx_2 \\ &\leq m_0^2 \int_0^l \gamma^2 (v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2) dx_2, \end{aligned} \tag{4.15}$$

and hence, by using the inequality (3.16), we obtain

$$\mathcal{I}''(x_1) - \overline{m}^2 \mathcal{I}(x_1) \geq 0, \quad x_1 > 0, \tag{4.16}$$

where

$$\bar{m}^2 = \frac{1}{m_0^2}\left(1 - \frac{\omega^2}{\omega_m^2}\right) = \frac{2\pi^2}{l^2}\left(1 - \frac{\omega^2}{\omega_m^2}\right). \tag{4.17}$$

To determinate the consequences of the second–order differential inequality (4.16), we write it in the following form

$$\frac{d}{dx_1}\left\{e^{\overline{m}x_1}\left[\mathcal{I}'(x_1) - \overline{m}\mathcal{I}(x_1)\right]\right\} \geq 0, \tag{4.18}$$

and then integrate it over $[0, x_1]$ to obtain

$$-\mathcal{I}'(x_1) + \overline{m}\mathcal{I}(x_1) \leq e^{-\overline{m}x_1}\left[-\mathcal{I}'(0) + \overline{m}\mathcal{I}(0)\right], \quad x_1 \geq 0. \tag{4.19}$$

On the basis of this relation, we further can note that a successive integration over $[x_1, \infty)$ of the relation (3.16) gives

$$-\mathcal{I}'(x_1) \geq \left(1 - \frac{\omega^2}{\omega_m^2}\right)\int_{D_{x_1}} \gamma^2\left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da, \ \ x_1 \geq 0, \tag{4.20}$$

and

$$\mathcal{I}(x_1) \geq \left(1 - \frac{\omega^2}{\omega_m^2}\right)\int_{x_1}^{\infty}\int_{D_\xi} \gamma^2\left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da_\xi d\xi, \ \ x_1 \geq 0. \tag{4.21}$$

Further, by using the estimate (4.19), from the relations (4.20) and (4.21), we deduce the following spatial estimates

$$\int_{D_{x_1}}\left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da \leq \frac{1}{\gamma^2\left(1 - \frac{\omega^2}{\omega_m^2}\right)}\left[-\mathcal{I}'(0) + \overline{m}\mathcal{I}(0)\right] e^{-\overline{m}x_1}, \quad x_1 \geq 0, \tag{4.22}$$

and

$$\int_{x_1}^{\infty}\int_{D_\xi}\left(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2\right) da_\xi d\xi \leq \frac{l}{\pi\sqrt{2}\gamma^2}\left(1 - \frac{\omega^2}{\omega_m^2}\right)^{-\frac{3}{2}}\left[-\mathcal{I}'(0) + \overline{m}\mathcal{I}(0)\right] e^{-\overline{m}x_1}, \quad x_1 \geq 0. \tag{4.23}$$

Thus, we can conclude that in the class of amplitudes $v(x)$ for which $\mathcal{I}'(x_1) \leq 0$ for all $x_1 \geq 0$ the measure

$$\mathcal{F}(x_1) = \int_{D_{x_1}}(v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2) da \tag{4.24}$$

decays toward zero faster than the exponential $e^{-\overline{m}x_1}$ when x_1 goes to infinity.

5. CONCLUSION

On the basis of the above analysis we can conclude that, for an amplitude $v(x)$, solution of the boundary value problem $\mathcal{P}$, we have the following alternative of Phragmèn-Lindelöf type: either the measure $\mathcal{E}(x_1)$ grows toward infinity faster than the exponential $e^{\tilde{m}x_1}$ when x_1 goes to infinity and then the energy

$$\mathcal{U}(v) = \int_S (v_{,11}^2 + 2v_{,12}^2 + v_{,22}^2) da \tag{5.1}$$

is unbounded, or the energy $\mathcal{U}(v)$ is bounded and then the measure $\mathcal{F}(x_1)$ decays toward zero faster than the exponential $e^{-\bar{m}x_1}$, provided the excited frequency ω is lower than the critical value ω_m defined by the relation (3.15).

6. THE KIRCHHOFF THEORY OF THIN PLATES

We consider here as a limit case the Kirchhoff theory of thin elastic plates, that is the case when β tends to zero. It can be seen from the relation (A7) that $r(\tau)$ decreases monotonically with increasing τ, and that

$$r(0^+) = \lim_{\tau\to 0^+} r(\tau) = \pi, \quad r(\infty) = \lim_{\tau\to\infty} r(\tau) = r_0, \tag{6.1}$$

where $r_0 = 2.365$ is the smallest positive root of the equation

$$\tan r = -\tanh r. \tag{6.2}$$

It follows then from the relations (A7) and (6.1) that $\lambda(t)$ is a decreasing function with respect to t, and that

$$\lambda(0^+) = \lim_{t\to 0^+} \lambda(t) = \frac{4\pi^2}{l^2}, \quad \lim_{t\to\infty} t\lambda(t) = \left(\frac{2r_0}{l}\right)^4. \tag{6.3}$$

In view of the relation (3.11) and by using the relation (6.3) it follows that

$$\lim_{\beta\to 0} \frac{\Lambda(\alpha,\beta)}{\beta^2} = \frac{1}{\alpha^2}\left(\frac{2r_0}{l}\right)^4, \tag{6.4}$$

and hence the relation (3.15) furnishes that

$$\omega_m^2 = \frac{\gamma^2}{\alpha^2}\left(\frac{2r_0}{l}\right)^4 = \frac{Eh^2}{12(1-\nu^2)\varrho}\left(\frac{4.73}{l}\right)^4. \tag{6.5}$$

To this end we recall the critical value established by Ciarletta [3] for the model of thin plates with transverse shear deformation

$$\omega_m^{*2} = \frac{h^2\pi^4}{4l^2}\frac{\mu}{\varrho(h^2\pi^2 + l^2)}, \tag{6.6}$$

that is

$$\omega_m^{*2} = \frac{Eh^2}{8(1+\nu)\varrho}\left(\frac{\pi}{l}\right)^4 \frac{1}{1+\frac{h^2}{l^2}\pi^2}. \tag{6.7}$$

Therefore, we have

$$\Phi = \frac{\omega_m^{*2}}{\omega_m^2} = 0.29191\frac{1-\nu}{1+\frac{h^2}{l^2}\pi^2}, \tag{6.8}$$

and because we have $h \ll l$ and $\frac{1}{2} < 1-\nu < 2$, it results that

$$\Phi < 0.58382. \tag{6.9}$$

This leads to the idea that for the Kirchhoff theory of thin plates we have an interval of frequencies larger than that of the Reissner–Mindlin model for which we can establish the spatial behaviour of the amplitudes.

7. APPENDIX

In [12] Knowles has established the following result: for any function $u \in C_0^2([0,l])$ and for any real number $t \geq 0$, we have

(A1) $$\int_0^l u_{,22}^2 dx_2 \geq \lambda(t) \int_0^t (u_{,2}^2 + tu^2) dx_2,$$

where

(A2) $$\lambda(t) = \frac{4}{l^2} \frac{r^4(\tau)}{\tau + r^2(\tau)}, \quad \tau = \frac{tl^2}{4},$$

and $r(\tau)$ is the smallest positive root of the equation

(A3) $$\tan r = -\sqrt{\frac{\tau}{\tau + r^2}} \tanh\left(r\sqrt{\frac{\tau}{\tau + r^2}}\right), \quad \tau \geq 0.$$

Moreover, $\lambda(t)$ is the largest possible constant in (A1) in the sense that if, for a given t, $\lambda(t)$ is replaced by a smaller constant, there is a $u \in C_0^2([0,l])$ for which (A1) fails to hold.

The proof of the result stated above is based on the fact that the variational problem of finding the extremals in $C_0^2([0,l])$ of the ratio

(A4) $$J\{u\} = \frac{\int_0^l u_{,22}^2 dx_2}{\int_0^l (u_{,2}^2 + tu^2) dx_2},$$

for fixed $t \geq 0$ leads formally to the eigenvalue problem

(A5) $$u_{,2222} + \lambda u_{,22} - \lambda t u = 0 \quad \text{on} \quad [0,l],$$

(A6) $$u(0) = u_{,2}(0) = u(l) = u_{,2}(l) = 0.$$

It can be proved that the eigenvalues λ are given by

(A7) $$\lambda(t) = \frac{4}{l^2} \frac{r^4(\tau)}{\tau + r^2(\tau)}, \quad \tau = \frac{tl^2}{4},$$

where r is a positive root of either of the equations

(A8) $$\tan r = \sqrt{\frac{\tau + r^2}{\tau}} \tanh\left(r\sqrt{\frac{\tau}{\tau + r^2}}\right),$$

(A9) $$\tan r = -\sqrt{\frac{\tau}{\tau + r^2}} \tanh\left(r\sqrt{\frac{\tau}{\tau + r^2}}\right).$$

It is shown that the smallest eigenvalue $\lambda(t)$ corresponds to the smallest positive root $r(\tau)$ of the equation (A9) and the corresponding eigenfunction has no zero in $(0,l)$ and realize the absolute minimum of $J\{u\}$ on $C_0^2([0,l])$.

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