



EQUIVALENT THEOREM ON LUPAŞ OPERATORS

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Abstract: The purpose of this present paper is to give an equivalent theorem on Lupaş operators with $\omega_{\phi,\lambda}^r(f, t)$, where $\omega_{\phi,\lambda}^r(f, t)$ is Ditzian-Totik modulus of smoothness for linear combination of Lupaş operators.

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Dedicatory: In memory of Professor Alexandru Lupaş.

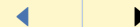
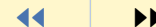
Equivalent Theorem on Lupaş Operators

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1. Introduction

Let $C[0, +\infty)$ be the set of continuous and bounded functions defined on $[0, +\infty)$. For $f \in C[0, +\infty)$ and $n \in \mathbb{N}$, the Lupaş operators are defined as

$$(1.1) \quad B_n(f, x) = \sum_{k=0}^{+\infty} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

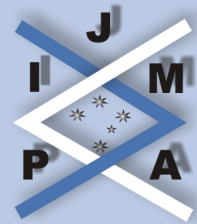
$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

Since the Lupaş operators cannot be used for the investigation of higher orders of smoothness, we consider combinations of these operators, which have higher orders of approximation. The linear combinations of Lupaş operators on $C[0, +\infty)$ are defined as (see [3, p.116])

$$(1.2) \quad B_{n,r}(f, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f, x), \quad r \in \mathbb{N},$$

where n_i and $C_i(n)$ satisfy:

- (i) $n = n_0 < \dots < n_{r-1} \leq A_n$;
- (ii) $\sum_{i=0}^{r-1} |C_i(n)| \leq C$;
- (iii) $B_{n,r}(1, x) = 1$;
- (iv) $B_{n,r}\left((t-x)^k, x\right) = 0$; $k = 1, 2, \dots, r-1$,



where constants A and C are independent of n .

Ditzian [1] used $\omega_{\phi^\lambda}^2(f, t)$ ($0 \leq \lambda \leq 1$) and studied an interesting direct result for Bernstein polynomials and unified the results with $\omega^2(f, t)$ and $\omega_\phi^2(f, t)$. In [2], $\omega_{\phi^\lambda}^r(f, t)$ was also used for polynomial approximation.

To state our results, we give some notations (c.f. [4]). Let $C[0, +\infty)$ be the set of continuous and bounded functions on $[0, +\infty)$. Our modulus of smoothness is given by

$$(1.3) \quad \omega_{\phi^\lambda}^r(f, t) = \sup_{0 < h \leq t} \sup_{x \pm (rh\phi^\lambda(x)/2) \in [0, +\infty)} \left| \Delta_{h\phi^\lambda(x)}^r f(x) \right|,$$

where

$$\Delta_h^1 f(x) = f\left(x - \frac{h}{2}\right) - f\left(x + \frac{h}{2}\right), \quad \Delta^{k+1} = \Delta^1(\Delta^k), \quad k \in \mathbb{N}$$

and our K -functional by

$$(1.4) \quad K_{\phi^\lambda}(f, t^r) = \inf \left\{ \|f - g\|_{C[0, +\infty)} + t^r \|\phi^{r\lambda} g^{(r)}\|_{C[0, +\infty)} \right\},$$

$$(1.5) \quad \tilde{K}_{\phi^\lambda}(f, t^r) = \inf \left\{ \|f - g\|_{C[0, +\infty)} + t^r \|\phi^{r\lambda} g^{(r)}\|_{C[0, +\infty)} + t^{r/(1-\lambda/2)} \|g^{(r)}\|_{C[0, +\infty)} \right\},$$

where the infimum is taken on the functions satisfying $g^{(r-1)} \in A \cdot C_{loc}$, $\phi(x) = \sqrt{x(1+x)}$ and $0 \leq \lambda \leq 1$. It is well known (see [1]) that

$$(1.6) \quad \omega_{\phi^\lambda}^r(f, t) \sim K_{\phi^\lambda}(f, t^r) \sim \tilde{K}_{\phi^\lambda}(f, t^r),$$

($x \sim y$ means that there exists $c > 0$ such that $c^{-1}y \leq x \leq cy$).



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To prove the inverse, we need the following notations. Let us denote

$$C_0 := \{f \in C[0, +\infty), f(0) = 0\},$$

$$\|f\|_0 := \sup_{x \in (0, +\infty)} |\delta_n^{\alpha(\lambda-1)}(x)f(x)|,$$

$$C_\lambda^0 := \{f \in C_0 : \|f\|_0 < +\infty\},$$

$$\|f\|_r := \sup_{x \in (0, +\infty)} |\delta_n^{r+\alpha(\lambda-1)}(x)f^{(r)}(x)|,$$

$$C_\lambda^r := \{f \in C_0 : f^{(r-1)} \in A \cdot C_{loc}, \|f\|_r < +\infty\},$$

where $\delta_n(x) = \phi(x) + \frac{1}{\sqrt{n}} \sim \max\left\{\phi(x), \frac{1}{\sqrt{n}}\right\}$, $0 \leq \lambda \leq 1$, $r \in \mathbb{N}$ and $0 < \alpha < r$.

Now, we state the main results.

If $f \in C[0, +\infty)$, $r \in \mathbb{N}$, $0 < \alpha < r$, $0 \leq \lambda \leq 1$, then the following statements are equivalent

$$(1.7) \quad |B_{n,r}(f, x) - f(x)| = O\left(\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha\right),$$

$$(1.8) \quad \omega_{\phi^\lambda}^r(f, t) = O(t^\alpha),$$

where $\delta_n(x) = \phi(x) + \frac{1}{\sqrt{n}} \sim \max\left\{\phi(x), \frac{1}{\sqrt{n}}\right\}$.

In this paper, we will consider the operators (1.2) and obtain an equivalent approximation theorem for these operators.

Throughout this paper C denotes a constant independent of n and x . It is not necessarily the same at each occurrence.



2. Basic Results

In this section, we mention some basis results, which will be used to prove the main results.

If $f \in C[0, +\infty)$, $r \in \mathbb{N}$, then by [3] we know that

$$B_n^{(r)}(f, x) = \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{+\infty} p_{n+r,k}(x) \Delta_{n-1}^r f\left(\frac{k}{n}\right).$$

Lemma 2.1. Let $f^{(r)} \in C[0, +\infty)$ and $0 \leq \lambda \leq 1$, then

$$(2.1) \quad \left| \phi^{r\lambda}(x) B_n^{(r)}(f, x) \right| \leq C \left\| \phi^{r\lambda} f^{(r)} \right\|_{\infty}.$$

Proof. In [3, Sec.9.7], we have

$$(2.2) \quad \Delta_{n-1}^r f\left(\frac{k}{n}\right) \leq C n^{-r+1} \int_0^{r/n} \left| f^{(r)}\left(\frac{k}{n} + u\right) \right| du,$$

$$(2.3) \quad \Delta_{n-1}^r f(0) \leq C n^{-r(2-\lambda)/2} \int_0^{r/n} |u^{r\lambda/2} f^{(r)}(u)| du.$$

Using (2.2), (2.3) and the Hölder inequality, we get

$$\begin{aligned} \left| \phi^{r\lambda}(x) B_n^{(r)}(f, x) \right| &\leq \left| \phi^{r\lambda}(x) \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{+\infty} p_{n+r,k}(x) \Delta_{n-1}^r f\left(\frac{k}{n}\right) \right| \\ &\leq \frac{(n+r-1)!}{(n-1)!} \left| \phi^{r\lambda}(x) p_{n+r,0}(x) \Delta_{n-1}^r f(0) \right. \\ &\quad \left. + \sum_{k=1}^{+\infty} \phi^{r\lambda}(x) p_{n+r,k}(x) \Delta_{n-1}^r f\left(\frac{k}{n}\right) \right| \end{aligned}$$

$$\leq C \|\phi^{r\lambda}(x)f^{(r)}\|_{\infty}.$$

□

Lemma 2.2. Let $f \in C[0, +\infty)$, $r \in \mathbb{N}$ and $0 \leq \lambda \leq 1$, then for $n > r$, we get

$$|\phi^{r\lambda}(x)B_n^{(r)}(f, x)| \leq Cn^{r/2}\delta_n^{-r(1-\lambda)}(x)\|f\|_{\infty}.$$

Proof. We consider $x \in [0, 1/n]$. Then we have $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, $\phi(x) \leq \frac{2}{\sqrt{n}}$. Using

$$\left(\frac{d}{dx}\right) p_{n,k}(x) = n(p_{n+1,k-1}(x) - p_{n+1,k}(x))$$

and

$$\int_0^{\infty} p_{n,k}(x)dx = \frac{1}{n-1},$$

we have

$$|\phi^{r\lambda}(x)B_n^{(r)}(f, x)| \leq Cn^{r/2}\delta_n^{-r(1-\lambda)}(x)\|f\|_{\infty}.$$

Now we consider the interval $x \in (1/n, +\infty)$, then $\phi(x) \sim \delta_n(x)$. Using Lemma 4.5 in [6], we get

$$(2.4) \quad |\phi^r(x)B_n^{(r)}(f, x)| \leq Cn^{r/2}\|f\|_{\infty}.$$

Therefore

$$\begin{aligned} |\phi^{r\lambda}(x)B_n^{(r)}(f, x)| &= \phi^{r(\lambda-1)}(x) |\phi^r(x)B_n^{(r)}(f, x)| \\ &\leq C\phi^{r(\lambda-1)}(x)n^{r/2}\|f\|_{\infty} \\ &\leq C\delta_n^{-r(\lambda-1)}(x)n^{r/2}\|f\|_{\infty}. \end{aligned}$$

□



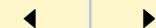
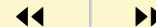
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Lemma 2.3. Let $r \in \mathbb{N}$, $0 \leq \beta \leq r$, $x \pm rt/2 \in I$ and $0 \leq t \leq 1/8r$, then we have the following inequality:

$$(2.5) \quad \int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \delta_n^{-\beta} \left(x + \sum_{j=1}^r u_j \right) du_1 \cdots du_r \leq C(\beta) t^r \delta_n^{-\beta}(x).$$

Proof. The result follows from [7, (4.11)]. □

Lemma 2.4. For $x, t, u \in (0, +\infty)$, $x < u < t$, $t, r \in \mathbb{N}$ and $\lambda \in [0, 1]$, then

$$(2.6) \quad B_n \left(\left| \int_x^t |t-u|^{r-1} \phi^{-r\lambda}(u) du \right|, x \right) \leq C n^{-r/2} \delta_n^r(x) \phi^{-r\lambda}(x).$$

Proof. When $r = 1$, then we have

$$(2.7) \quad \left| \int_x^t \phi^{-\lambda}(u) du \right| \leq |t-x| \{ x^{-\lambda/2} (1+t)^{-\lambda/2} + (1+x)^{-\lambda/2} t^{-\lambda/2} \}.$$

From [3, (9.5.3)],

$$(2.8) \quad B_n((t-x)^{2r}, x) \leq C n^{-r} \phi^{2r}(x).$$

Applying the Hölder inequality, we get

$$(2.9) \quad \begin{aligned} B_n \left(\left| \int_x^t \phi^{-\lambda}(u) du \right|, x \right) &\leq \{ B_n((t-x)^4, x) \}^{\frac{1}{4}} \left[x^{-\lambda/2} \{ B_n((1+t)^{-2\lambda/3}, x) \}^{\frac{3}{4}} \right. \\ &\quad \left. + (1+x)^{-\lambda/2} \{ B_n(t^{-2\lambda/3}, x) \}^{\frac{3}{4}} \right] \\ &\leq C n^{-1/2} \delta_n(x) \left[x^{-\lambda/2} \{ B_n((1+t)^{-2\lambda/3}, x) \}^{\frac{3}{4}} \right. \\ &\quad \left. + (1+x)^{-\lambda/2} \{ B_n(t^{-2\lambda/3}, x) \}^{\frac{3}{4}} \right]. \end{aligned}$$



Applying the Hölder inequality, we get

$$(2.10) \quad B_n(t^{-2\lambda/3}, x) \leq \left(\sum_{k=0}^{+\infty} p_{n,k}(x) \left(\frac{k}{n} \right)^{-1} \right)^{\frac{2}{3\lambda}} \leq Cx^{-2\lambda/3}.$$

Similarly,

$$(2.11) \quad B_n((1+t)^{-2\lambda/3}, x) \leq C(1+x)^{-2\lambda/3}.$$

Combining (2.9) to (2.11), we obtain (2.6).

When $r > 1$, then we have

$$\frac{|t-u|^2}{\phi^2(u)} \leq \frac{|t-x|^2}{\phi^2(x)} \quad (\text{Trivial for } t < u < x),$$

otherwise

$$|t-u|x \leq |t-x|u$$

and

$$\frac{u|t-u|}{\phi^2(u)} \leq \frac{x|t-x|}{\phi^2(x)} \quad (\text{for } t < u < x).$$

Thus

$$(2.12) \quad \frac{|t-u|^{r-2}}{\phi^{(r-2)\lambda}(u)} \leq \frac{|t-x|^{r-2}}{\phi^{(r-2)\lambda}(x)}, \quad r > 2$$

because

$$(2.13) \quad \frac{|t-u|}{\phi^2(u)} \leq \frac{|t-x|}{x} \left(\frac{1}{1+x} + \frac{1}{1+t} \right) \quad (\text{Trivial for } t < u < x),$$



otherwise

$$\frac{(u-t)}{u} \leq \frac{(x-t)}{t}$$

and

$$\frac{1}{1+u} \leq \frac{1}{1+x} + \frac{1}{1+t},$$

$$(2.14) \quad \frac{|t-u|}{\phi^{2\lambda}(u)} \leq \frac{|t-x|}{x^\lambda} \left(\frac{1}{1+x} + \frac{1}{1+t} \right)^\lambda.$$

Because the function t^λ ($0 \leq \lambda \leq 1$) is subadditive, using (2.12) and (2.14), we obtain

$$(2.15) \quad \frac{|t-u|^{r-1}}{\phi^{r\lambda}(u)} \leq \frac{|t-x|^{r-1}}{\phi^{(r-2)\lambda}(x)x^\lambda} \left\{ \left(\frac{1}{1+x} \right)^\lambda + \left(\frac{1}{1+t} \right)^\lambda \right\}.$$

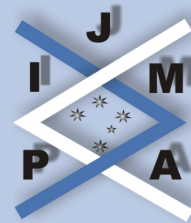
Since $B_n((1+t)^{-r}, x) \leq C(1+x)^{-r}$ ($r \in \mathbb{N}$, $x \in [0, +\infty)$), using the Hölder inequality, we have

$$(2.16) \quad B_n((1+t)^{-2\lambda}, x) \leq C(1+x)^{-2\lambda} \quad (x \in [0, +\infty), \lambda \in [0, 1]).$$

Using (2.8), (2.15), (2.16) and the Hölder inequality, we get

$$\begin{aligned} B_n \left(\left| \int_x^t |t-u|^{r-1} \phi^{-r\lambda}(u) du \right|, x \right) &\leq B_n^{1/2}(|t-u|^{2r}, x) \\ &\quad \cdot \left(\phi^{-r\lambda}(x) + x^{-\lambda} \phi^{-(r-2)\lambda}(x) B_n^{1/2}((1+t)^{-2\lambda}, x) \right) \\ &\leq C n^{-r/2} \delta_n^r(x) \phi^{-r\lambda}(x). \end{aligned}$$

□



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Lemma 2.5. *If $r \in \mathbb{N}$ and $0 < \alpha < r$ then*

$$(2.17) \quad \|B_n f\|_r \leq C n^{r/2} \|f\|_0 \quad (f \in C_\lambda^0),$$

$$(2.18) \quad \|B_n f\|_r \leq C \|f\|_r \quad (f \in C_\lambda^r).$$

Proof. For $x \in [0, \frac{1}{n}]$ and $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, according to Lemma 2.2, we get

$$|\delta_n^{r+\alpha(\lambda-1)}(x) B_n^{(r)}(f, x)| \leq C n^{r/2} \|f\|_0.$$

On the other hand, for $x \in (\frac{1}{n}, +\infty)$ and $\delta_n(x) \sim \phi(x)$, according to [3], we can obtain

$$(2.19) \quad B_n^{(r)}(f, x) = \phi^{-2r} \sum_{i=0}^r V_i(n\phi^2(x)) n^i \sum_{k=0}^{+\infty} p_{n,k}(x) \left(\frac{k}{n} - x\right)^i \Delta_{1/n}^r f\left(\frac{k}{n}\right),$$

where $V_i(n\phi^2(x))$ is polynomial in $n\phi^2(x)$ of degree $(r-i)/2$ with constant coefficients and therefore,

$$(2.20) \quad |\phi^{-2r} V_i(n\phi^2(x)) n^i| \leq C \left(\frac{n}{\phi^2(x)}\right)^{(r+i)/2}, \quad \text{for any } x \in \left(\frac{1}{n}, +\infty\right).$$

Since

$$(2.21) \quad \left| \Delta_{1/n}^r f\left(\frac{k}{n}\right) \right| \leq C \|f\|_0 \phi^{\alpha(\lambda-1)}(x),$$

using the Hölder inequality, we get

$$(2.22) \quad \left| \sum_{k=0}^{+\infty} p_{n,k}(x) \left(\frac{k}{n} - x\right)^i \Delta_{1/n}^r f\left(\frac{k}{n}\right) \right| \leq C \left(\frac{\phi^2(x)}{n}\right)^{i/2} \|f\|_0 \phi^{\alpha(\lambda-1)}(x).$$

From (2.19) to (2.22) we can deduce (2.17) easily. Similarly, like Lemma 2.1 we can obtain (2.18). □



3. Direct Results

Theorem 3.1. Let $f \in C[0, +\infty)$, $r \in \mathbb{N}$ and $0 \leq \lambda \leq 1$, then

$$(3.1) \quad |B_{n,r}(f, x) - f(x)| \leq C\omega_{\phi^\lambda}^r(f, n^{-1/2}\delta_n^{1-\lambda}(x)).$$

Proof. Using (1.5) and (1.6) and taking $d_n = d_n(x, \lambda) = n^{-1/2}\delta_n^{1-\lambda}(x)$, we can choose $g_n = g_{n,x,\lambda}$ for fixed x and λ satisfying:

$$(3.2) \quad \|f - g_n\| \leq C\omega_{\phi^\lambda}^r(f, d_n),$$

$$(3.3) \quad d_n^r \|\phi^{r\lambda} g_n^{(r)}\| \leq C\omega_{\phi^\lambda}^r(f, d_n),$$

$$(3.4) \quad d_n^{r/(1-(\lambda/2))} \|g_n^{(r)}\| \leq C\omega_{\phi^\lambda}^r(f, d_n).$$

Now

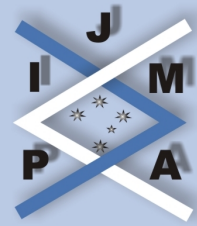
$$(3.5) \quad \begin{aligned} |B_{n,r}(f, x) - f(x)| &\leq |B_{n,r}(f, x) - B_{n,r}(g_n, x)| + |f - g_n(x)| \\ &\quad + |B_{n,r}(g_n, x) - g_n(x)| \\ &\leq C\|f - g_n\|_\infty + |B_{n,r}(g_n, x) - g_n(x)|. \end{aligned}$$

By using Taylor's formula:

$$g_n(t) = g_n(x) + (t - x)g_n'(x) + \cdots + \frac{(t - x)^{r-1}}{(r - 1)!} g_n^{(r-1)}(x) + R_r(g_n, t, x),$$

where

$$R_r(g_n, t, x) = \frac{1}{(r - 1)!} \int_x^t (t - u)^{r-1} g_n^{(r)}(u) du.$$



Using (i)-(iv) of (1.2) and Lemma 2.4, we obtain

$$\begin{aligned}
 & |B_{n,r}(g_n, x) - g_n(x)| \\
 &= \left| B_{n,r} \left(\frac{1}{(r-1)!} \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du, x \right) \right| \\
 (3.6) \quad &\leq C \|\phi^{r\lambda} g_n^{(r)}\| \left(B_{n,r} \left(\left| \int_x^t \frac{|t-u|^{r-1}}{\phi^{r\lambda}(u)} du \right|, x \right) \right) \\
 &\leq C \phi^{-r\lambda}(x) n^{-r/2} \delta_n^r(x) \|\phi^{r\lambda} g_n^{(r)}\|.
 \end{aligned}$$

Again using (i)-(iv) of (1.2) and (2.12), we get

$$\begin{aligned}
 & |B_{n,r}(g_n, x) - g_n(x)| \\
 &\leq C \|\delta_n^{r\lambda} g_n^{(r)}\| \left| B_{n,r} \left(\int_x^t \frac{|t-u|^{r-1}}{\delta_n^{r\lambda}(u)} du, x \right) \right| \\
 (3.7) \quad &\leq C \|\delta_n^{r\lambda} g_n^{(r)}\| n^{r\lambda/2} \left| B_{n,r}((t-x)^{2r}, x) \right|^{1/2} \\
 &\leq C n^{-r/2} \delta_n^r(x) n^{r\lambda/2} \|\delta_n^{r\lambda} g_n^{(r)}\|.
 \end{aligned}$$

We will take the following two cases:

Case-I: For $x \in [0, 1/n]$, $\delta_n(x) \sim \frac{1}{\sqrt{n}}$. Then by (3.2) – (3.5) and (3.7), we have

$$\begin{aligned}
 & |B_{n,r}(f, x) - f(x)| \\
 &\leq C (\|f - g_n\| + d_n^r \|\delta_n^{r\lambda} g_n^{(r)}\|) \\
 (3.8) \quad &\leq C (\|f - g_n\| + d_n^r \|\phi^{r\lambda} g_n^{(r)}\| + d_n^r n^{-r\lambda/2} \|g_n^{(r)}\|) \\
 (3.9) \quad &\leq C (\|f - g_n\| + d_n^r \|\phi^{r\lambda} g_n^{(r)}\| + d_n^{r/(1-(\lambda/2))} \|g_n^{(r)}\|) \\
 &\leq C \omega_{\phi^\lambda}^r(f, d_n).
 \end{aligned}$$

Case-II: For $x \in (1/n, +\infty)$, $\delta_n(x) \sim \phi(x)$. Then by (3.2) – (3.3) and (3.5) – (3.6), we get

$$(3.10) \quad |B_{n,r}(f, x) - f(x)| \leq C(\|f - g_n\| + d_n^r \|\phi^{r\lambda} g_n^{(r)}\|) \leq C\omega_{\phi^\lambda}^r(f, d_n).$$

□



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4. Inverse Results

Theorem 4.1. Let $f \in C[0, +\infty)$, $r \in \mathbb{N}$, $0 < \alpha < r$, $0 \leq \lambda \leq 1$. Then

$$(4.1) \quad |B_{n,r}(f, x) - f(x)| = O(d_n^\alpha).$$

implies

$$(4.2) \quad \omega_{\phi_\lambda}^r(f, t) = O(t^\alpha),$$

where $d_n = n^{-1/2} \delta_n^{1-\lambda}(x)$.

Proof. Since $B_n(f, x)$ preserves the constant, hence we may assume $f \in C_0$. Suppose that (4.1) holds. Now we introduce a new K -functional as

$$K_\lambda^\alpha(f, t^r) = \inf_{g \in C_\lambda^r} \{ \|f - g\|_0 + t^r \|g\|_r \}.$$

Choosing $g \in C_\lambda^r$ such that

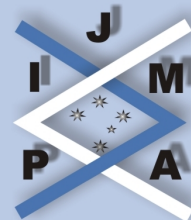
$$(4.3) \quad \|f - g\|_0 + n^{-r/2} \|g\|_r \leq 2K_\lambda^\alpha(f, n^{-r/2}).$$

By (4.1), we can deduce that

$$\|B_{n,r}(f, x) - f(x)\|_0 \leq Cn^{-\alpha/2}.$$

Thus, by using Lemma 2.5 and (4.3), we obtain

$$\begin{aligned} K_\lambda^\alpha(f, t^r) &\leq \|f - B_{n,r}(f)\|_0 + t^r \|B_{n,r}(f)\|_r \\ &\leq Cn^{-\alpha/2} + t^r (\|B_{n,r}(f - g)\|_r + \|B_{n,r}(g)\|_r) \\ &\leq C \left(n^{-\alpha/2} + t^r (n^{r/2} \|f - g\|_0 + \|g\|_r) \right) \\ &\leq C \left(n^{-\alpha/2} + \frac{t^r}{n^{-r/2}} K_\lambda^\alpha(f, n^{-r/2}) \right) \end{aligned}$$



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which implies that by [3, 7]

$$(4.4) \quad K_\lambda^\alpha(f, t^r) \leq Ct^\alpha.$$

On the other hand, since $x + (j - \frac{r}{2})t\phi^\lambda(x) \geq 0$, therefore

$$\left| \left(j - \frac{r}{2} \right) t\phi^\lambda(x) \right| \leq x$$

and

$$x + \left(j - \frac{r}{2} \right) t\phi^\lambda(x) \leq 2x,$$

so that

$$(4.5) \quad \delta_n \left(x + \left(j - \frac{r}{2} \right) t\phi^\lambda(x) \right) \leq 2\delta_n(x).$$

Thus, for $f \in C_\lambda^0$, we get

$$(4.6) \quad \begin{aligned} \left| \Delta_{t\phi^\lambda(x)}^r f(x) \right| &\leq \|f\|_0 \left(\sum_{j=0}^r \binom{r}{j} \delta_n^{\alpha(1-\lambda)} \left(x + \left(j - \frac{r}{2} \right) t\phi^\lambda(x) \right) \right) \\ &\leq 2^{2r} \delta_n^{\alpha(1-\lambda)}(x) \|f\|_0. \end{aligned}$$

From Lemma 2.3, for $g \in C_\lambda^r$, $0 < t\phi^\lambda(x) < 1/8r$, $x \pm rt\phi^\lambda(x)/2 \in [0, +\infty)$, we have

$$\begin{aligned} &\left| \Delta_{t\phi^\lambda(x)}^r g(x) \right| \\ &\leq \left| \int_{-(t/2)\phi^\lambda(x)}^{(t/2)\phi^\lambda(x)} \cdots \int_{-(t/2)\phi^\lambda(x)}^{(t/2)\phi^\lambda(x)} g^{(r)} \left(x + \sum_{j=1}^r u_j \right) du_1 \cdots du_r \right| \end{aligned}$$



$$(4.7) \quad \leq \|g\|_r \int_{-(t/2)\phi^\lambda(x)}^{(t/2)\phi^\lambda(x)} \cdots \int_{-(t/2)\phi^\lambda(x)}^{(t/2)\phi^\lambda(x)} \delta_n^{-r+\alpha(1-\lambda)} \left(x + \sum_{j=1}^r u_j \right) du_1 \dots du_r$$

$$(4.8) \quad \leq Ct^r \delta_n^{-r+\alpha(1-\lambda)}(x) \|g\|_r \cdot v$$

Using (4.4), (4.6) and (4.7), for $0 < t\phi^\lambda(x) < 1/8r$, $x \pm rt\phi^\lambda(x)/2 \in [0, +\infty)$ and choosing appropriate g , we get

$$\begin{aligned} |\Delta_{t\phi^\lambda(x)}^r f(x)| &\leq |\Delta_{t\phi^\lambda(x)}^r (f - g)(x)| + |\Delta_{t\phi^\lambda(x)}^r g(x)| \\ &\leq C\delta_n^{\alpha(1-\lambda)}(x) \left\{ \|f - g\|_0 + t^r \delta_n^{r(\lambda-1)}(x) \|g\|_r \right\} \\ &\leq C\delta_n^{\alpha(1-\lambda)}(x) K_\lambda^\alpha \left(f, \frac{t^r}{\delta_n^{r(1-\lambda)}(x)} \right) \\ &\leq Ct^r. \end{aligned}$$

□

Remark 1. Very recently Gupta and Deo [5] have studied two dimensional modified Lupaş operators. In the same manner we can obtain an equivalent theorem with $\omega_{\phi^\lambda}^r(f, t)$.

References

- [1] Z. DITZIAN, Direct estimate for Bernstein polynomials, *J. Approx. Theory*, **79** (1994), 165–166.
- [2] Z. DITZIAN AND D. JIANG, Approximations by polynomials in $C[-1, 1]$, *Canad. J. Math.*, **44** (1992), 924–940.
- [3] Z. DITZIAN AND V. TOTIK, *Moduli of Smoothness*, Springer-Verlag, New York, (1987).
- [4] S. GUO, H. TONG AND G. ZHANG, Stechkin-Marchaud-type inequalities for Baskakov polynomials, *J. Approx. Theory*, **114** (2002), 33–47.
- [5] V. GUPTA AND N. DEO, On the rate of convergence for bivariate Beta operators, *General Math.*, **13**(3) (2005), 107–114.
- [6] M. HEILMANN, Direct and converse results for operators of Baskakov-Durrmeyer type, *Approx. Theory and its Appl.*, **5**(1) (1989), 105–127.
- [7] D. ZHOU, On a paper of Mazhar and Totik, *J. Approx. Theory*, **72** (1993), 209–300.



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