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EQUIVALENT THEOREM ON LUPAŞ OPERATORS

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In memory of Professor Alexandru Lupaş.

ABSTRACT. The purpose of this present paper is to give an equivalent theorem on Lupaş operators with $\omega_{\phi^\lambda}^r(f,t)$, where $\omega_{\phi^\lambda}^r(f,t)$ is Ditzian-Totik modulus of smoothness for linear combination of Lupaş operators.

Key words and phrases: Lupaş operators, Linear combinations.

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1. INTRODUCTION

Let $C[0,+\infty)$ be the set of continuous and bounded functions defined on $[0,+\infty)$. For $f \in C[0,+\infty)$ and $n \in \mathbb{N}$, the Lupaş operators are defined as

$$B_n(f,x) = \sum_{k=0}^{+\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

Since the Lupaş operators cannot be used for the investigation of higher orders of smoothness, we consider combinations of these operators, which have higher orders of approximation. The

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linear combinations of Lupaş operators on $C[0,+\infty)$ are defined as (see [3, p.116])

$$B_{n,r}(f,x)=\sum_{i=0}^{r-1}C_i(n)B_{n_i}(f,x), \quad r\in\mathbb{N}, \tag{1.2}$$

where n_i and $C_i(n)$ satisfy:

(i) $n=n_0<\cdots<n_{r-1}\leq A_n$;
(ii) $\sum_{i=0}^{r-1}\left|C_i(n)\right|\leq C$;
(iii) $B_{n,r}(1,x)=1$;
(iv) $B_{n,r}\left((t-x)^k,x\right)=0;\ k=1,2,\dots,r-1,$

where constants A and C are independent of n.

Ditzian [1] used $\omega^2_{\phi^\lambda}(f,t)$ $(0\leq\lambda\leq 1)$ and studied an interesting direct result for Bernstein polynomials and unified the results with $\omega^2(f,t)$ and $\omega^2_\phi(f,t)$. In [2], $\omega^r_{\phi^\lambda}(f,t)$ was also used for polynomial approximation.

To state our results, we give some notations (c.f. [4]). Let $C[0,+\infty)$ be the set of continuous and bounded functions on $[0,+\infty)$. Our modulus of smoothness is given by

$$\omega^r_{\phi^\lambda}(f,t)=\sup_{0<h\leq t}\ \sup_{x\pm(rh\phi^\lambda(x)/2)\in[0,+\infty)}\left|\triangle^r_{h\phi^\lambda(x)}f(x)\right|, \tag{1.3}$$

where

$$\triangle^1_h f(x)=f\left(x-\frac{h}{2}\right)-f\left(x+\frac{h}{2}\right), \quad \triangle^{k+1}=\triangle^1(\triangle^k), \quad k\in\mathbb{N}$$

and our $K-$functional by

$$K_{\phi^\lambda}(f,t^r)=\inf\left\{\|f-g\|_{C[0,+\infty)}+t^r\left\|\phi^{r\lambda}g^{(r)}\right\|_{C[0,+\infty)}\right\}, \tag{1.4}$$

$$\tilde{K}_{\phi^\lambda}(f,t^r)=\inf\Big\{\|f-g\|_{C[0,+\infty)}+t^r\left\|\phi^{r\lambda}g^{(r)}\right\|_{C[0,+\infty)} + t^{r/(1-\lambda/2)}\left\|g^{(r)}\right\|_{C[0,+\infty)}\Big\}, \tag{1.5}$$

where the infimum is taken on the functions satisfying $g^{(r-1)}\in A\cdot C_{loc},\ \phi(x)=\sqrt{x(1+x)}$ and $0\leq\lambda\leq 1$. It is well known (see [1]) that

$$\omega^r_{\phi^\lambda}(f,t)\sim K_{\phi^\lambda}(f,t^r)\sim\tilde{K}_{\phi^\lambda}(f,t^r), \tag{1.6}$$

($x\sim y$ means that there exists $c>0$ such that $c^{-1}y\leq x\leq cy$).

To prove the inverse, we need the following notations. Let us denote

$$\begin{aligned}
C_0 &:= \{f\in C[0,+\infty),\ f(0)=0\},\\
\|f\|_0 &:= \sup_{x\in(0,+\infty)}\left|\delta_n^{\alpha(\lambda-1)}(x)f(x)\right|,\\
C^0_\lambda &:= \left\{f\in C_0 : \|f\|_0<+\infty\right\},\\
\|f\|_r &:= \sup_{x\in(0,+\infty)}\left|\delta_n^{r+\alpha(\lambda-1)}(x)f^{(r)}(x)\right|,\\
C^r_\lambda &:= \left\{f\in C_0 : f^{(r-1)}\in A\cdot C_{loc},\ \|f\|_r<+\infty\right\},
\end{aligned}$$

where $\delta_n(x) = \phi(x) + \frac{1}{\sqrt{n}} \sim \max\left\{\phi(x), \frac{1}{\sqrt{n}}\right\}, \;\; 0 \le \lambda \le 1, \; r \in \mathbb{N}$ and $0 < \alpha < r$.

Now, we state the main results.

If $f \in C[0, +\infty), \; r \in \mathbb{N}, \; 0 < \alpha < r, \; 0 \le \lambda \le 1$, then the following statements are equivalent

$$|B_{n,r}(f,x) - f(x)| = O\left((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^{\alpha}\right), \tag{1.7}$$

$$\omega^r_{\phi^\lambda}(f,t) = O(t^\alpha), \tag{1.8}$$

where $\delta_n(x) = \phi(x) + \frac{1}{\sqrt{n}} \sim \max\left\{\phi(x), \frac{1}{\sqrt{n}}\right\}$.

In this paper, we will consider the operators (1.2) and obtain an equivalent approximation theorem for these operators.

Throughout this paper C denotes a constant independent of n and x. It is not necessarily the same at each occurrence.

2. Basic Results

In this section, we mention some basis results, which will be used to prove the main results.

If $f \in C[0, +\infty), \; r \in \mathbb{N}$, then by [3] we know that

$$B_n^{(r)}(f,x) = \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{+\infty} p_{n+r,k}(x) \triangle^r_{n^{-1}} f\left(\frac{k}{n}\right).$$

Lemma 2.1. *Let* $f^{(r)} \in C[0, +\infty)$ *and* $0 \le \lambda \le 1$, *then*

$$\left|\phi^{r\lambda}(x) B_n^{(r)}(f,x)\right| \le C \left\|\phi^{r\lambda} f^{(r)}\right\|_\infty. \tag{2.1}$$

Proof. In [3, Sec.9.7], we have

$$\triangle^r_{n^{-1}} f\left(\frac{k}{n}\right) \le C n^{-r+1} \int_0^{r/n} \left| f^{(r)}\left(\frac{k}{n} + u\right)\right| du, \tag{2.2}$$

$$\triangle^r_{n^{-1}} f(0) \le C n^{-r(2-\lambda)/2} \int_0^{r/n} \left|u^{r\lambda/2} f^{(r)}(u)\right| du. \tag{2.3}$$

Using (2.2), (2.3) and the Hölder inequality, we get

$$\begin{aligned}
\left|\phi^{r\lambda}(x) B_n^{(r)}(f,x)\right| &\le \left|\phi^{r\lambda}(x) \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{+\infty} p_{n+r,k}(x) \triangle^r_{n^{-1}} f\left(\frac{k}{n}\right)\right| \\
&\le \frac{(n+r-1)!}{(n-1)!} \left| \phi^{r\lambda}(x) p_{n+r,0}(x) \triangle^r_{n^{-1}} f(0) \right. \\
&\qquad \left. + \sum_{k=1}^{+\infty} \phi^{r\lambda}(x) p_{n+r,k}(x) \triangle^r_{n^{-1}} f\left(\frac{k}{n}\right)\right| \\
&\le C \left\|\phi^{r\lambda}(x) f^{(r)}\right\|_\infty.
\end{aligned}$$

□

Lemma 2.2. *Let* $f \in C[0, +\infty)$, $r \in \mathbb{N}$ *and* $0 \le \lambda \le 1$, *then for* $n > r$, *we get*

$$\left|\phi^{r\lambda}(x) B_n^{(r)}(f,x)\right| \le C n^{r/2} \delta_n^{-r(1-\lambda)}(x) \left\| f \right\|_\infty.$$

Proof. We consider $x \in [0, 1/n]$. Then we have $\delta_n(x) \sim \frac{1}{\sqrt{n}},\ \phi(x) \le \frac{2}{\sqrt{n}}$. Using

$$\left(\frac{d}{dx}\right) p_{n,k}(x) = n\big(p_{n+1,k-1}(x) - p_{n+1,k}(x)\big)$$

and

$$\int_0^\infty p_{n,k}(x)dx = \frac{1}{n-1},$$

we have

$$\left|\phi^{r\lambda}(x) B_n^{(r)}(f,x)\right| \le C n^{r/2} \delta_n^{-r(1-\lambda)}(x) \left\| f \right\|_\infty.$$

Now we consider the interval $x \in (1/n, +\infty)$, then $\phi(x) \sim \delta_n(x)$. Using Lemma 4.5 in [6], we get

$$\left|\phi^{r}(x) B_n^{(r)}(f,x)\right| \le C n^{r/2} \left\| f \right\|_\infty. \tag{2.4}$$

Therefore

$$\begin{aligned} \left|\phi^{r\lambda}(x) B_n^{(r)}(f,x)\right| &= \phi^{r(\lambda-1)}(x) \left|\phi^{r}(x) B_n^{(r)}(f,x)\right| \\ &\le C \phi^{r(\lambda-1)}(x) n^{r/2} \left\| f \right\|_\infty \\ &\le C \delta_n^{-r(\lambda-1)}(x) n^{r/2} \left\| f \right\|_\infty. \end{aligned}$$

□

Lemma 2.3. *Let* $r \in \mathbb{N}$, $0 \le \beta \le r$, $x \pm rt/2 \in I$ *and* $0 \le t \le 1/8r$, *then we have the following inequality:*

$$\int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \delta_n^{-\beta}\left(x + \sum_{j=1}^r u_j\right) du_1 \dots du_r \le C(\beta) t^r \delta_n^{-\beta}(x). \tag{2.5}$$

Proof. The result follows from [7, (4.11)]. □

Lemma 2.4. *For* $x, t, u \in (0, +\infty)$, $x < u < t$, $t, r \in \mathbb{N}$ *and* $\lambda \in [0, 1]$, *then*

$$B_n\left(\left|\int_x^t \left|t-u\right|^{r-1} \phi^{-r\lambda}(u) du\right|, x\right) \le C n^{-r/2} \delta_n^r(x) \phi^{-r\lambda}(x). \tag{2.6}$$

Proof. When $r = 1$, then we have

$$\left|\int_x^t \phi^{-\lambda}(u) du\right| \le \left|t - x\right| \left\{x^{-\lambda/2}(1+t)^{-\lambda/2} + (1+x)^{-\lambda/2} t^{-\lambda/2}\right\}. \tag{2.7}$$

From [3, (9.5.3)],

$$B_n\left((t-x)^{2r}, x\right) \le C n^{-r} \phi^{2r}(x). \tag{2.8}$$

Applying the Hölder inequality, we get

$$(2.9)\qquad B_n\left(\left|\int_x^t \phi^{-\lambda}(u)du\right|, x\right) \le \left\{B_n\big((t-x)^4, x\big)\right\}^{\frac14}\left[x^{-\lambda/2}\left\{B_n\big((1+t)^{-2\lambda/3}, x\big)\right\}^{\frac34}\right.$$
$$\left. +(1+x)^{-\lambda/2}\left\{B_n(t^{-2\lambda/3}, x)\right\}^{\frac34}\right]$$
$$\le Cn^{-1/2}\delta_n(x)\left[x^{-\lambda/2}\left\{B_n\big((1+t)^{-2\lambda/3}, x\big)\right\}^{\frac34}\right.$$
$$\left. +\,(1+x)^{-\lambda/2}\left\{B_n(t^{-2\lambda/3}, x)\right\}^{\frac34}\right].$$

Applying the Hölder inequality, we get

$$(2.10)\qquad B_n(t^{-2\lambda/3}, x) \le \left(\sum_{k=0}^{+\infty} p_{n,k}(x)\left(\frac{k}{n}\right)^{-1}\right)^{\frac{2}{3\lambda}} \le Cx^{-2\lambda/3}.$$

Similarly,

$$(2.11)\qquad B_n((1+t)^{-2\lambda/3}, x) \le C(1+x)^{-2\lambda/3}.$$

Combining (2.9) to (2.11), we obtain (2.6).

When $r > 1$, then we have

$$\frac{|t-u|^2}{\phi^2(u)} \le \frac{|t-x|^2}{\phi^2(x)} \quad (\text{Trivial for } \ t<u<x),$$

otherwise

$$|t-u|x \le |t-x|u$$

and

$$\frac{u|t-u|}{\phi^2(u)} \le \frac{x|t-x|}{\phi^2(x)} \quad (for\ \ t<u<x).$$

Thus

$$(2.12)\qquad \frac{|t-u|^{r-2}}{\phi^{(r-2)\lambda}(u)} \le \frac{|t-x|^{r-2}}{\phi^{(r-2)\lambda}(x)}, \qquad r>2$$

because

$$(2.13)\qquad \frac{|t-u|}{\phi^2(u)} \le \frac{|t-x|}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right) \quad (\text{Trivial for } \ t<u<x),$$

otherwise

$$\frac{(u-t)}{u} \le \frac{(x-t)}{t}$$

and

$$\frac{1}{1+u} \le \frac{1}{1+x}+\frac{1}{1+t},$$

$$(2.14)\qquad \frac{|t-u|}{\phi^{2\lambda}(u)} \le \frac{|t-x|}{x^\lambda}\left(\frac{1}{1+x}+\frac{1}{1+t}\right)^\lambda.$$

Because the function t^λ $(0\le\lambda\le1)$ is subadditive, using (2.12) and (2.14), we obtain

$$(2.15)\qquad \frac{|t-u|^{r-1}}{\phi^{r\lambda}(u)} \le \frac{|t-x|^{r-1}}{\phi^{(r-2)\lambda}(x)x^\lambda}\left\{\left(\frac{1}{1+x}\right)^\lambda+\left(\frac{1}{1+t}\right)^\lambda\right\}.$$

Since $B_n\big((1+t)^{-r},x)\big) \le C(1+x)^{-r}$ $(r \in \mathbb{N},\ x \in [0,+\infty))$, using the Hölder inequality, we have

$$B_n\big((1+t)^{-2\lambda},x)\big) \le C(1+x)^{-2\lambda} \quad \big(x \in [0,+\infty)\,,\lambda \in [0,1]\big). \tag{2.16}$$

Using (2.8), (2.15), (2.16) and the Hölder inequality, we get

$$\begin{aligned} B_n\left(\left|\int_x^t \left|t-u\right|^{r-1}\phi^{-r\lambda}(u)du\right|,x\right) &\le B_n^{1/2}\big(\left|t-u\right|^{2r},x\big) \\ &\quad .\Big(\phi^{-r\lambda}(x) + x^{-\lambda}\phi^{-(r-2)\lambda}(x)B_n^{1/2}\big((1+t)^{-2\lambda},x\big)\Big) \\ &\le Cn^{-r/2}\delta_n^r(x)\phi^{-r\lambda}(x). \end{aligned}$$

□

Lemma 2.5. *If* $r \in \mathbb{N}$ *and* $0 < \alpha < r$ *then*

$$\left\|B_n f\right\|_r \le C n^{r/2}\left\|f\right\|_0 \quad (f \in C_\lambda^0), \tag{2.17}$$

$$\left\|B_n f\right\|_r \le C\left\|f\right\|_r \qquad (f \in C_\lambda^r). \tag{2.18}$$

Proof. For $x \in [0,\frac{1}{n}]$ and $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, according to Lemma 2.2, we get

$$\left|\delta_n^{r+\alpha(\lambda-1)}(x)B_n^{(r)}(f,x)\right| \le Cn^{r/2}\left\|f\right\|_0.$$

On the other hand, for $x \in \left(\frac{1}{n},+\infty\right)$ and $\delta_n(x) \sim \phi(x)$, according to [3], we can obtain

$$B_n^{(r)}(f,x) = \phi^{-2r}\sum_{i=0}^{r} V_i\big(n\phi^2(x)\big)n^i\sum_{k=0}^{+\infty} p_{n,k}(x)\left(\frac{k}{n}-x\right)^i \triangle_{1/n}^r f\left(\frac{k}{n}\right), \tag{2.19}$$

where $V_i\big(n\phi^2(x)\big)$ is polynomial in $n\phi^2(x)$ of degree $(r-i)/2$ with constant coefficients and therefore,

$$\left|\phi^{-2r}V_i(nx)n^i\right| \le C\left(\frac{n}{\phi^2(x)}\right)^{(r+i)/2}, \ \text{ for any } \ x \in \left(\frac{1}{n},+\infty\right). \tag{2.20}$$

Since

$$\left|\triangle_{1/n}^r f\Big(\frac{k}{n}\Big)\right| \le C\left\|f\right\|_0\phi^{\alpha(\lambda-1)}(x), \tag{2.21}$$

using the Hölder inequality, we get

$$\left|\sum_{k=0}^{+\infty} p_{n,k}(x)\left(\frac{k}{n}-x\right)^i \triangle_{1/n}^r f\left(\frac{k}{n}\right)\right| \le C\left(\frac{\phi^2(x)}{n}\right)^{i/2}\left\|f\right\|_0\phi^{\alpha(\lambda-1)}(x). \tag{2.22}$$

From (2.19) to (2.22) we can deduce (2.17) easily. Similarly, like Lemma 2.1 we can obtain (2.18). □

3. DIRECT RESULTS

Theorem 3.1. *Let* $f \in C[0,+\infty)$, $r \in \mathbb{N}$ *and* $0 \le \lambda \le 1$, *then*

$$\left|B_{n,r}(f,x) - f(x)\right| \le C\omega^r_{\phi^\lambda}\left(f, n^{-1/2}\delta_n^{1-\lambda}(x)\right). \tag{3.1}$$

Proof. Using (1.5) and (1.6) and taking $d_n = d_n(x,\lambda) = n^{-1/2}\delta_n^{1-\lambda}(x)$, we can choose $g_n = g_{n,x,\lambda}$ for fixed x and λ satisfying:

$$\left\|f - g_n\right\| \le C\omega^r_{\phi^\lambda}(f, d_n), \tag{3.2}$$

$$d_n^r\left\|\phi^{r\lambda}g_n^{(r)}\right\| \le C\omega^r_{\phi^\lambda}(f, d_n), \tag{3.3}$$

$$d_n^{r/(1-(\lambda/2))}\left\|g_n^{(r)}\right\| \le C\omega^r_{\phi^\lambda}(f, d_n). \tag{3.4}$$

Now

$$\begin{aligned}\left|B_{n,r}(f,x) - f(x)\right| &\le \left|B_{n,r}(f,x) - B_{n,r}(g_n,x)\right| + \left|f - g_n(x)\right| \\ &\quad + \left|B_{n,r}(g_n,x) - g_n(x)\right| \\ &\le C\left\|f - g_n\right\|_\infty + \left|B_{n,r}(g_n,x) - g_n(x)\right|.\end{aligned} \tag{3.5}$$

By using Taylor's formula:

$$g_n(t) = g_n(x) + (t-x)g_n'(x) + \cdots + \frac{(t-x)^{r-1}}{(r-1)!}g_n^{(r-1)}(x) + R_r(g_n,t,x),$$

where

$$R_r(g_n,t,x) = \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1}g_n^{(r)}(u)du.$$

Using (i)-(iv) of (1.2) and Lemma 2.4, we obtain

$$\begin{aligned}\left|B_{n,r}(g_n,x) - g_n(x)\right| &= \left|B_{n,r}\left(\frac{1}{(r-1)!}\int_x^t (t-u)^{r-1}g_n^{(r)}(u)du, x\right)\right| \\ &\le C\left\|\phi^{r\lambda}g_n^{(r)}\right\|\left(B_{n,r}\left(\left|\int_x^t \frac{|t-u|^{r-1}}{\phi^{r\lambda}(u)}du\right|, x\right)\right) \\ &\le C\phi^{-r\lambda}(x)n^{-r/2}\delta_n^r(x)\left\|\phi^{r\lambda}g_n^{(r)}\right\|.\end{aligned} \tag{3.6}$$

Again using (i)-(iv) of (1.2) and (2.12), we get

$$\begin{aligned}\left|B_{n,r}(g_n,x) - g_n(x)\right| &\le C\left\|\delta_n^{r\lambda}g_n^{(r)}\right\|\left|B_{n,r}\left(\int_x^t \frac{|t-u|^{r-1}}{\delta_n^{r\lambda}(u)}du, x\right)\right| \\ &\le C\left\|\delta_n^{r\lambda}g_n^{(r)}\right\|n^{r\lambda/2}\left|B_{n,r}\left((t-x)^{2r}, x\right)\right|^{1/2} \\ &\le Cn^{-r/2}\delta_n^r(x)n^{r\lambda/2}\left\|\delta_n^{r\lambda}g_n^{(r)}\right\|.\end{aligned} \tag{3.7}$$

We will take the following two cases:

Case-I: For $x \in [0, 1/n],\ \delta_n(x) \sim \frac{1}{\sqrt{n}}$. Then by (3.2) – (3.5) and (3.7), we have

$$\begin{aligned}
\left|B_{n,r}(f,x) - f(x)\right| &\le C\big(\left\|f - g_n\right\| + d_n^r\left\|\delta_n^{r\lambda} g_n^{(r)}\right\|\big) \\
&\le C\big(\left\|f - g_n\right\| + d_n^r\left\|\phi^{r\lambda} g_n^{(r)}\right\| + d_n^r n^{-r\lambda/2}\left\|g_n^{(r)}\right\|\big) \quad (3.8)\\
&\le C\big(\left\|f - g_n\right\| + d_n^r\left\|\phi^{r\lambda} g_n^{(r)}\right\| + d_n^{r/(1-(\lambda/2))}\left\|g_n^{(r)}\right\|\big) \quad (3.9)\\
&\le C\omega_{\phi^\lambda}^r\left(f, d_n\right).
\end{aligned}$$

Case-II: For $x \in (1/n, +\infty),\ \delta_n(x) \sim \phi(x)$. Then by (3.2) – (3.3) and (3.5) – (3.6), we get

$$\left|B_{n,r}(f,x) - f(x)\right| \le C\big(\left\|f - g_n\right\| + d_n^r\left\|\phi^{r\lambda} g_n^{(r)}\right\| \le C\omega_{\phi^\lambda}^r\left(f, d_n\right). \quad (3.10)$$

□

4. INVERSE RESULTS

Theorem 4.1. *Let* $f \in C\left[0, +\infty\right),\ r \in \mathbb{N},\ 0 < \alpha < r,\ 0 \le \lambda \le 1$. *Then*

$$\left|B_{n,r}(f,x) - f(x)\right| = O(d_n^\alpha). \quad (4.1)$$

implies

$$\omega_{\phi^\lambda}^r(f,t) = O(t^\alpha), \quad (4.2)$$

where $d_n = n^{-1/2}\delta_n^{1-\lambda}(x)$.

Proof. Since $B_n(f,x)$ preserves the constant, hence we may assume $f \in C_0$. Suppose that (4.1) holds. Now we introduce a new K-functional as

$$K_\lambda^\alpha(f, t^r) = \inf_{g \in C_\lambda^r}\left\{\left\|f - g\right\|_0 + t^r\left\|g\right\|_r\right\}.$$

Choosing $g \in C_\lambda^r$ such that

$$\left\|f - g\right\|_0 + n^{-r/2}\left\|g\right\|_r \le 2K_\lambda^\alpha(f, n^{-r/2}). \quad (4.3)$$

By (4.1), we can deduce that

$$\|B_{n,r}(f,x) - f(x)\|_0 \le Cn^{-\alpha/2}.$$

Thus, by using Lemma 2.5 and (4.3), we obtain

$$\begin{aligned}
K_\lambda^\alpha(f, t^r) &\le \left\|f - B_{n,r}(f)\right\|_0 + t^r\left\|B_{n,r}(f)\right\|_r \\
&\le Cn^{-\alpha/2} + t^r\big(\left\|B_{n,r}(f - g)\right\|_r + \left\|B_{n,r}(g)\right\|_r\big) \\
&\le C\Big(n^{-\alpha/2} + t^r\big(n^{r/2}\left\|f - g\right\|_0 + \left\|g\right\|_r\big)\Big) \\
&\le C\Big(n^{-\alpha/2} + \frac{t^r}{n^{-r/2}}K_\lambda^\alpha\big(f, n^{-r/2}\big)\Big)
\end{aligned}$$

which implies that by [3, 7]

$$K_\lambda^\alpha\big(f, t^r\big) \le Ct^\alpha. \quad (4.4)$$

On the other hand, since $x + (j - \frac{r}{2})t\phi^{\lambda}(x) \geq 0$, therefore

$$\left|\left(j - \frac{r}{2}\right)t\phi^{\lambda}(x)\right| \leq x$$

and

$$x + \left(j - \frac{r}{2}\right)t\phi^{\lambda}(x) \leq 2x,$$

so that

$$\delta_n\left(x + \left(j - \frac{r}{2}\right)t\phi^{\lambda}(x)\right) \leq 2\delta_n(x). \tag{4.5}$$

Thus, for $f \in C_{\lambda}^{0}$, we get

$$\begin{aligned} \left|\triangle_{t\phi^{\lambda}(x)}^{r} f(x)\right| &\leq \left\|f\right\|_0 \left(\sum_{j=0}^{r} \binom{r}{j} \delta_n^{\alpha(1-\lambda)}\left(x + \left(j - \frac{r}{2}\right)t\phi^{\lambda}(x)\right)\right) \\ &\leq 2^{2r}\delta_n^{\alpha(1-\lambda)}(x)\left\|f\right\|_0. \end{aligned} \tag{4.6}$$

From Lemma 2.3, for $g \in C_{\lambda}^{r},\ 0 < t\phi^{\lambda}(x) < 1/8r,\ x \pm rt\phi^{\lambda}(x)/2 \in [0, +\infty)$, we have

$$\begin{aligned} \left|\triangle_{t\phi^{\lambda}(x)}^{r} g(x)\right| &\leq \left|\int_{-(t/2)\phi^{\lambda}(x)}^{(t/2)\phi^{\lambda}(x)} \cdots \int_{-(t/2)\phi^{\lambda}(x)}^{(t/2)\phi^{\lambda}(x)} g^{(r)}\left(x + \sum_{j=1}^{r} u_j\right) du_1 \dots du_r\right| \\ &\leq \left\|g\right\|_r \int_{-(t/2)\phi^{\lambda}(x)}^{(t/2)\phi^{\lambda}(x)} \cdots \int_{-(t/2)\phi^{\lambda}(x)}^{(t/2)\phi^{\lambda}(x)} \delta_n^{-r+\alpha(1-\lambda)}\left(x + \sum_{j=1}^{r} u_j\right) du_1 \dots du_r \\ &\leq Ct^r\delta_n^{-r+\alpha(1-\lambda)}(x)\left\|g\right\|_r. \end{aligned} \tag{4.7}$$

Using (4.4), (4.6) and (4.7), for $0 < t\phi^{\lambda}(x) < 1/8r,\ x \pm rt\phi^{\lambda}(x)/2 \in [0, +\infty)$ and choosing appropriate g, we get

$$\begin{aligned} \left|\triangle_{t\phi^{\lambda}(x)}^{r} f(x)\right| &\leq \left|\triangle_{t\phi^{\lambda}(x)}^{r}(f-g)(x)\right| + \left|\triangle_{t\phi^{\lambda}(x)}^{r} g(x)\right| \\ &\leq C\delta_n^{\alpha(1-\lambda)}(x)\left\{\left\|f-g\right\|_0 + t^r\delta_n^{r(\lambda-1)}(x)\left\|g\right\|_r\right\} \\ &\leq C\delta_n^{\alpha(1-\lambda)}(x)K_{\lambda}^{\alpha}\left(f, \frac{t^r}{\delta_n^{r(1-\lambda)}(x)}\right) \\ &\leq Ct^r. \end{aligned}$$

□

Remark 4.2. Very recently Gupta and Deo [5] have studied two dimensional modified Lupaş operators. In the same manner we can obtain an equivalent theorem with $\omega_{\phi^{\lambda}}^{r}(f,t)$.

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