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GENERALIZED CO-COMPLEMENTARITY PROBLEMS IN p-UNIFORMLY SMOOTH BANACH SPACES

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ABSTRACT. The objective of this paper is to study the iterative solutions of a class of generalized co-complementarity problems in p-uniformly smooth Banach spaces, with the devotion of sunny retraction mapping, p-strongly accretive, p-relaxed accretive and Lipschitzian (or more generally uniformly continuous) mappings. Our results are new and represents a significant improvement of previously known results. Some special cases are also discussed.

Key words and phrases: Generalized co-complementarity problems, Iterative algorithm, Sunny retraction, Sunny nonexpansive mapping, p-strongly accretive, p-relaxed accretive mapping, Lipschitzian mapping, Hausdorff metric, p-uniformly smooth Banach spaces.

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1. INTRODUCTION

The theory of complementarity problems initiated by Lemke [19] and Cottle and Dantzing [10] in the early sixties and later developed by other mathematicians see for example [6, 9, 11, 14, 17, 22] plays an important role and is fundamental in the study of a wide class of problems arising in optimization, game theory, economics and engineering sciences [3, 6, 8, 11, 15].

On the other hand, the accretive operators are of interest because several physically resolvent problems can be modeled by nonlinear evolution systems involving operators of the accretive type. Very closely related to the accretive operators is the class of dissipative operators, where an operator T is said to be dissipative if and only if $(-T)$ is accretive. The concepts of strictly strongly and m-(or sometimes hyper-) dissipativity are similarly defined.

These classes of operators have attracted a lot of interest because of their involvement in evolution systems modeling several real life problems. Consequently several authors have studied the existence, uniqueness and iterative approximations of solutions of nonlinear equations involving such operators, see [5, 12, 18] and the references cited therein.

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It is our purpose in this article to establish the strong convergence of the iterative algorithm to a solution of the generalized co-complimentarity problems in $p-$uniformly smooth Banach spaces when the operators are accretive, strictly accretive, strongly accretive, relaxed accretive and Lipschitzian. Our iteration processes are simple and independent of the geometry of E and iteration parameters can be chosen at the start of the iteration process. Consequently, most important results known in this connection will be special cases of our problem.

2. Background of Problem Formulations

Throughout this article, we assume that E is a real Banach space whose norm is denoted by $\|\cdot\|$, $E^\star$ its topological dual space. $CB(E)$ denotes the family of all nonempty closed and bounded subsets of E. $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$\max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(A,y)\right\} = D(A,B),$$

where

$$d(x,B) = \inf_{y\in B} d(x,y) \quad \text{and} \quad d(A,y) = \inf_{x\in A} d(x,y),$$

d is the metric on E induced by the norm $\|\cdot\|$. As usual, $\langle\cdot,\cdot\rangle$ is the generalized duality pairing between E and $E^\star$. For $1<p<\infty$, the mapping $J_p : E \to 2^{E^\star}$ defined by

$$J_p(x) = \{f^\star \in E^\star : \langle x, f^\star\rangle = \|f\|\cdot\|x\| \text{ and } \|f\| = \|x\|^{p-1}\} \quad \text{for all } x\in E,$$

is called the duality mapping with gauge function $\phi(t) = t^{p-1}$. In particular for $p = 2$, the duality mapping J_2 with gauge function $\phi(t) = t$ is called the normalized duality mapping. It is known that $J_p(x) = \|x\|^{p-2}J_2(x)$ for all $x \neq 0$ and J_p is single valued if $E^\star$ is strictly convex. If $E = H$ is a Hilbert space, then J_2 becomes the identity mapping on H.

Proposition 2.1 ([7]). *Let E be a real Banach space. For $1 < p < \infty$, the duality mapping $J_p : E \to 2^{E^\star}$ has the following basic properties:*

(1) $J_p(x) \neq \emptyset$ *for all* $x \in E$ *and* $D(J_p)$ *(the domain of* J_p*)* $= E$*,*
(2) $J_p(x) = \|x\|^{p-2}J_2(x)$ *for all* $x \in E$*,* $(x \neq 0)$*,*
(3) $J_p(\alpha x) = \alpha^{p-1}J_p(x)$ *for all* $\alpha \in [0,\infty)$*,*
(4) $J_p(-x) = -J_p(x)$*,*
(5) J_p *is bounded i.e., for any bounded subset* $A \subset E$*,* $J_p(A)$ *is a bounded subset in* $E^\star$*,*
(6) J_p *can be equivalently defined as the subdifferential of the functional* $\varphi(x) = p^{-1}\|x\|^p$ *(Asplund* [2]*), i.e.,*

$$J_p(x) = \partial\varphi(x) = \{f \in E^\star :\ \varphi(y) - \varphi(x) \geq (f, y-x),\ \textit{for all } y \in E\}$$

(7) E *is a uniformly smooth Banach space (equivalently,* $E^\star$ *is a uniformly convex Banach space) if and only if* J_p *is single valued and uniformly continuous on any bounded subset of* E *(Xu and Roach* [24]*).*

Definition 2.1. Let E be a real Banach space and K a nonempty subset of E. Let $T : K \to 2^E$ be a multivalued mapping

(1) T is said to be accretive if for any $x, y \in K$, $u \in T(x)$ and $v \in T(y)$ there exists $j_2 \in J_2(x-y)$ such that

$$\langle u - v,\ j_2\rangle \geq 0,$$

or equivalently, there exists $j_p \in J_p(x-y)$, $1 < p < \infty$, such that

(2) T is said to be strongly accretive if for any $x, y \in K$, $u \in T(x)$ and $v \in T(y)$ there exists $j_2 \in J_2(x-y)$ such that

$$\langle u - v,\ j_2\rangle \geq k\|x-y\|^2,$$

or equivalently, there exists $j_p \in J_p(x-y)$, $1 < p < \infty$ such that

$$\langle u - v,\ j_p\rangle \geq k\|x-y\|^p,$$

for some constant $k > 0$.

The concept of a single-valued accretive mapping was introduced independently by Browder [5] and Kato [18] in 1967. An early fundamental result in the theory of accretive mappings which is due to Browder states that the following initial value problem,

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0,$$

is solvable if T is locally Lipschitzian and accretive on E.

More precisely, let $N : E \times E \to E$ and $m, g : E \to E$ be the single-valued mappings and $F, G, T : E \to CB(E)$ the multivalued mappings. Let X be a fixed closed convex cone of E. Define $K : E \to 2^E$ by

$$K(z) = m(z) + X \quad \text{for all } x \in E,\ z \in T(x).$$

We shall study the following generalized co-complementarity problem (**GCCP**):

Find $x \in E$, $u \in F(x)$, $v \in G(x)$, $z \in T(x)$ such that $g(x) \in K(z)$ and

$$N(u, v) \in (J(K(z) - g(x)))^\star, \tag{2.1}$$

where $(J(K(z) - g(x)))^\star$ is the dual cone of the set $J(K(z) - g(x))$.

2.1. **Special Cases.**

(i) If E is a Hilbert space, F, T are identity mappings and $N(u, v) = Bx + Av$, where B, A are single-valued mappings, then Problem (2.1) reduces to a problem of finding $x \in E$, $v \in G(x)$ such that $g(x) \in K(z)$ and

$$B(x) + A(v) \in (K(x) - g(x))^\star \tag{2.2}$$

considered by Jou and Yao [16].

(ii) If G and g are identity mappings, then (2.2) reduces to finding $x \in K(x)$ such that

$$B(x) + A(y) \in (K(x) - x)^\star \tag{2.3}$$

which is called a strongly nonlinear quasi complementarity problem, studied by Noor [22].

(iii) If m is a zero mapping, then (2.3) is equivalent to finding $x \in E$ such that

$$Bx + Ax \in E^\star \quad \text{and} \quad \langle Bx + Ax,\ x\rangle = 0, \tag{2.4}$$

which is known as the mildly nonlinear complementarity problem, studied by Noor [21].

(iv) If A is zero mapping, then (2.4) is equivalent to a problem of finding $x \in E$ such that $Bx \in E^\star$ and

$$\langle Bx,\ x\rangle = 0, \tag{2.5}$$

considered by Habetler [14] and Karamardian [17].

3. THE CHARACTERIZATION OF PROBLEM AND SOLUTIONS

In this section, we briefly consider some basic concepts and results, which will be used throughout the paper. The real Banach space E is said to be uniformly smooth if its modulus of smoothness $\rho_E(\tau)$ defined by

$$\rho_E(\tau) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \ \|x\| = 1, \|y\| \leq \tau \right\}$$

satisfies $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. It follows that E is uniformly smooth if and only if J_p is single-valued and uniformly continuous on any bounded subset of E and there exists a complete duality between uniform convexity and uniform smoothness. E is uniformly convex (smooth) if and only if $E^\star$ is uniformly smooth (convex). Recall that E is said to have the modulus of smoothness of power type $p > 1$ (and E is said to be p-uniformly smooth) if there exists a constant $c > 0$ such that

$$\rho_E(\tau) \leq c\tau^p \quad \text{for } 0 < \tau < \infty.$$

Remark 3.1. It is known that all Hilbert spaces and Banach spaces, L_p, l_p and W^p_m $(1 < p < \infty)$ are uniformly smooth and

$$\rho_E(\tau) < \begin{cases} \dfrac{1}{p}\,\tau^p, & 1 < p < 2 \\ \\ \dfrac{p-1}{2}\,\tau^2, & p \leq 2 \end{cases} \qquad (E = L_p, l_p \text{ or } W^p_m)$$

therefore E is a p-uniformly smooth Banach space with modulus of smoothness of power type $p < 1$ and J_p will always represent the single-valued duality mapping.

Definition 3.1 ([4, 13])**.** Let E be a p-uniformly smooth Banach space and let Ω a nonempty closed convex subset of E. A mapping $Q_\Omega : E \to \Omega$ is said to be

(i) retraction on Ω if $Q^p_\Omega = Q_\Omega$;

(ii) nonexpansive retraction if it satisfies the inequality

$$\|Q_\Omega(x) - Q_\Omega(y)\| \leq \|x - y\|, \quad \text{for all } x, y \in E;$$

(iii) sunny retraction if for all $x \in E$ and for all $-\infty < t < \infty$

$$Q_\Omega(Q_\Omega(x) + t(x - Q_\Omega(x))) = Q_\Omega(x).$$

The following characterization of a sunny nonexpansive retraction mapping can be found in [4, 13].

Lemma 3.2 ([4, 13])**.** *Q_Ω is sunny nonexpansive retraction if and only if for all $x, y \in E$,*

$$\langle x - Q^\star_\Omega,\ J(Q_\Omega\, x - y)\rangle \geq 0.$$

Lemma 3.3. *Let E be a real Banach space and $J_p : E \to 2^{E^\star}$, $1 < p < \infty$ a duality mapping. Then, for any given $x, y \in E$, we have*

$$\|x+y\|^p \leq \|x\|^p + p\,\langle y, j_p\rangle, \quad \textit{for all } j_p \in J_p(x+y).$$

Proof. From Proposition 2.1, it follows that $J_p(x) = \partial\varphi(x)$ (subdifferential of ψ), where $\psi(x) = p^{-1}\|x\|^p$. Also, it follows from the definition of the subdifferential of ψ that

$$\psi(x) - \psi(x+y) \geq \langle x - (x+y),\ j_p\rangle,$$

for all $j_p \in J_p(x+y)$. Substituting $\psi(x)$ by $p^{-1}\|x\|^p$, we have

$$\|x+y\|^p \leq \|x\|^p + p\,\langle y, j_p\rangle, \quad \text{for all } j_p \in J_p(x+y).$$

This completes the proof. □

Theorem 3.4 ([9]). *Let E be a Banach space, Ω a nonempty closed convex subset of E, and $m : E \to E$. Then for all $x, y \in E$, we have*

$$Q_{\Omega+m(z)}(x) = m(z) + Q_\Omega(x - m(z)).$$

We mention the following characterization theorem for the solution of a generalized co-complementarity problem which can be easily proved by using Lemma 3.2 and the argument of [1, Theorem 8.1].

Theorem 3.5. *Let E be a real p-uniformly smooth Banach space and X a closed convex cone in E. Let $F, G, T : E \to CB(E)$ be the multivalued mappings, $m, g : E \to E$ the two single-valued mappings and $N : E \times E \to E$ the nonlinear mapping. Let $K : E \to 2^E$ and $K(z) = m(z) + X$ for $x \in E$. Then the following statements are equivalent:*

(i) $x \in E$, $u \in F(x)$, $v \in G(x)$ *and* $z \in T(x)$ *are solutions of the Problem (2.1), i.e.,* $g(x) \in K(z)$ *and*

$$N(u, v) \in (J(K(z) - g(x)))^\star.$$

(ii) $x \in E$, $u \in F(x)$, $v \in G(x)$ *and* $z \in T(x)$ *and* $\tau > 0$

$$g(x) = Q_{K(z)}[g(x) - \tau N(u, v)].$$

Combining Theorem 3.4 and 3.5, we have the following result.

Theorem 3.6. *Let E be a p-uniformly smooth Banach space and X a closed convex cone in B. Let $m, g : E \to E$ be the two single-valued mappings, $F, G, T : E \to CB(E)$ the multivalued mappings and $N : E \times E \to E$ a nonlinear mapping. Then the following statements are equivalent:*

(i) $x \in E$, $u \in F(x)$, $v \in G(x)$ *and* $z \in T(x)$ *are solutions of the Problem (2.1),*

(ii) $x = x - g(x) + m(z) + Q_X[g(x) - \tau N(u, v) - m(z)]$, *for some* $\tau > 0$.

The following inequality will be used in our main results.

Lemma 3.7. *Let E be a real Banach space and $j_p : E \to 2^{E^\star}$, $1 < p < \infty$ a duality mapping. Then, for any given $x, y \in E$, we have*

$$\langle x - y,\, j_p(x) - j_p(y)\rangle \le 2d^p \rho_E\left(\frac{4\|x - y\|}{d}\right),$$

where

$$d^p = \left(\frac{\|x\|^2 + \|y\|^2}{2}\right).$$

Proof. The proof of the above inequalities are the generalized form of the proof of Theorem 3.4, and hence will be omitted. □

4. ITERATIVE ALGORITHMS AND PERTINENT CONCEPTS

We now propose the following iterative algorithm for computing the approximate solution of (**GCCP**).

Algorithm 4.1. Let $g, m : E \to E$ be the two single-valued mappings, $F, G, T : E \to CB(E)$ the multivalued mappings and $N : E \times E \to E$ a nonlinear mapping.

For any given $x_0 \in E$, $u_0 \in F(x_0)$, $v_0 \in G(x_0)$ and $z_0 \in T(x_0)$, let

$$x_1 = x_0 - g(x_0) + m(z_0) + Q_X[g(x_0) - \tau N(u_0, v_0) - m(z_0)]$$

where $\tau > 0$ is a constant.

Since $u_0 \in F(x_0) \in CB(E)$, $v_0 \in G(x_0) \in CB(E)$ and $z_0 \in T(x_0) \in CB(E)$, by Nadler's Theorem [20], there exists $u_1 \in F(x_1)$, $v_1 \in G(x_1)$ and $z_1 \in T(x_1)$ such that

$$\|u_0 - u_1\| \leq (1+1)D(F(x_0),\, F(x_1)),$$
$$\|v_0 - v_1\| \leq (1+1)D(G(x_0),\, G(x_1)),$$
$$\|z_0 - z_1\| \leq (1+1)D(T(x_0),\, T(x_1)),$$

where D is a Hausdorff metric on $CB(E)$.

Let

$$x_2 = x_1 - g(x_1) + m(z_1) + Q_X[g(x_1) - \tau N(u_1, v_1) - m(z_1)].$$

Since $u_1 \in F(x_1) \in CB(E)$, $v_1 \in G(x_1) \in CB(E)$ and $z_1 \in T(x_1) \in CB(E)$, there exists $u_2 \in F(x_2)$, $v_2 \in G(x_2)$ and $z_2 \in T(x_2)$ such that

$$\|u_1 - u_2\| \leq (1+2^{-1})D(F(x_1),\, F(x_2)),$$
$$\|v_1 - v_2\| \leq (1+2^{-1})D(G(x_1),\, G(x_2)),$$
$$\|z_1 - z_2\| \leq (1+2^{-1})D(T(x_1),\, T(x_2)).$$

By induction, we can obtain $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ as

$$x_{n+1} = x_n - g(x_n) + m(z_n) + Q_X[g(x_n) - \tau N(u_n, v_n) - m(z_n)]. \tag{4.1}$$

$$u_n \in F(x_n); \|u_n - u_{n+1}\| \leq \left(1 + (1+n)^{-1}\right) D(F(x_n),\, F(x_{n+1})),$$
$$v_n \in G(x_n); \|v_n - v_{n+1}\| \leq \left(1 + (1+n)^{-1}\right) D(G(x_n),\, G(x_{n+1})),$$
$$z_n \in T(x_n); \|z_n - z_{n+1}\| \leq \left(1 + (1+n)^{-1}\right) D(T(x_n),\, T(x_{n+1})),$$

$n \geq 0$, where $\tau > 0$ is a constant.

These iteration processes have been extensively investigated by various authors for approximating either the fixed point of nonlinear mappings or solutions of nonlinear equations in Banach spaces or variational inequalities, variational inclusions, or complementarity problems in Hilbert spaces.

Definition 4.1. A single valued mapping $g : E \to E$ is said to be

(i) p-strongly accretive if for all $x, y \in E$ there exists $j_p \in J_p(x-y)$ such that
$$\langle g(x) - g(y),\, j_p(x-y)\rangle \geq \kappa \|x-y\|^p$$
for some real constant $\kappa \in (0,1)$ and $1 < p < \infty$.

(ii) Lipschitz continuous if for any $x, y \in E$, there exists constant $\beta > 0$, such that
$$\|g(x) - g(y)\| \leq \beta \|x - y\|.$$

Definition 4.2. A multivalued mapping $F : E \to CB(E)$ is said to be D-Lipschitz continuous if for any $x, y \in E$,

$$D(F(x),\, F(y)) \leq \mu \|x - y\|$$

for $\mu > 0$ and $D(\cdot,\cdot)$ is Hausdorff metric defined on $CB(E)$.

Definition 4.3. Let $F : E \to CB(E)$ be a multivalued mapping. A nonlinear mapping $N : E \times E \to E$ is said to be relaxed accretive with respect to the first argument of map F, if there exists a constant $\alpha > 0$ such that

$$\langle N(u_n, \cdot) - N(u_{n-1}, \cdot),\, j_p(x_n - x_{n-1})\rangle \geq -\alpha \|x_n - x_{n-1}\|^p;$$

and N is Lipschitz continuous with respect to the first argument if

$$\|N(u,\cdot) - N(y,\cdot)\| \leq \sigma \|x - y\|, \quad \text{for } x, y \in E,$$

where $\sigma > 0$ is a constant.

Similarly, we define the Lipschitz continuity of N with respect to second argument.

5. Main Results

In this section, we show that if E is a p-uniformly smooth Banach space, then the iterative process converges strongly to the given problem (2.1).

Theorem 5.1. *Let E be a p-uniformly smooth real Banach space with $\rho_E(\tau) \le c\tau^p$ for some $c > 0$, $0 < \tau < \infty$ and $1 < p < \infty$. Let X be a closed convex cone of E. Let $m, g : E \to E$ be the two single-valued mappings, $F, G, T : E \to CB(E)$ the multivalued mappings. Let $K : E \to 2^E$ such that $K(z) = m(z) + X$ for all $x \in E$, $z \in T(x)$ and the following conditions hold.*

(i) *g and m are Lipschitz continuous;*
(ii) *g is strongly accretive;*
(iii) *F, G and T are D-Lipschitz continuous;*
(iv) *N is Lipschitz continuous with respect to the first as well as the second argument;*
(v) *N is p-relaxed accretive with respect to the first argument with mapping $F : E \to CB(E)$;*
(vi)
$$q + (1 + p\alpha\tau + pc2^{2p+1}\tau^p\sigma^p\eta^p)^{1/p} + \tau\delta\xi < 1$$
and
$$q = 2(1 - p\kappa + c2^{2p+1}p\beta^p)^{1/p} + 2\mu\rho. \tag{5.1}$$
Then for any $x_0 \in E$, $u_0 \in F(x_0)$, $v_0 \in G(x_0)$ and $z_0 \in T(x_0)$ the sequences x_n, u_n, v_n and z_n generated by Algorithm 4.1, converge strongly to some $x \in E$, $u \in F(x)$, $v \in G(x)$ and $z \in T(x)$, which solve the problem (2.1).

Proof. By the iterative schemes (4.1) and Definition 3.1, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|x_n - g(x_n) + m(z_n) + Q_X[g(x_n) - \tau N(u_n, v_n) - m(z_n)] \\
&\qquad - x_{n-1} + g(x_{n-1}) - m(z_{n-1}) - Q_X[g(x_{n-1}) - \tau N(u_{n-1}, v_{n-1}) - m(z_{n-1})]\| \\
&\le \|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + \|m(z_n) - m(z_{n-1}\| \\
&\quad + \|Q_X[g(x_n) - \tau N(u_n, v_n) - m(z_n)] \\
&\qquad - Q_X[g(x_{n-1}) - \tau N(u_{n-1}, v_{n-1}) - m(z_{n-1})]\| \\
&\le 2\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + 2\|m(z_n) - m(z_{n-1})\| \\
&\quad + \|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_{n-1}))\| \\
&\le 2\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\| + 2\|m(z_n) - m(z_{n-1})\| \\
&\quad + \|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_n))\| \\
&\quad + \tau\|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\|.
\end{aligned} \tag{5.2}$$

By Lemmas 3.3, 3.7, p-strongly accretive, Lipschitz continuity of g and $j_p \in J_p(x + y)$, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^p \\
&\le \|x_n - x_{n-1}\|^p + p\langle -(g(x_n) - g(x_{n-1})),\, j_p\rangle \\
&\le \|x_n - x_{n-1}\|^p - p\langle g(x_n) - g(x_{n-1}),\, j_p(x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - x_{n-1}\|^p - p\langle g(x_n) - g(x_{n-1}),\, j_p(x_n - x_{n-1})\rangle \\
&\qquad - p\langle g(x_n) - g(x_{n-1}), j_p(x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))) - j_p(x_n - x_{n-1})\rangle \\
&\le \|x_n - x_{n-1}\|^p - p^k\|x_n - x_{n-1}\|^p + 2pd^p\rho_E\left(\frac{4\|g(x_n) - g(x_{n-1})\|}{d}\right) \\
&\le \|x_n - x_{n-1}\|^p - p^k\|x_n - x_{n-1}\|^p + \frac{2pd^p c4^p\|g(x_n) - g(x_{n-1})\|^p}{d^p} \\
&\le \|x_n - x_{n-1}\|^p - pk\|x_n - x_{n-1}\|^p + 2^{2p+1}cp\beta^p\|x_n - x_{n-1}\|^p \\
&\le (1 - pk + c2^{2p+1}p\beta^p)\|x_n - x_{n-1}\|^p. \qquad (5.3)
\end{aligned}$$

By the Lipschitz continuity of m and D-Lipschitz continuity of T, we have

$$\begin{aligned}
\|m(z_n) - m(z_{n-1})\| &\le \mu\|z_n - z_{n-1}\| \\
&\le \mu(1 + n^{-1})D(T(x_n),\, T(x_{n-1})) \\
&\le \mu(1 + n^{-1})\rho\|x_n - x_{n-1}\|. \qquad (5.4)
\end{aligned}$$

Since F is η-Lipschitz continuous, G is ξ-Lipschitz continuous and N is Lipschitz continuous with respect to the first and second arguments with positive constants σ and δ respectively. Using a similar argument to that of Xiaolin He [23], we have for every $u_n,\, u_n' \in F(x_n)$, $N(u_n, v) = N(u_n', v)$. On the other hand $u_{n-1} \in F(x_{n-1})$, and from the definition of Hausdorff metric and compactness of $F(x_n)$, there is a $u_n' \in F(x_n)$such that

$$\|u_n' - u_{n-1}\| \le D(F(x_n),\, F(x_{n-1})).$$

Hence

$$\begin{aligned}
\|N(u_n, v_n) - N(u_{n-1}, v_n)\| &= \|N(u_n', v_n) - N(u_{n-1}, v_n)\| \\
&\le \sigma\|u_n' - u_{n-1}\| \\
&\le \sigma(1 + n^{-1})D(F(x_n), F(x_{n-1})) \\
&\le \sigma(1 + n^{-1})\eta\|x_n - x_{n-1}\|. \qquad (5.5)
\end{aligned}$$

Similarly, we get

$$\|N(u_{n-1}, v_n) - N(u_{n-1}, v_{n-1})\| \le \delta\xi(1 + n^{-1})\|x_n - x_{n-1}\|. \qquad (5.6)$$

By using Lemma 3.7, the p-relaxed accretive mapping and (5.5), we have

$$\begin{aligned}
&\|x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_n))\|^p \\
&\le \|x_n - x_{n-1}\|^p \\
&\qquad - p\tau\langle N(u_n, v_n) - N(u_{n-1}, v_n),\, j_p(x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_n)))\rangle \\
&\le \|x_n - x_{n-1}\|^p - p\tau\langle N(u_n, v_n) - N(u_{n-1}, v_n),\, j_p(x_n - x_{n-1})\rangle \\
&\qquad - p\langle \tau(N(u_n, v_n) - N(u_{n-1}, v_n)), \\
&\qquad j_p(x_n - x_{n-1} - \tau(N(u_n, v_n) - N(u_{n-1}, v_n))) - j_p(x_n - x_{n-1})\rangle \\
&\le \|x_n - x_{n-1}\|^p + p\tau\alpha\|x_n - x_{n-1}\|^p + 2pd^p\rho_E\left(\frac{\tau 4\|N(u_n, v_n) - N(u_{n-1}, v_n)\|}{d}\right) \\
&\le \|x_n - x_{n-1}\|^p + p\alpha\tau\|x_n - x_{n-1}\|^p + 2p\tau^p c\, 4^p\|N(u_n, v_n) - N(u_{n-1}, v_n)\|^p \\
&\le \|x_n - x_{n-1}\|^p + p\alpha\tau\|x_n - x_{n-1}\|^p + p\tau^p c\, 2^{2p+1}(1 + n^{-1})^p\sigma^p\eta^p\|x_n - x_{n-1}\|^p \\
&\le (1 + p\alpha\tau + pc2^{2p+1}(1 + n^{-1})^p\tau^p\sigma^p\eta^p)\|x_n - x_{n-1}\|^p. \qquad (5.7)
\end{aligned}$$

Now from (5.2) – (5.7), we get

$$\begin{aligned}
\|x_n - x_{n-1}\| &\le 2(1 - pk + c2^{2p+1}p\beta^p)^{1/p}\|x_n - x_{n-1}\| + 2\mu\rho(1 + n^{-1})\|x_n - x_{n-1}\| \\
&\quad + (1 + p\alpha\tau + pc2^{2p+1}(1 + n^{-1})^p\tau^p\sigma^p\eta^p)^{1/p}\|x_n - x_{n-1}\| \\
&\quad + \tau\delta\xi(1 + n^{-1})\|x_n - x_{n-1}\| \\
&\le [2(1 - pk + c2^{2p+1}p\beta^p)^{1/p} + 2\mu\rho(1 + n^{-1}) \\
&\quad + (1 + p\alpha\tau + pc2^{2p+1}(1 + n^{-1})^p\tau^p\sigma^p\eta^p)^{1/p} + \tau\delta\xi(1 + n^{-1})] \\
&\le [q_n + (1 + p\alpha\tau + pc2^{2p+1}(1 + n^{-1})^p\tau^p\sigma^p\eta^p)^{1/p} \\
&\quad + \tau\delta\xi(1 + n^{-1})]\|x_n - x_{n-1}\| \\
&\le \theta_n\|x_n - x_{n-1}\|, \qquad (5.8)
\end{aligned}$$

where

$$\theta_n = q_n + (1 + p\alpha\tau + pc2^{2p+1}(1 + n^{-1})^p\tau^p\sigma^p\eta^p)^{1/p} + \tau\delta\xi(1 + n^{-1})$$

and

$$q_n = 2(1 - pk + c2^{2p+1}p\beta^p)^{1/p} + 2\mu\rho\left(1 + n^{-1}\right). \qquad (5.9)$$

Letting

$$\theta = q + (1 + p\alpha\tau + pc2^{2p+1}\tau^p\sigma^p\eta^p)^{1/p} + \tau\delta\xi \qquad (5.10)$$

and

$$q = 2(1 - pk + c2^{2p+1}p\beta^p)^{1/p} + 2\mu\rho.$$

We know that $\theta_n \to \theta$ as $n \to \infty$. From condition (5.1), it follows that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently large. Consequently $\{x_n\}$ is a Cauchy sequence and this converges to some $x \in E$. By Algorithm 4.1 and the D-Lipschitz continuity of F, G and T, it follows that

$$\begin{aligned}
\|u_n - u_{n-1}\| &\le (1 + n^{-1})D(F(x_n),\, F(x_{n-1})) \\
&\le (1 + n^{-1})\,\eta\|x_n - x_{n-1}\|,
\end{aligned}$$

$$\begin{aligned}
\|v_n - v_{n-1}\| &\le (1 + n^{-1})D(G(x_n),\, G(x_{n-1})) \\
&\le (1 + n^{-1})\,\xi\|x_n - x_{n-1}\|,
\end{aligned}$$

$$\begin{aligned}
\|z_n - z_{n-1}\| &\le (1 + n^{-1})D(T(x_n),\, T(x_{n-1})) \\
&\le (1 + n^{-1})\,\rho\|x_n - x_{n-1}\|,
\end{aligned}$$

which means that $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ are all Cauchy sequences in E. Therefore there exist $u \in E$, $v \in E$ and $z \in E$ such that $u_n \to u$, $v_n \to v$ and $z_n \to z$ as $n \to \infty$. Since g, m, F, G, T, N and Q_X are all continuous, we have

$$x = x - g(x) + Q_X[g(x) - \tau N(u, v) - m(z)].$$

Finally, we prove that $u \in F(x)$. In fact, since $u_n \in F(x_n)$ and

$$\begin{aligned}
d(u_n,\, F(x)) &\le \max\left\{ d(u_n, F(x)),\, \sup_{u \in F(x)} d(T(x_n),\, u) \right\} \\
&\le \max\left\{ \sup_{y \in F(x_n)} d(y,\, F(x)),\, \sup_{u \in F(x)} d(F(x_n), u) \right\} \\
&= D(F(x_n),\, F(x)),
\end{aligned}$$

We have

$$\begin{aligned} d(u,\, F(x)) &\le \|u - u_n\| + d(u_n,\, F(x)) \\ &\le \|u - u_n\| + D(F(x_n),\, F(x)) \\ &\le \|u - u_n\| + \eta\|x_n - x\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $d(u, F(x)) = 0$. Since $F(x) \in CB(E)$, it follows that $u \in F(x)$. Similarly, we can prove that $G(x) \in CB(E)$ i.e., $v \in G(x)$ and $T(x) \in CB(E)$ i.e., $z \in T(x)$. Hence by Theorem 3.6, we get the conclusion. □

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