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ON SOME SPECTRAL RESULTS RELATING TO THE RELATIVE VALUES OF MEANS

C.E.M. PEARCE

SCHOOL OF APPLIED MATHEMATICS,
ADELAIDE UNIVERSITY,
ADELAIDE SA 5005, AUSTRALIA.
cpearce@maths.adelaide.edu.au

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ABSTRACT. In the case of two positive numbers, the geometric mean is closer to the harmonic than to the arithmetic mean. We derive some spectral results relating to corresponding properties with more than two positive numbers.

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1. INTRODUCTION

Let A, G, H denote respectively the arithmetic, geometric and harmonic means of n positive real numbers $x_1, \dots, x_n$, which are not all equal. It is well–known that $H < G < A$. Scott [3] has shown in the case $n = 2$ that G is closer to H than to A, so that

$$\frac{A-G}{A-H} > \frac{1}{2}. \tag{1.1}$$

He showed by a counterexample that this need not be the case when $n > 2$.

Subsequently Lord [1] and Pearce and Pečarič [2] addressed the question of the behaviour of the quotient

$$f_n(x_1, \dots, x_n) := \frac{A-G}{A-H}$$

in the case of general n. Several generalisations and extensions of (1.1) were obtained. The following are pertinent to the present article.

Since

$$f_n(ax_1, \dots, ax_n) = f(x_1, \dots, x_n)$$

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for $a > 0$, it suffices to consider the values taken by f_n when $\mathbf{x} = (x_1, \dots, x_n)$ lies on the intersection

$$\mathbf{K} := \left\{ x \in \mathbb{R}^n : \ x_i \geq 0 \text{ for } \ 1 \leq i \leq n \ \text{ and } \ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

of the nonnegative orthant and the surface of the unit hypersphere. The function f_n is clearly well–defined and continuous on the interior of $\mathbf{K}$ except at $\mathbf{e} := \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right)$, where it is undefined since A, G and H all coincide. In fact this singularity is removable. It is shown in [1] that defining $f_n(\mathbf{x}) = 1$ for boundary points of $\mathbf{K}$ (where some but not all values x_i vanish) and $f_n(\mathbf{e}) = \frac{1}{2}$ makes f_n continuous on the whole of $\mathbf{K}$. Since $\mathbf{K}$ is compact, f_n possesses and realises an infimum α_n. Further, the range of f_n constitutes the interval $[\alpha_n, 1]$, the sequence $(\alpha_n)_2^\infty$ is strictly decreasing to limit zero and $\alpha_n > \frac{1}{n}$ for $n \geq 3$. The seminal paper of Scott gives $\alpha_2 = \frac{1}{2}$.

In this article we continue the development of [1] and [2] and derive some striking structural results, principally as follows. In Section 2, Theorem 2.1, we show that if $\mathbf{x}$ is such that $f_n(\mathbf{x}) = \alpha_n$, then $\{x_1, \dots , x_n\}$ contains precisely two distinct values. In Section 3, Theorem 3.3, we show that if $f_n(\mathbf{x}) = \alpha_n$, then the smaller of the two distinct components of $\mathbf{x}$ must occur with multiplicity one. We conclude in Section 4 by giving characterisations of α_n and some related infima arising naturally in our analysis.

We postpone consideration of asymptotics to a subsequent article.

2. The Dichotomy Theorem

Theorem 2.1. *For $n > 2$, any set $\{x_1, \dots, x_n\}$ for which $f_n(x) = \alpha_n$ contains precisely two distinct values.*

Proof. First suppose that $S_1 := \sum_{i=1}^n x_i$ and $S_n := \prod_{i=1}^n x_i$ are fixed. Subject to these constraints, the mimimum of f_n correspond to an extremum of $\sum_{i=1}^n \frac{1}{x_i}$ and satisfies

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0 \ \text{ for } \ i = 1, \dots, n,$$

where $\mathcal{L}$ denotes the Lagrangian

$$\mathcal{L} := \sum_{i=1}^{n} \frac{1}{x_i} - \lambda \left(\sum_{i=1}^{n} x_i - S_1 \right) - \mu \left(\prod_{i=1}^{n} x_i - S_n \right).$$

Then

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{1}{x_i^2} - \lambda - \mu \prod_{j \neq i} x_j = 0 \quad (i = 1, \dots, n),$$

that is,

$$\frac{1}{x_i} + \lambda x_i + \mu S_n = 0 \quad (i = 1, \dots, n).$$

Hence each x_i must be equal to one of the two solutions of the quadratic

$$\lambda x^2 + \mu S_n x + 1 = 0.$$

For a minimum, these solutions must be distinct, since $f_n(\mathbf{e}) = \frac{1}{2}$ while $\alpha_n < \frac{1}{2}$ for $n \geq 3$. □

For $j = 1, 2, \dots, n$ and fixed $n > 2$, define

$$\mathcal{V}_j = \{\mathbf{x} : \ \{x_1, \dots, x_n\} \text{ contains precisely } j \text{ distinct values} \},$$

$$\mathcal{V}_j^* = \{f_n(\mathbf{x}) : \ \mathbf{x} \in \mathcal{V}_j\}$$

and

$$\delta_j = \inf \mathcal{V}_j^*.$$

An immediate implication of Theorem 2.1 is the following result.

Corollary 2.2. *We have*

$$\delta_2 = \delta_3 = \cdots = \delta_n = \alpha_n.$$

For $j > 1$, the set $\mathcal{V}_j^*$ contains its infimum only for $j = 2$.

Proof. If $1 \le j \le n-1$, any element of $\mathcal{V}_j$ can be approximated arbitrarily closely by elements of $\mathcal{V}_{j+1}$, but not conversely. Since $\mathbf{K}$ is compact and f_n continuous, we must therefore have that $\delta_{j+1} \le \delta_j$. Thus

$$\delta_n \le \delta_{n-1} \le \ \dots \ \le \delta_2 \le \delta_1 = \frac{1}{2}.$$

On the other hand, by Theorem 2.1

$$\delta_2 = \alpha_n = \inf \{f_n(\mathbf{x})\} = \min\{\delta_1, \delta_2, \dots, \delta_n\}.$$

The first part of the corollary follows.

The second part follows by invoking Theorem 2.1 again. □

3. COMPARISON RESULTS

In the remaining sections of the paper we examine more closely the central case when $\{x_1, \dots, x_n\}$ contains only two distinct values, that is $\mathbf{x} \in \mathcal{V}_2$. We may assume without loss of generality an ordering

$$x_1 \le x_2 \le \cdots \le x_n.$$

We decompose

$$\mathcal{V}_2 = \bigcup_{k=1}^{n-1} \mathcal{U}_k,$$

where

$$\mathcal{U}_k = \{\mathbf{x} : x_1 = x_2 = \cdots = x_k < x_{k+1} = \cdots = x_n\} \quad (1 \le k < n).$$

For $\mathbf{x} \in \mathcal{U}_k$ we have for the k equal points denoted by x and the rest by y that

$$\frac{A-G}{A-H} = \frac{\frac{k}{n}x + \left(1 - \frac{k}{n}\right) y - x^{k/n} y^{1-k/n}}{\frac{k}{n}x + \left(1 - \frac{k}{n}\right) y - n \Big/ \left(\frac{k}{x} + \frac{n-k}{y}\right)}.$$

If we set $\beta = k/n$ and $u = x/y$, this gives

$$f_n(\mathbf{x}) = \frac{\beta u + 1 - \beta - u^\beta}{\beta u + 1 - \beta - 1 \big/ \left(\frac{\beta}{u} + 1 - \beta\right)}$$

with $\beta \in \left\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\right\}$ and $0 < u < 1$.

This may be rearranged as

$$f_n(\mathbf{x}) = 1 - \frac{u}{(u-1)^2} g(u, \beta), \tag{3.1}$$

where

$$g(u, \beta) = \frac{u^\beta - 1}{\beta} + \frac{u^{-(1-\beta)} - 1}{1 - \beta}. \tag{3.2}$$

We shall find it convenient to have alternative sets of variables and functions. Set $v = u^{1/n}$. Then for $\mathbf{x} \in \mathcal{U}_k$ we put

$$h_k(n, v) = f_n(\mathbf{x}) \quad \text{and} \quad \phi_k(v) = g\left(u, \frac{k}{n}\right).$$

Proposition 3.1. *For fixed $n \geq 3$ and $v \in (0, 1)$, the sequence $(h_k(n, v))_{k=1}^{n-1}$ is strictly increasing.*

Proof. By virtue of the representation (3.1), (3.2), it suffices to prove that the sequence $(\phi_k(v))_{k=1}^{n-1}$ is strictly decreasing. To show that $\phi_k(v) > \phi_{k+1}(v)$, we need to establish the inequality

$$\frac{v^k - 1}{k} + \frac{v^{-(n-k)} - 1}{n-k} > \frac{v^{k+1} - 1}{k+1} + \frac{v^{-(n-k-1)} - 1}{n-k-1},$$

which on multiplication by v^{n-k} becomes

$$\Theta(v) < 0,$$

where Θ is the polynomial

$$\Theta(v) = \frac{v^{n+1}}{k+1} - \frac{v^n}{k} + v^{n-k}\left[\frac{1}{k} + \frac{1}{n-k} - \frac{1}{k+1} - \frac{1}{n-k-1}\right] + \frac{v}{n-k-1} - \frac{1}{n-k}. \tag{3.3}$$

Since $n + 1 > n > n - k > 1 > 0$, (3.3) expresses Θ in descending powers of v. The coefficients taken in sequence have exactly three changes in sign, regardless of whether the expression in brackets is positive, negative or zero. Hence by Descartes' rule of signs the polynomial equation

$$\Theta(w) = 0 \tag{3.4}$$

has at most three positive solutions.

Now by elementary algebra we have that

$$\Theta(1) = \Theta'(1) = \Theta''(1) = 0,$$

so that $w = 1$ is a triple zero of $\Theta(w)$. Hence $\Theta(w)$ has no zeros on $(0, 1)$ and therefore must have constant sign on $(0, 1)$. Because $\Theta(0) < 0$, we thus have $\Theta(w) < 0$ throughout $(0, 1)$ and we are done. □

For $1 \leq k < n$, put

$$\mathcal{U}_k^* = \{f_n(\mathbf{x}) : \mathbf{x} \in \mathcal{U}_k\}$$

and

$$\varepsilon_k = \inf\, \mathcal{U}_k^*.$$

Lemma 3.2. *For each $n \geq 3$ we have*

$$\varepsilon_k \begin{cases} < \dfrac{1}{2} & \text{for} \quad 1 \leq k < \dfrac{n}{2} \\[2mm] = \dfrac{1}{2} & \text{for} \quad \dfrac{n}{2} \leq k \leq n-1. \end{cases}$$

Proof. Since $v = 1$ gives $f_n = \frac{1}{2}$ and $v = 0$ gives $f_n = 1$, a necessary and sufficient condition that $\varepsilon_k < \frac{1}{2}$ is that there should exist $v \in (0,1)$ for which

$$\frac{nv^n}{(1-v^n)^2}\left[\frac{v^k-1}{k} + \frac{v^{-(n-k)}-1}{n-k}\right] > \frac{1}{2}$$

or

$$\Omega(v) < 0, \tag{3.5}$$

where

$$\begin{aligned}\Omega(v) &= v^{2n} - \frac{2n}{k}v^{n+k} + 2v^n\left[\frac{n}{k} + \frac{n}{n-k} - 1\right] - v^k\frac{2n}{n-k} + 1\\ &= v^{2n} - \frac{2n}{k}v^{n+k} + 2v^n\left[\frac{n}{k} + \frac{k}{n-k}\right] - v^k\frac{2n}{n-k} + 1.\end{aligned}$$

The polynomial Ω has four changes of sign in its coefficients, and so has at most four positive zeros. We may verify readily that

$$\Omega(1) = \Omega'(1) = \Omega''(1) = 0, \tag{3.6}$$

while

$$\Omega'''(1) = 2n^2(n-2k). \tag{3.7}$$

If $\frac{n}{2} < k \le n-1$, then $\Omega(v)$ has a triple zero at $v = 1$ and so can have at most one zero on $(0,1)$. Since $\Omega(0) > 0$, condition (3.5) can thus be satisfied if and only if there is such a zero, in which case $\Omega(1-\Delta) < 0$ for all $\Delta > 0$ sufficiently small. But by Taylor's theorem

$$\begin{aligned}\Omega(1-\Delta) &= \Omega(1) - \Delta\Omega'(1) + \frac{\Delta^2}{2!}\Omega''(1) - \frac{\Delta^3}{3!}\Omega'''(1) + 0(\Delta^4)\\ &\approx -\frac{\Delta^3}{3}n^2(n-2k), \end{aligned}\tag{3.8}$$

which is positive.

Hence we must have $\varepsilon_k \ge \frac{1}{2}$. But since $\mathbf{e}$ can be approximated arbitrarily closely by elements of $\mathcal{U}_k$ by letting $v \to 1$, we must have $\varepsilon_k \le f_n(\mathbf{e}) = \frac{1}{2}$. Thus $\varepsilon_k = \frac{1}{2}$.

If $k = \frac{n}{2}$, then $\Omega(v)$ has exactly four positive zeros, all at $v = 1$, so Ω has constant sign on $(0,1)$. Since $\Omega(0) > 0$, we thus have $\Omega(v) > 0$ on $(0,1)$. Arguing as in the previous paragraph, we derive again that $\varepsilon_k = \frac{1}{2}$.

Finally, if $k < \frac{n}{2}$, we have by (3.8) that $\Omega(1-\Delta) < 0$ for $\Delta > 0$ sufficiently small, so that condition (3.5) is satisfied. This completes the proof. □

Theorem 3.3. *The sequence* $(\varepsilon_k)_{1\le k<\frac{n}{2}}$ *is strictly increasing.*

Proof. The desired result is equivalent to $(\xi_k)_{1\le k<\frac{n}{2}}$ being strictly decreasing, where

$$\xi_k = \sup_{u\in(0,1)} \frac{u}{(1-u)^2}\phi_k(u) = 1 - \varepsilon_k.$$

By Proposition 3.1,

$$\frac{u}{(1-u)^2}\phi_k(u) > \frac{u}{(1-u)^2}\phi_{k+1}(u)$$

for each $u \in (0,1)$, so that

$$\xi_k \ge \xi_{k+1} \quad \text{for} \quad 1 \le k \le n-1.$$

Further, ξ_k is realised for some choice of u, for $u = u_k$, say, and arguing as in Lemma 3.2 we must have $u_k \in (0,1)$ for $1 \le k < \frac{n}{2}$.

To show the inequalities are strict, suppose if possible that equality holds for some value of k, so that

$$\frac{u_k}{(1-u_k)^2}\phi_k(u_k) = \frac{u_{k+1}}{(1-u_{k+1})^2}\phi_{k+1}(u_{k+1}). \tag{3.9}$$

By Proposition 3.1,

$$\frac{u_{k+1}}{(1-u_{k+1})^2}\phi_k(u_{k+1}) > \frac{u_{k+1}}{(1-u_{k+1})^2}\phi_{k+1}(u_{k+1}),$$

so that by (3.9)

$$\frac{u_{k+1}}{(1-u_{k+1})^2}\phi_k(u_{k+1}) > \frac{u_k}{(1-u_k)^2}\phi_k(u_k) = \xi_k,$$

contradicting the definition of ξ_k. □

4. CHARACTERISATION OF ε_k

In the previous section we saw that for $1 \le k < \frac{n}{2}$ the supremum ξ_k is realised for some $u = u_k \in (0,1)$. We now consider the determination of u_k. For convenience we again employ $v_k = u_k^{1/n}$.

Theorem 4.1. (i) *For $1 \le k < \frac{n}{2}$, $v = v_k$ is the unique solution on $(0,1)$ of the equation*

$$\Phi_k(v) = 0, \tag{4.1}$$

where

$$\Phi_k(v) = (v^n - 1)\left[v^n \frac{n+k}{k} - v^{n-k}\left(\frac{n}{k} + \frac{n}{n-k}\right) + \frac{k}{n-k}\right] - 2nv^n\left[\frac{v^n}{k} - v^{n-k}\left(\frac{1}{k} + \frac{1}{n-k}\right) + \frac{1}{n-k}\right].$$

(ii) *If $v \in (0,1)$, then $v < v_k$ or $v > v_k$ according as $\Phi_k(v) < 0$ or $\Phi_k(v) > 0$.*

Proof. Since f_n achieves a minimum at $v = v_k \in (0,1)$, we have that

$$\frac{d}{dv}\left\{\frac{nv^n}{(v^n-1)^2} \cdot \left[\frac{v^k - 1}{k} + \frac{v^{-(n-k)} - 1}{n-k}\right]\right\} = 0$$

for $v = v_k$, this value of v corresponding to a local maximum of the differentiated expression. The left–hand side is the quotient of

$$n(v^n-1)^2\left[v^{n+k-1}\frac{n+k}{k} - v^{n-1}\left(\frac{n}{k} + \frac{n}{n-k}\right) + v^{k-1}\frac{k}{n-k}\right] - 2n^2(v^n-1)^2 v^{n-1}\left[\frac{v^{n+k}}{k} - v^n\left(\frac{1}{k} + \frac{1}{n-k}\right) + \frac{v^k}{n-k}\right]$$

by $(v^n-1)^4$. Removing this denominator and the factor $n(v^n-1)v^{k-1}$ from the numerator gives that $v = v_k$ satisfies (4.1). Statement (i) will therefore follow if it can be shown that (4.1) has a unique solution on $(0,1)$. Uniqueness gives that the differentiated expression has positive gradient for $v < v_k$ and negative gradient for $v > v_k$. Statement (ii) will then follow, since the term cancelled is negative.

It therefore remains only to show that $\Phi_k(v)$ has a unique zero on $(0,1)$. This we do as follows. The polynomial $\Phi_k(v)$ may be written in descending powers of v as

$$-v^{2n}\frac{n-k}{k}+v^{2n-k}\left(\frac{n}{k}+\frac{n}{n-k}\right)-v^{n}\left(\frac{2n-k}{n-k}+\frac{n+k}{k}\right)+v^{n-k}\left(\frac{n}{k}+\frac{n}{n-k}\right)-\frac{k}{n-k},$$

the coefficients of which exhibit four changes of sign. Hence by Descartes' rule of signs, $\Phi_k(v)$ has at most four positive zeros.

By elementary algebra,

$$\Phi_k(1)=\Phi_k'(1)=\Phi_k''(1)=0,\ \Phi_k'''(1)=n^2(2k-n), \tag{4.2}$$

so that $\Phi_k(v)$ has a triple zero at $v=1$. Hence $\Phi_k(v)$ has at most one zero on $(0,1)$.

Now $\Phi_k(v)<0$ and for $\Delta>0$ small

$$\Phi_k(1-\Delta)=-\frac{\Delta^3}{3!}\Phi_k'''(1)+0(\Delta^4)>0,$$

by Taylor's theorem and (4.2). Hence $\Phi_k(v)$ has a zero on $(0,1)$ and this must be unique. □

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